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Generalization of Abel's theorem and some finiteness property of zero-cycles on surfaces

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Introduction

The starting point of the theory of non-singular projective curves is the following well-known theorem of Abel:

Let C be a non-singular projective curve, and $J(C)$ be its jacobian

$$J(C) = H^0(C, \Omega_C^1)\backslash H^1(C, \mathbb{C})/H^1(C, \mathbb{Z}) = (H^0(C, \Omega_C^1))^*/H^1(C, \mathbb{Z}),$$

which is an abelian variety. To a divisor D of degree zero on C is associated a point $\gamma(D)$ of the jacobian by integration, and so-called Abel's theorem states that the image of D in the jacobian vanishes if and only if D is rationally equivalent to zero, i.e., D is a divisor of a rational function of C, in other words, it gives an algebraic condition for $\gamma(D) = 0$, whereas the jacobian is defined complex-analytically.

We put $\mathrm{Pic}^0\, C$ = the divisors of degree 0 on C modulo rational equivalence. We have then an injection $\mathrm{Pic}^0\, C \to J(C)$ and the map is bijection (so called Jacobi's inversion problem). Let C' be another curve, and Z be an (algebraic) 1-cycle on the product $C \times C'$. The cycle Z induces a map $\mathrm{Pic}^0\, C \to \mathrm{Pic}^0\, C'$, a correspondence, together with a homomorphism of abelian varieties $J(C) \to J(C')$. The category of direct sums of $\mathrm{Pic}^0\, C$ with the direct sums of the above maps as morphisms is, therefore, equivalent to the subcategory of the category of abelian varieties, consisting of direct sums of jacobian varieties. These are additive categories, and for each of these, we consider the category tensored with $\mathbb{Q}$, i.e. the category having the same ones as objects and for sets of morphisms those tensored with $\mathbb{Q}$, and take its pseudo-abelian envelope, i.e., the category for objects added formally the direct summands. We denote by $\mathscr{C}(1)_{\mathrm{curve}}$ the category thus obtained from that of $\mathrm{Pic}^0\, C$. The category obtained from that of jacobians is no other than that of abelian varieties up to isogeny, which is equivalent to the category $\mathrm{Hdg}(1)$ of polarizable $\mathbb{Q}$-Hodge structures of weight 1. Thus, $\mathscr{C}(1)_{\mathrm{curve}}$ is equivalent to the category $\mathrm{Hdg}(1)$.

The following conjecture of Bloch [5] can be regarded as the weight 2 counterparts of the above equivalence:

For any smooth projective variety V over $\mathbb{C}$, there exists a filtration on the Chow group of 0-cycles $\mathrm{CH}_0(V)$, at least the beginning of which is given by

$$F^1\mathrm{CH}_0(V) = \mathrm{Ker}(\deg : \mathrm{CH}_0(V) \to Z),$$
$$F^2\mathrm{CH}_0(V) = \mathrm{Ker}(F^1\mathrm{CH}_0(V) \to \mathrm{Alb}(V)).$$

Let S be a surface and let z be a cycle on $V \times S$ with $\dim z = m = \dim V$. Then z induces a map

$$z : \mathrm{CH}_0(V) \to \mathrm{CH}_0(S).$$

The above filtration being functorial for correspondences, we get also

$$[z] : \mathrm{gr}^{\cdot}\mathrm{CH}_0(V) \to \mathrm{gr}^{\cdot}\mathrm{CH}_0(S).$$

CONJECTURE ([5], 1.8). The map [z] depends only upon the cohomology class $\{z\} \in H^4(V \times S)$.

Moreover,

METACONJECTURE ([5], 1.10). There is an equivalence of category between a suitable category of polarized Hodge structures of weight 2 and a category built up from $\mathrm{gr}^2\mathrm{CH}_0(S)$.

The aim of this article is two-fold: to give a condition for the vanishing of cycles in the intermediate jacobian, and to construct filtrations on the Chow groups which satisfy the above conjectures.

The basic notion we introduce is that of *product* of adequate equivalence relations. An adequate equivalence relation E consists of subgroups $\mathrm{ECH}(V)$ of the Chow ring $\mathrm{CH}(V)$ which are stable under the correspondences. For adequate equivalence relations E, E', the product denoted by $E * E'$ is the minimum adequate equivalence relation satisfying the condition

$$x \in E\mathrm{CH}(W), \quad \text{and} \quad y \in E'\mathrm{CH}(V) \Rightarrow x \times y \in E * E'\mathrm{CH}(W \times V).$$

As a trivial but useful consequence, the filtration H^{*l} given by powers of homological equivalence relation H has the following property (cf. the above conjecture):

If we denote its associate graded by $\mathrm{gr}^l_H\mathrm{CH}(V)$, the map induced by an algebraic cycle $z \in \mathrm{CH}^{p+q}(W \times V)$

$$[z]: \mathrm{gr}_H^l \mathrm{CH}_q(W) \to \mathrm{gr}_H^l \mathrm{CH}^p(V)$$

depends only on the homology class of z.

We state our main results:

Let A denote the algebraic equivalence relation, and $J_a^p(V)$ the algebraic part of the p-th intermediate jacobian. We have the Abel-Jacobi map $\mathrm{ACH}^p(V) \to J_a^p(V)$. Then,

THEOREM ($=6.2+6.4$. Cf. [16], p. 534.). *The kernel of the Abel-Jacobi map is, up to finite group, equal to the product of algebraic equivalence and homological equivalence*:

$$A * \mathrm{HCH}^p(V) = \mathrm{Ker}(\mathrm{ACH}^p(V) \to J_a^p(V)) \quad (\textit{up to finite group}).$$

Moreover these coincide precisely for $p = 1, 2, \dim V$.

The significance of the theorem is that the left hand side is nothing to do with the intermediate jacobian. (Note also that homological equivalence is defined algebraically by virtue of etale cohomology.) One might hope that $J_a^p(V)$ can be constructed as was done for Picard variety.

To state the second of our main results, first, consider the additive subcategory of the category of abelian groups whose objects are direct sum of $\mathrm{gr}_H^2 \mathrm{CH}_0(S)$ for S surfaces and whose morphisms from $\mathrm{gr}_H^2 \mathrm{CH}_0(S)$ to $\mathrm{gr}_H^2 \mathrm{CH}_0(S')$ are induced by algebraic cycles $z \in \mathrm{CH}^2(S \times S')$. We then get $\mathbb{Q}$-additive category having the same objects as above and morphisms tensored with rationals $\mathbb{Q}$. We denote its pseudo-abelian envelope by $\mathscr{C}(2)_{\mathrm{surf}}$.

On the other hand, for a surface S, we denote by $\mathrm{gr}^0 H^2(S, \mathbb{Q})$ the quotient of $H^2(S, \mathbb{Q})$ by its (rational) Neron-Severi group. The group $\mathrm{gr}^0 H^2(S, \mathbb{Q})$ has a Hodge structure of weight 2. Then we proceed as above: we consider the subcategory of the category of polarizable $\mathbb{Q}$-Hodge structures of weight 2 whose objects are direct sums of $\mathrm{gr}^0 H^2(S, \mathbb{Q})$ and whose morphisms are induced by (rational) algebraic cycles. We denote by $\mathscr{M}_2$ its pseudo-abelian envelope. In fact, we can define the category of motives for *surfaces* as planned by Grothendieck (see [10]) and there are motives corresponding to $\mathrm{gr}^0 H^2(S, \mathbb{Q})$. The category of direct summands of these objects are equivalent to $\mathscr{M}_2$ and is semi-simple and abelian. As the metaconjecture part of our results,

THEOREM ($=7.5$). *We can define the functor* $\mathrm{gr}_H^2 \mathrm{CH}_0(S) \mapsto \mathrm{gr}^0 H^2(S, \mathbb{Q})$ *which gives an (anti-)equivalence of the categories* $\mathscr{C}(2)_{\mathrm{surf}}$ *and* $\mathscr{M}_2$. *In particular, the category* $\mathscr{M}_2$ *is semi-simple abelian.*

We shall explain the organization.

In section 1, we introduce the notion of product of adequate equivalence relations explained above. The readers who are interested only in generalization of Abel's theorem can proceed directly to section 6. (For notation, however, see §5.1.)

Sections 2 and 3 are preliminaries: in section 2, we define the fundamental class (or cohomology class) of families of subschemes, following [15], and prove that the subfunctor of product of Hilbert schemes corresponding to the pairs of subschemes having the same fundamental classes is representable. Section 3 is concerning the Chow schemes by [1], i.e., families of cycles on a scheme over a base scheme (of characteristic zero), and we show that the direct image morphism for a proper morphism is defined on the whole of the Chow scheme, when we add the cycle "zero" to the Chow scheme.

In section 4, we show that on a smooth projective variety over an algebraically closed *uncountable* field of characteristic zero, for a family of cycles $\{Z(s)\}_{s\in S}$, if at each closed point s, $Z(s)$ is equivalent to zero with respect to a power of homological equivalence, so is generically. This is an analogue of [11], 5.6.

From section 5 on, the ground field is assumed to be the field of complex numbers.

In section 5, we generalize the theorem 3.2 of [12], which, in particular, says, in Severi's terminology [14], that a family of 0-cycles on a surface in a class of cube of homological equivalence is *a circolazione algebrica.* Further, we introduce an additive category $\mathscr{C}(l)$ constructed from the powers of homological equivalence and define a functor from $\mathscr{C}(l)$ to the category $\mathrm{Hdg}(l)$ of effective polarizable $\mathbb{Q}$-Hodge structures of weight l. We also prove that $\mathrm{gr}_H^2\mathrm{CH}_0$ of smooth projective varieties are objects of $\mathscr{C}(2)$.

In section 6, we prove above-mentioned generalization of Abel's theorem, and section 7 is devoted to the proof of the metaconjecture.

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1. Products of adequate equivalence relations

1.1. Let k be an algebraically closed field, and we work in the category of smooth projective varieties. First recall the definition of adequate equivalence relation.

DEFINITION 1.1.1 ([13]). An adequate equivalence relation E is an equivalence relation on cycles such that

(i) it is compatible with addition of cycles;
(ii) Let X be a cycle on V, and $W_1, \ldots, W_k$ a finite number of subvarieties on V. Then there exists a cycle X' equivalent to X such that X' and W intersect properly;
(iii) If Z is a cycle on $V \times W$, if X is a cycle on V equivalent to zero, and if $Z(X) = \mathrm{pr}_{W*}(Z \cdot X \times W)$ is defined, then the cycle $Z(X)$ on W is equivalent to zero.

1.1.2. It is well-known that the rational equivalence relation, which we denote by 0, is the finest adequate equivalence relation and the numerical equivalence relation is the non-trivial coarsest one. We denote the trivial adequate equivalence relation that all cycles are equivalent by I. The cycles on V modulo rational equivalence is called the Chow ring $\mathrm{CH}(V)$ of V and it has a ring structure by intersection, and is graded by codimension. The codimension p part will be denoted by $\mathrm{CH}^p(V)$.

1.2. Let E be an adequate equivalence relation and $E\mathrm{CH}(V) := \{$cycles on V E-equivalent to zero$\}$/rational equivalence. Then $E\mathrm{CH}(V)$ has the following properties:

(i) $E\mathrm{CH}(V)$ is a graded submodule of $\mathrm{CH}(V)$;
(ii) IF $x \in E\mathrm{CH}(V)$ and if $z \in \mathrm{CH}(V \times W)$, then

$$z(x) := \mathrm{pr}_{W_*}(z \cdot x \times 1) \in E\mathrm{CH}(W).$$

PROPOSITION 1.2.1. *Giving an adequate equivalence relation E is equivalent to assigning* $E\mathrm{CH}(V) \subset \mathrm{CH}(V)$ *to each V which satisfies the condition* (i) *and* (ii) *of* 1.2.

Let E and E' be adequate equivalence relations. Then we define the adequate equivalence relations $E + E'$, $E \cap E'$ by

$$(E + E')\mathrm{CH}(V) := E\mathrm{CH}(V) + E'\mathrm{CH}(V),$$

$$(E \cap E')\mathrm{CH}(V) := E\mathrm{CH}(V) \cap E'\mathrm{CH}(V).$$

We shall denote $E \subset E'$ if $E\mathrm{CH}(V) \subset E'\mathrm{CH}(V)$ for all V.

1.3. For adequate equivalence relations E and E', we shall define their *product* denoted by $E * E'$ as follows:

$(E * E')\mathrm{CH}(V)$ is a submodule of $\mathrm{CH}(V)$ generated by the elements of the form $\mathrm{pr}_{V_*}(x \cdot y)$, where $x \in E\mathrm{CH}(T \times V)$, $y \in E'\mathrm{CH}(T \times V)$, T is a (smooth projective) variety, $\mathrm{pr}_V \colon T \times V \to V$ is the projection.

LEMMA 1.3.1. *$E * E'$ satisfies the conditions of* 1.2. (i), (ii) *and hence defines an adequate equivalence relation. A cycle Z on V is $E * E'$-equivalent to zero if and* *only if Z is a sum of cycles of the forms $\mathrm{pr}_{V*}(X \cdot Y)$, where X is a cycle on $T \times V$ E-equivalent to zero and Y is a cycle on $T \times V$ E'-equivalent to zero and the cycles X and Y intersect properly, and where T is a variety and $\mathrm{pr}_V: T \times V \to V$ is the projection.*

By linearity, it is sufficient to show that if $z \in \mathrm{CH}(V \times W)$ and $x \in E\mathrm{CH}(V \times T)$ and $y \in E'\mathrm{CH}(V \times T)$, then

$$z(\mathrm{pr}_V(x \cdot y)) \in (E * E')\mathrm{CH}(W).$$

$$\begin{aligned} z(\mathrm{pr}_V(x \cdot y)) &= \mathrm{pr}_W(z \cdot \mathrm{pr}_V(x \cdot y) \times 1_W) \\ &= \mathrm{pr}_W(1_T \times z \cdot x \times 1_W \cdot y \times 1_W) \end{aligned}$$

and $1_T \times z \cdot x \times 1_W \in E\mathrm{CH}(T \times V \times W)$, $y \times 1_W \in E'\mathrm{CH}(T \times V \times W)$. The latter part results from the moving lemma.

LEMMA 1.4. *Let E, E', E'' be adequate equivalence relations. Then the following are equivalent:*

(i) *if $x \in E\mathrm{CH}(V)$ and $y \in E'\mathrm{CH}(V)$, then $x \cdot y \in E''\mathrm{CH}(V)$ for arbitrary V.*
(ii) *if $x \in E\mathrm{CH}(V)$ and $y \in E'\mathrm{CH}(W)$ then $x \times y \in E''\mathrm{CH}(V \times W)$ for arbitrary V and W.*
(iii) $E * E' \subset E''$.

It is clear that (i) implies (iii) and (ii) implies (i). To see that (iii) implies (ii), let $T := \mathrm{Spec}\, k$. Then

$$1_T \times x \times 1_W \in E\mathrm{CH}(T \times V \times W),\ 1_T \times 1_V \times y \in E'\mathrm{CH}(T \times V \times W),$$

and

$$x \times y = \mathrm{pr}_{V \times W}(1_T \times x \times 1_W \cdot 1_T \times 1_V \times y).$$

1.5. For adequate equivalence relations E, E', E'', we have

$$(E + E') + E'' = E + (E' + E''),$$

$$(E * E') * E'' = E * (E' * E''),$$

$$E + E' = E' + E, \qquad E * E' = E' * E,$$

$$E + O = E, \qquad E * I = E$$

$$E * (E' + E'') = E * E' + E * E'',$$

$$E' \subset E'' \quad \text{implies} \quad E * E' \subset E * E''.$$

Let $E^{*l} := E * \cdots * E$ (l times for $l > 0$, and $E^{*0} = I$), and set

$$\mathrm{gr}_E^l \mathrm{CH}(V) := E^{*l}\mathrm{CH}(V)/E^{*(l+1)}\mathrm{CH}(V).$$

By virtue of lemma 1.4, we have

LEMMA 1.5.1. *The ring structure of* CH(V) *defines the bigraded ring structure on* $\mathrm{gr}_E^{\cdot}\mathrm{CH}^{\cdot}(V)$. *In particular,* $z \in \mathrm{CH}^{p+q}(V \times W)$ *defines the map*

$$[z]: \mathrm{gr}_E^l \mathrm{CH}_q(V) \to \mathrm{gr}_E^l \mathrm{CH}^p(W),\ x \mapsto z(x)$$

and it depends only on the class of z *in* $\mathrm{gr}_E^0 \mathrm{CH}(V \times W)$.

REMARKS 1.6.1. Let E, E' be adequate equivalence relations. Then $z \in \mathrm{CH}(V)$ is in $(E * E')\mathrm{CH}(V)$ if and only if there exists a finite number of $x_1, \ldots, x_k \in E\mathrm{CH}(T \times V)$ and $y_1, \ldots, y_k \in E'\mathrm{CH}(T \times V)$ such that

$$z = \sum \mathrm{pr}_{V*}(x_i \cdot y_i).$$

In fact, the following formula shows that we can take the variety T common to all of terms in the sum: if $x \in E\mathrm{CH}(T \times V)$ and $y \in E'\mathrm{CH}(T \times V)$, then for any variety T', and a point t' of T', we have

$$\mathrm{pr}_{V*}(x \cdot y) = \mathrm{pr}'_{V*}(t' \times x \cdot 1_{T'} \times y),$$

where $\mathrm{pr}'_V: T' \times T \times V \to V$ is the projection, and

$$t' \times x \in E\mathrm{CH}(T' \times T \times V)$$

and

$$1_{T'} \times y \in E'\mathrm{CH}(T' \times T \times V).$$

1.6.2. More generally, let $E_1, \ldots, E_l$ be adequate equivalence relations and Z a cycle of codimension p on V. Then Z is $(E_1 * \cdots * E_l)$-equivalent to zero if and only if there exist a variety W, a (projective) morphism $f: W \to V$, cycles X_{ij} of codimension p_{ij} on W, E_i-equivalent to zero ($1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k$) such that $X_{i1}, \ldots, X_{ik}$ intersect properly on W,

$$\sum_j p_{ij} = p - \dim V + \dim W \quad \text{for all } i,$$

and that

$$Z = \sum_j f_*(X_{1j} \cdot \cdots \cdot X_{lj}).$$

By 1.4, it is clear that Z is $(E_1 * \cdots * E_l)$-equivalent to zero. To see the converse, by induction, it suffices to consider the case $l = 3$. Let $u \in (E_1 * E_2)\mathrm{CH}(T \times V)$, $v \in E_3\mathrm{CH}(T \times V)$. By linearity, we may assume that $u = \mathrm{pr}_{T \times V*}(x \cdot y)$, where $x \in E_1\mathrm{CH}(T' \times T \times V)$ and $y \in E_2\mathrm{CH}(T' \times T \times V)$. Then,

$$\begin{aligned} \mathrm{pr}_{V_*}(u \cdot v) &= \mathrm{pr}_{V_*}(\mathrm{pr}_{T \times V_*}(x \cdot y) \cdot v) \\ &= \mathrm{pr}'_{V_*}(x \cdot y \cdot 1_{T'} \times v), \end{aligned}$$

where $\mathrm{pr}'_V \colon T' \times T \times V \to V$ is the projection and $1_{T'} \times v \in E_3\mathrm{CH}(T' \times T \times V)$.

1.7. Let E be an adequate equivalence relation. We *define* the adequate equivalence relation $\langle E \rangle_0$ as the equivalence relation generated by 0-cycles E-equivalent to zero. More precisely,

$$\langle E \rangle_0 \mathrm{CH}(V) = \sum z(\mathrm{CH}_0(T)),$$

where T runs over all smooth projective varieties, and z runs over the cycles on $T \times V$. It is clear that $\langle E \rangle_0 \mathrm{CH}(V)$ defines an adequate equivalence relation.

LEMMA 1.7.1. *Let E, E' be adequate equivalence relations.*

(i) $\langle E \rangle_0 \subset E$, *and* $\langle E' \rangle_0 \subset E$ *if and only if* $E'\mathrm{CH}_0(V) \subset E\,\mathrm{CH}_0(V)$ *for every variety V.*
(ii) $\langle E \rangle_0 * \langle E' \rangle_0 \subset \langle E * E' \rangle_0$.

Proof. (i) is trivial, and (ii) follows from the formula

$$z(x) \times z'(x') = (z \times z')(x \times x')$$

for $z \in \mathrm{CH}(V \times T)$, $x \in E\,\mathrm{CH}_0(T)$, $z' \in \mathrm{CH}(V' \times T')$, and $x' \in E'\mathrm{CH}_0(T')$.

EXAMPLE 1.8. We work in the category of varieties over the complex numbers $\mathbb{C}$. We denote by $H_{\mathbb{Q}}$ the $\mathbb{Q}$-homological equivalence in $H^{\bullet}(V, \mathbb{Q})$ and $H = H_{\mathbb{Z}}$ the homological equivalence in $H^{\bullet}(V, \mathbb{Z})$, which are both adequate equivalence relations. We have a filtration of CH by powers of H:

$$I = H^{*0} \supset H = H^{*1} \supset H^{*2} \supset H^{*3} \supset \cdots \supset H^{*l} \supset H^{*(l+1)} \supset \cdots. \tag{1.8.1}$$

We set

$$\mathrm{Gr}^l\mathrm{CH}(V) = \langle H^{*l} \rangle_0 \mathrm{CH}(V) / (\langle H^{*l} \rangle_0 \cap H^{*(l+1)})\mathrm{CH}(V) \tag{1.8.2}$$

By 1.7.1, $\mathrm{Gr}^{\bullet}\,\mathrm{CH}^{\bullet}(V)$ has a bigraded ring structure, and for $z \in \mathrm{CH}(T \times V)$, the induced map

$$[z]: \mathrm{Gr}^l\mathrm{CH}(T) \to \mathrm{Gr}^l\mathrm{CH}(V)$$

depends only on the cohomology class of z. For 0-cycles, notice that $\mathrm{Gr}^l\mathrm{CH}_0(V)$ is the associated graded to the filtration 1.8.1.

EXAMPLE 1.9. Let $ACH(V)$ denote the classes of cycles which are algebraically equivalent to zero. Then $ACH(V)$ defines an adequate equivalence relation, and A^{*l} is nothing but the l-cubic equivalence relation [13]. Note that $A = \langle H \rangle_0 = \langle H_{\mathbb{Q}} \rangle_0$.

LEMMA 1.10. *Let E and E' be adequate equivalence relations, and assume that $E'\mathrm{CH}(V)$ are divisible for all V. Then $E * E'\mathrm{CH}(V)$ are also divisible. In particular, $A * ECH(V)$ is divisible for each smooth projective variety V.*

EXAMPLE 1.11. Let $T^p(V)$ denote the Griffiths intermediate jacobian; we have the Abel-Jacobi map

$$c^p: HCH^p(V) \to T^p(V)$$

and the image of the restriction to $ACH^p(V)$ is, by definition, $J^p_a(V)$. For $z \in \mathrm{CH}^{p+q}(W \times V)$, the diagram

$$\begin{array}{ccc}
HCH_q(W) & \xrightarrow[z(?)]{} & HCH^p(V) \\
\downarrow & & \downarrow \\
T_q(W) & \longrightarrow & T^p(V)
\end{array}$$

commutes, where the map below is induced by the fundamental class $\{z\} \in H^{2p+2q}(W \times V, \mathbb{Z})$. It follows that

$$\bar{J}CH^p(V) = \mathrm{Ker}(HCH^p(V) \to T^p(V))$$
$$JCH^p(V) = \mathrm{Ker}(ACH^p(V) \to J^p_a(V))$$

define adequate equivalence relations $\bar{J}$ and J. We have $J = \bar{J} \cap A$. It also follows from the diagram above that $c^p(H * H_{\mathbb{Q}}CH^p(V)) = 0$, which shows that $H^{*2} \subset H_{\mathbb{Q}} * H \subset \bar{J}$. In particular,

$$\langle H^{*2} \rangle_0 \subset \langle H_{\mathbb{Q}} * H \rangle_0 \subset \langle \bar{J} \rangle_0 \subset \bar{J} \cap \langle H \rangle_0 = \bar{J} \cap A = J.$$

Hence we have a surjective canonical map

$$\gamma^p: \mathrm{Gr}^1 CH^p(V) \to J_a^p(V).$$

For $p = 1$, $JCH^1(V) = \bar{J}CH^1(V) = 0$, hence $H^{*2}CH^1(V) = 0$, and we have a bijection

$$\gamma^1: \mathrm{Gr}^1 CH^1(V) \to J_a^1(V) = \mathrm{Pic}^0(V).$$

2. Fundamental classes for Hilbert scheme

2.1. Let S be a locally noetherian scheme and $f: X \to S$ be a compactifiable morphism, F an etale sheaf on S. For an integer n, we define

$$H_n(X/S, F) := H^0(S, R^{-n}f_*Rf^!F).$$

If $g: Y \to S$ is a compactifiable morphism and $h: X \to Y$ is a proper S-morphism, we have

$$h_*: H_n(X/S, F) \to H_n(Y/S, F)$$

induced by adjunction $Rf_*Rf^!F = Rg_*Rh_*Rh^!Rg^!F \to Rg_*Rg^!F$. It is clear that $h \mapsto h_*$ is functorial. For a morphism $\varphi: S' \to S$, we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[\varphi]{} & S. \end{array}$$

Then we obtain $\varphi^*: H_n(X/S, F) \to H_n(X'/S', \varphi^*F)$, i.e.,

$$H^0(S, R^{-n}f_*Rf^!F) \to H^0(S, \varphi_*R^{-n}f'_*Rf'^!\varphi^*F) \text{ as follows:}$$

By [3], 2.3.1, we have $\varphi'^*Rf^!F \to Rf'^!\varphi^*F$, or $Rf^!F \to R\varphi'_*Rf'^!\varphi^*F$. Applying Rf_*, we get

$$Rf_*Rf^!F \to Rf_*R\varphi'_*Rf'^!\varphi^*F = R\varphi_*Rf'_*Rf'^!\varphi^*F.$$

By Leray spectral sequence, we obtain

$$R^{-n}f_*Rf^!F \to \varphi_* R^{-n}f'_*Rf'^!\varphi^*F.$$

The following diagram is commutative:

$$\begin{tikzcd} H_n(X/S, F) \arrow[r, "h_*"] \arrow[d, "\varphi^*"'] & H_n(Y/S, F) \arrow[d, "\varphi^*"] \\ H_n(X'/S', \varphi^*F) \arrow[r, "h'_*"'] & H_n(Y'/S', \varphi^*F). \end{tikzcd}$$

2.2. Let $g: Z \to S$ be a flat morphism of pure relative dimension r, and e be a prime integer invertible in S. By definition,

$$H_{2r}(Z/S, \mathbb{Z}_e(-r)) = H^0(S, R^{-2r}g_*Rg^!\mathbb{Z}_e(-r)).$$

On the other hand, we have

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}_e, R^{-2r}g_*Rg^!\mathbb{Z}_e(-r)) &= \mathrm{Hom}(\mathbb{Z}_e, Rg_*Rg^!\mathbb{Z}_e(-r)[-2r]) \\ &= \mathrm{Hom}(g^*\mathbb{Z}_e, Rg^!\mathbb{Z}_e(-r)[-2r]) \\ &= \mathrm{Hom}(Rg_!g^*\mathbb{Z}_e(r)[2r], \mathbb{Z}_e) \\ &= \mathrm{Hom}(R^{2r}g_!g^*\mathbb{Z}_e(r), \mathbb{Z}_e). \end{aligned}$$

We have the trace map ([3], §2)

$$\mathrm{Tr}_g: R^{2r}g_!g^*\mathbb{Z}_e(r) \to \mathbb{Z}_e,$$

hence the corresponding map $\mathrm{Tr}'_g: \mathbb{Z}_e \to R^{-2r}g_*Rg^!\mathbb{Z}_e(-r)$. Therefore we get $H^0(S, \mathbb{Z}_e) \to H_{2r}(Z/S, \mathbb{Z}_e(-r))$. Suppose Z is a closed subscheme of X over $S: j: Z \hookrightarrow X$. The image of $1 \in H^0(S, \mathbb{Z}_e)$ by

$$H^0(S, \mathbb{Z}_e) \to H_{2r}(Z/S, \mathbb{Z}_e(-r)) \xrightarrow{j_*} H_{2r}(X/S, \mathbb{Z}_e(-r))$$

will be called the *fundamental class of* Z/S and denoted by $\{Z/S\}$. For $\varphi: S' \to S$, and $Z' = Z \times_S S'$, the base-change of Z, we have

$$\varphi^*\{Z/S\} = \{Z'/S'\} \in H_{2r}(X'/S', \mathbb{Z}_e(-r)).$$

If X is smooth of pure relative dimension m over S, denoting $p = m - r$, we have

$$\begin{aligned}\{Z/S\} \in H_{2r}(X/S, \mathbb{Z}_e(-r)) &= H^0(S, R^{-2r}f_*Rf^!\mathbb{Z}_e(-r)) \\ &= H^0(S, R^{-2r}f_*\mathbb{Z}_e(p)[2m]) \\ &= H^0(S, R^{2p}f_*\mathbb{Z}_e(p)).\end{aligned}$$

2.3. Suppose X is smooth projective over S of pure relative dimension m, and let $\mathrm{Hilb}_r(X/S)$ denote the set of subschemes flat of pure relative dimension r over S.
We have

$$\{\ /S\}: \mathrm{Hilb}_r(X/S) \to H^0(S, R^{2p}f_*\mathbb{Z}_e(p)),\ Z \mapsto \{Z/S\},$$

and set

$$\mathrm{Hilb}_r(X/S)^{\times 2,H} = \mathrm{Ker}(\mathrm{Hilb}_r(X/S) \times \mathrm{Hilb}_r(X/S) \underset{\{/S\}\mathrm{pr}_2}{\overset{\{/S\}\mathrm{pr}_1}{\rightrightarrows}} H^0(S, R^{2p}f_*\mathbb{Z}_e(p)).$$

It is clear that, for $\varphi: S' \to S$,

$$\varphi^*: \mathrm{Hilb}_r(X/S)^{\times 2,H} \to \mathrm{Hilb}_r(X'/S)^{\times 2,H},\ (Z_1, Z_2) \mapsto (Z'_1, Z'_2)$$

defines a function $\mathrm{Hilb}_{X/S,r}^{\times 2,H}$ on locally noetherian schemes over S.

PROPOSITION 2.4. *With above hypotheses, the functor* $\mathrm{Hilb}_{X/S,r}^{\times 2,H}$ *is representable by an open subscheme of the product of Hilbert schemes*

$$\mathrm{Hilb}_{X/S,r} \times_S \mathrm{Hilb}_{X/S,r}.$$

It is enough to show that if $(Z_1, Z_2) \in \mathrm{Hilb}_r(X/S) \times \mathrm{Hilb}_r(X/S)$ and if, for $s \in S$, $((Z_1)_s, (Z_2)_s) \in \mathrm{Hilb}_r(X_s/s)^{\times 2,H}$, then there exists an open neighbourhood U of s such that

$$((Z_1)_U, (Z_2)_U) \in \mathrm{Hilb}_r(X_U/U)^{\times 2,H}.$$

Let $\sigma = \{Z_1/S\} - \{Z_2/S\} \in H^0(S, R^{2p}f_*\mathbb{Z}_e(p))$. *If* $\bar{s}$ is a geometric point of s, the pull back of σ in $H^{2p}(X_{\bar{s}}, \mathbb{Z}_e(p))$ vanishes. It suffices to see that there exists an open neighbourhood U of s where $\sigma = 0$.

LEMMA 2.4.1. *Let $f: X \to S$ be a smooth proper morphism and s a geometric point of S, $\sigma \in H^0(S, R^nf_*\mathbb{Z}_e(k))$. If the pull back of σ in $H^n(X_s, \mathbb{Z}_e(k))$ is zero, then $\sigma = 0$ on the connected component of S containing s.*

We have $\sigma = (\sigma_\nu) \in (H^0(S, R^n f_*(\mathbb{Z}/e\nu(k))))_\nu$ and the hypothesis means that $(\sigma_\nu)_s = 0$ in $H^n(X_s, \mathbb{Z}/e^\nu(k)) = (R^n f_*(\mathbb{Z}/e^\nu(k)))_s$ for any ν, since f is proper. The morphism f is smooth proper, hence $R^n f_*(\mathbb{Z}/e^\nu(k))$ is a locally constant. Let U be an etale neighbourhood of s where $R^n f_*(\mathbb{Z}/e^\nu(k))$ is constant. Then $\sigma_\nu \mid U = 0 \Leftrightarrow (\sigma_\nu)_{\bar{t}} = 0$ at some geometric point $\bar{t}$ of U. It follows that $\sigma_\nu = 0$ on the connected component of S containing s, and $\sigma = 0$ on it.

REMARKS 2.4.2. If S is the spectre of an algebraically closed field k, and Z is a closed subscheme of pure dimension r of a smooth projective variety V over k, then $\{Z/k\} \in H^{2p}(V, \mathbb{Z}_e(p))$ is the fundamental class of the cycle associated to the subscheme Z, cf. [15], 3.3.4.

2.4.3. The homological equivalence relation we have considered above is the $\mathbb{Z}_e$-homological equivalence. We can also consider the $\mathbb{Q}_e$-homological equivalence and in that case, the proposition remains true. In fact, with the notation of proof of the proposition, if the pull back of σ in $H^{2p}(X_{\bar{s}}, \mathbb{Q}_e(p))$ vanishes, then $k \cdot \sigma = 0$ in $H^{2p}(X_{\bar{s}}, \mathbb{Z}_e(p))$ with $k \neq 0$, hence $k \cdot \sigma$ vanishes in a neighbourhood of s with $\mathbb{Z}_e$-coefficient, hence σ vanishes there with $\mathbb{Q}_e$-coefficient.

2.4.4. Let E be a set consisting of some prime integer invertible in S. We could consider the intersection of $\mathbb{Z}_e$-homological equivalence, i.e.,

$$\begin{aligned}&\mathrm{Hilb}_r(X/S)^{\times 2, H, E}\\&\quad = \{(Z_1, Z_2);\ \{Z_1/S\} = \{Z_2/S\} \text{ in } H^0(S, R^{2p} f_* \mathbb{Z}_e(p)) \text{ for } e \in E\}.\end{aligned}$$

In view of lemma 2.4.1, the functor $S \mapsto \mathrm{Hilb}_r(X/S)^{\times 2, H, E}$ is also representable by an open subscheme of $\mathrm{Hilb}_{X/S,r} \times_S \mathrm{Hilb}_{X/S,r}$. Moreover, we can replace the equivalent relation by the mixture of the type considered in 2.4.3.

3. Direct image morphism of Chow schemes

3.1. Let S be a locally noetherian scheme. Recall that a morphism $h: X \to S$ of finite type is called of pure relative dimension r if $X_s = h^{-1}(s)$ is of pure dimension r for every $s \in h(X)$. We set

$$X(r) = \{x \in X;\ \dim_x h^{-1}(h(x)) \geqslant r\}.$$

Then $X(r)$ is a closed subset of X.

Note that h is of pure relative dimension r if and only if $X(r) = X$, provided that all the fibres of h are of dimension $\leqslant r$.

PROPOSITION 3.2. *Let X, Y be S-schemes of finite type, $f: X \to Y$ be a proper surjective S-morphism with Y irreducible and $X \to S$ of pure relative dimension r. Suppose that there exists $s \in S$ such that* $\dim Y_s = r$. *Then $Y \to S$ is of pure relative dimension r.*

The conclusion is equivalent to $Y = Y(r)$. If f is finite, then $f_s: X_s \to Y_s$ is also finite, and it is clear that $Y = Y(r)$. In general case, let Y° be the maximum open subscheme of Y such that $f^\circ = f_{(Y^\circ)}: X^\circ \to Y^\circ$ is finite; then $Y^\circ \neq \phi$. In fact, consider $X_s \to Y_s$. If $x \in X_s$ is the generic point of a component of X_s such that $\dim \overline{f(x)} = r$, then the restriction $\bar{x} \to \overline{f(x)}$ of f_s is generically finite (the bars denote the closure in the fibres), and $f(x) \in Y^\circ$. Since $X_s \to Y_s$ is surjective, such an x exists by hypothesis, hence $Y^\circ \neq \phi$. Let $g: Y \to S$, and $g^\circ: Y^\circ \to S$ be its restriction. Then $Y^\circ(r) := \{y \in Y^\circ; \dim_y g^{\circ -1}(g^\circ(y)) \geqslant r\} = Y^\circ \cap Y(r)$. For $y \in Y^\circ$, $\dim_y g^{\circ -1}(g^\circ(y)) = \dim_y(Y^\circ \cap g^{-1}(g(y))) = \dim_y g^{-1}(g(y))$. Since $f^\circ: X^\circ \to Y^\circ$ is finite, $Y^\circ(r) = Y^\circ$, so that $Y^\circ = Y^\circ(r) = Y^\circ \cap Y(r) \subset Y(r) \subset Y$. Since the closure of Y° is Y, $Y(r) = Y$.

LEMMA 3.3. *Let S be a locally noetherian scheme and $f: X \to Y$ be a proper S-morphism, and suppose that $X \to S$ is of pure relative dimension r. Then there exist closed subsets Y_1, Y_2 of Y such that $f(X) = Y_1 \cup Y_2$ and $Y_1 \to S$ is of pure relative dimension r, and* $\dim(Y_2)_s < r$ *for any $s \in S$.*

We can suppose X reduced, and replacing f by $X \to f(X)$, we may assume f is surjective. If Y is a union of closed subsets Y_λ, then for $y \in Y_s \subset Y$, since

$$\dim_y Y_s = \sup_\lambda \dim_y (Y_\lambda)_s \leqslant r,$$

$Y(r)$ is the union of $Y_y(r)$. Let $X = \bigcup_\lambda X_\lambda$ is the decomposition into irreducible components. Then $Y = \bigcup_\lambda f(X_\lambda)$. Consider $X_\lambda \to f(X_\lambda)$, and we have either $f(X_\lambda)(r) = f(X_\lambda)$ or $f(X_\lambda)(r) = \phi$ by 3.2. It will suffice to put $Y_1 = f(X)(r)$ and

$$Y_2 := \bigcup f(X_\lambda)$$

where the union is over those X_λ with $f(X_\lambda)(r) = \phi$.

3.4. Let S be an affine scheme of characteristic zero, and X be a smooth projective S-scheme of pure relative dimension m. Then for an integer p, $0 \leqslant p \leqslant m$, we have the Chow scheme $C^p_{X/S}$ of cycles of relative codimension p on X/S ([1]), while $C^p_{X/S}$ is, in fact, only an algebraic space in general. If X is a subscheme of $S \times P^N$ for some N, then, $C^p_{X/S}$ is embedded in $C^{N-m+p}_{S \times P^N/S} = C^{N-m+p}_{P^N} \times S$, and since $(C^{N-m+p}_{P^N})_{\mathrm{red}}$ is the usual Chow variety of P^N, $C^p_{X/S}$ is an S-scheme, a countable union of proper S-schemes.

We set

$$\bar{C}^p_{X/S} = C^p_{X/S} \coprod o(S),\ o(S) = S.$$

Intuitively, $o(S)$ corresponds to the cycle "zero" of codimension p. We shall show that, for a proper S-morphism $f: X \to Y$ of smooth projective S-scheme, we can define the direct image morphism

$$f_*: \bar{C}^p_{X/S} \to \bar{C}^{p+n-m}_{Y/S}$$

of Chow schemes, where n is the relative dimension of Y/S. To do this, it suffices to define a morphism as functors.

Let S' be an S-scheme, and put

$$X' = X \times_S S', \quad Y' = Y \times_S S',$$

$$f' = f \times \mathrm{id}_{S'}: X' \to Y',$$

$$C^p(X'/S') = C^p_{X/S}(S'),$$

$$\bar{C}^p(X'/S') = \bar{C}^p_{X/S}(S') = C^p(X'/S') \coprod \{\mathrm{id}_{S'}\}.$$

Recall that an element of $C^p(X'/S')$ is a pair (Z, c) of a closed subset $Z \subset X'$ of pure relative dimension $r = m - p$ over S', and an element $c \in H^p_Z(X', \Omega^p_{X'/S'})$ which satisfy some conditions (cr. [1], 4.1, 4.2). By Lemma 3.3, $Z' = f'(Z) = Z'_1 \cup Z'_2$, where Z'_1 is of pure relative dimension r over S, and Z'_2 is of relative dimension $<r$.

Note that

$$R\Gamma_{f'^{-1}(Z')}(X', ?) = R\Gamma_{Z'}(Y', Rf'_* ?).$$

Putting $d = m - n$, we have

$$\begin{aligned}
&\mathrm{Hom}(Rf'_*\Omega^p_{X'/S'}, \Omega^{p-d}_{Y'/S'}[-d]) \\
&\quad = \mathrm{Hom}(\Omega^p_{X'/S'}, Rf'^!\Omega^{p-d}_{Y'/S'}[-d]) \\
&\quad = \mathrm{Hom}(\Omega^p_{X'/S'}, Rf'^!\Omega^n_{Y'/S'}[n] \otimes (f'^*\Omega^r_{Y'/S'})^\vee[-m]) \\
&\quad = \mathrm{Hom}(\Omega^p_{X'/S'}, \Omega^m_{X'/S'}[m] \otimes (f'^*\Omega^r_{Y'/S'})^\vee[-m]) \\
&\quad = \mathrm{Hom}(\Omega^p_{X'/S'} \otimes f'^*\Omega^r_{Y'/S'}, \Omega^m_{X'/S'}),
\end{aligned}$$

and the canonical map

$$\Omega^p_{X'/S'} \otimes f'^*\Omega^r_{Y'/S'} \to \Omega^p_{X'/S'} \otimes \Omega^r_{X'/S'} \to \Omega^m_{X'/S'},$$

hence, we get

$$Rf'_*\Omega^p_{X'/S'} \to \Omega^{p-d}_{Y'/S'}[-d].$$

Therefore we obtain

$$\begin{aligned} H^p_Z(X', \Omega^p_{X'/S'}) &\to H^p_{f'^{-1}(Z')}(X', \Omega^p_{X'/S'}) \\ &= H^p_{Z'}(Y', Rf'_*\Omega^p_{X'/S'}) \\ &\to H^{p-d}_{Z'}(Y', \Omega^{p-d}_{Y'/S'}). \end{aligned}$$

LEMMA 3.5. *The canonical map*

$$H^{p-d}_{Z'_1}(Y', \Omega^{p-d}_{Y'/S'}) \to H^{p-d}_{Z'}(Y', \Omega^{p-d}_{Y'/S'})$$

is an isomorphism.

SUBLEMMA 3.5.1. (cf. [2]) *Let $g: Y \to S$ be a morphism of relative dimension $< r$ of locally noetherian schemes, then we have*

$$R^i g^! O_S = 0 \quad \text{for } i \leqslant -r.$$

The question is local on Y. For any $z \in Y$, we have a commutative diagram

$$\begin{tikzcd} z \in U \arrow[r, "j"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ A_S^{r-1} \arrow[r, "a"'] & S \end{tikzcd}$$

where j is an open immersion, and h is a quasi-finite morphism. By Zariski's Main theorem, there is a finite morphism $\bar{h}: V \to A_S^{r-1}$ and an open immersion $k: U \to V$ such that $h = \bar{h} \cdot k$. We have

$$\begin{aligned} R^i g^! O_S | U &= R^i (g \cdot j)^! O_S \\ &= R^i (a \cdot \bar{h} \cdot k)^! O_S \\ &= R^{i+r-1} \bar{h}^! \Omega^{r-1}_{A_S^{r-1}/S} | U, \\ R\bar{h}^! \Omega^{r-1}_{A_S^{r-1}/S} &= \bar{h}^* R\,\mathbf{Hom}_{O_{A^{r-1}}}(\bar{h}_* O_V, \Omega^{r-1}_{A_S^{r-1}/S}), \end{aligned}$$

where $\tilde{h}: (V, O_V) \to (A_S^{r-1}, h_* O_V)$. Since $\tilde{h}$ is flat, we have

$$R^i \tilde{h}^! \Omega^{r-1}_{A_S^{r-1}/S} = 0 \text{ for } i < 0.$$

Therefore we have $R^i g^! O_S = 0$ for $i + r - 1 < 0$, i.e., for $i < -r + 1$.

LEMMA 3.5.2. *Let $g: Y \to S$ be a smooth morphism of locally noetherian schemes of pure relative dimension n, E a locally free O_Y-Module of finite rank and $Z \subset Y$ a closed subscheme of relative dimension $< r$ over S and set $p' = n - r$. Then we have*

$$\mathrm{Ext}^i(O_Z, E) = 0, \quad \textit{and} \quad H^i_Z(Y, E) = 0 \textit{ for } i \leqslant p'.$$

Let $j: Z \to Y$ denote the closed immersion. We get

$$\begin{aligned}
\mathrm{Ext}^i(O_Z, E) &= \mathrm{Hom}(O_Z, E[i]) \\
&= \mathrm{Hom}(O_Z, \Omega^n_{Y/S} \otimes \mathbf{Hom}(\Omega^n_{Y/S}, E)[i]) \\
&= \mathrm{Hom}(O_Z \otimes \mathbf{Hom}(E, \Omega^n_{Y/S}), \Omega^n_{Y/S}[n][i-n]) \\
&= \mathrm{Hom}(Rj_* j^* \mathbf{Hom}(E, \Omega^n_{Y/S}), Rg^! O_S[i-n]) \\
&= \mathrm{Hom}(j^* \mathbf{Hom}(E, \Omega^n_{Y/S}), R(g \cdot j)^! O_S[i-n]),
\end{aligned}$$

and we have a spectral sequence

$$E_2^{a, i-a} = \mathrm{Ext}^a(j^* \mathbf{Hom}(E, \Omega^n_{Y/S}), R^{i-a-n}(g \cdot j)^! O_S) \Rightarrow \mathrm{Ext}^i(O_Z, E).$$

By sublemma 3.5.1, $R^{i-a-n}(g \cdot j)^! O_S = 0$ for $i - a - n \leqslant -r$, i.e., for $i - a \leqslant p'$. Since $E_2^{a, i-a} = 0$ unless $a \geqslant 0$ and $i - a > p'$, $\mathrm{Ext}^i(O_Z, E) = 0$ for $i \leqslant p'$. It follows that $H^i_Z(X, E) = 0$ for $i \leqslant p'$.

The proof of lemma 3.5 is now easy: we have an exact sequence

$$\begin{aligned}
&H^{p-d-1}_{Z' \backslash Z_1'}(Y' \backslash Z_1', \Omega^{p-d}_{Y'/S'}) \to H^{p-d}_{Z_1'}(Y', \Omega^{p-d}_{Y'/S'}) \\
&\quad \to H^{p-d}_{Z'}(Y', \Omega^{p-d}_{Y'/S'}) \to H^{p-d}_{Z' \backslash Z_1'}(Y' \backslash Z_1', \Omega^{p-d}_{Y'/S'})
\end{aligned}$$

and the both extremes vanish by virtue of sublemma 3.5.2, because $Z' \backslash Z_1' \subset Z_2'$.

3.6. We *define*

$$f'_*: \bar{C}^p(X'/S') \to \bar{C}^p(Y'/S')$$

as follows: the image of $\mathrm{id}_{S'}(=o(S)(S'))$ is $\mathrm{id}_{S'} \in \bar{C}^p(Y'/S')$. For $(Z, c) \in C^p(X'/S')$, we have

$$\bar{f}'_*: H^p_Z(X', \Omega^p_{X'/S'}) \to H^{p-d}_{Z'}(Y', \Omega^{p-d}_{Y'/S'}) \xleftarrow{\sim} H^{p-d}_{Z'_1}(Y', \Omega^{p-d}_{Y'/S'}).$$

If $Z'_1 \neq \phi$, we put

$$f'_*((Z, c)) = (Z'_1, \bar{f}'_*(c)),$$

and otherwise,

$$f'_*((Z, c)) = \mathrm{id}_{S'}.$$

PROPOSITION 3.7. *Under the above hypothesis.* $f'_*((Z, c)) \in \bar{C}^p(Y'/S')$, *and we have a morphism of functors*

$$f_*: \bar{C}^p_{X/S} \to \bar{C}^{p-d}_{Y/S}.$$

It suffices to see $f'_*((Z, c)) \in \bar{C}^p(Y'/S')$.

Let $z' \in Z'_1$, and (U', B', φ') be a projection of Z'_1 around z'. It is also a projection of Z'_1 around z'' for any generization $z'' \in Z'_1$ of z', hence $(Z'_1, \bar{f}'_*(c))$ is a Chow class at z' if it is a Chow class at z''. Let Z_1 be the pull-back of Z'_1 by $Z \subset X' \xrightarrow{f} Y'$, and let Z''_1 be the closed set of Z'_1 of points y such that the fiber of $Z_1 \to Z'_1$ over y has positive dimension. Take an irreducible component of Z'_1; it is not contained in Z'_2, nor in Z''_1, i.e., z' has its generization $z'' \in Z'_1 \backslash (Z''_1 \cup Z'_2)$. To show $(Z'_1, \bar{f}'_*(c))$ is a Chow class at z'', set $Y'' = Y' \backslash (Z''_1 \cup Z'_2)$, $X'' = f'^{-1}(Y'')$, $f'': X'' \to Y''$ the base-change of f'. Then, we have

$$\begin{aligned} &\bar{f}''_*(\text{restriction of } c \text{ to } H^p_{Z \cap X''}(X'', \Omega^p_{X'/S'})) \\ &\quad = \text{restriction of } \bar{f}'_*(c) \text{ in } H^{p-d}_{Z'_1 \cap Y''}(Y'', \Omega^{p-d}_{Y''/S'}), \end{aligned}$$

and $Z \cap X''$ is finite over Y''. In that case, the proof can be found in [1], 6.3.

3.8. With the notations and hypotheses in 3.2, let (Z, c) and (Z', c') be Chow classes. We have the sum $c + c'$ of c and c' by the natural maps

$$H^p_Z(X, \Omega^p_{X/S}) \to H^p_{Z \cup Z'}(X, \Omega^p_{X/S}),$$

and

$$H^p_{Z'}(X, \Omega^p_{X/S}) \to H^p_{Z \cup Z'}(X, \Omega^p_{X/S}),$$

respectively and $(Z \cup Z', c + c')$ is a Chow class, hence we get a morphism of functors $+: C^p(X/S) \times C^p(X/S) \to C^p(X/S)$. We extend it to the morphism of functors

$$+: \bar{C}^p(X/S) \times \bar{C}^p(X/S) \to \bar{C}^p(X/S)$$

as follows: it coincides with $+$ above on $C^p(X/S) \times C^p(X/S)$, and the first projection on $C^p(X/S) \times \{\mathrm{id}_S\}$, the second projection on $\{\mathrm{id}_S\} \times C^p(X/S)$, and the image of $(\mathrm{id}_S, \mathrm{id}_S)$ is id_S. Therefore we obtain the morphism of algebraic spaces

$$+: \bar{C}^p_{X/S} \times_S \bar{C}^p_{X/S} \to \bar{C}^p_{X/S}.$$

4. Genericity Theorem

4.1. In this section, the ground field k is supposed to be algebraically closed of characteristic zero and *uncountable*. Recall that we denote by $H_{\mathbb{Q}}$ the $\mathbb{Q}$-homological equivalence relation and we have the adequate equivalence relations $H_{\mathbb{Q}}^{*l}$ (See 1.5). The purpose of this section is to prove the following

THEOREM 4.2. *Let V be a smooth projective variety of dimension m, S a smooth variety, l an integer and Z a cycle on $S \times V$ of codimension p. Assume that for an arbitrary closed point $s \in S$, the cycle $Z(s)$ is defined, and is $H_{\mathbb{Q}}^{*l}$-equivalent to zero. Then there exist a smooth variety T, a dominant morphism $e: T \to S$, a smooth projective morphism $\pi: \mathscr{F} \to T$, cycles X_{ij} of codimension p_{ij} on $\mathscr{F} \times V (1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k)$ such that*

(i) $\sum_j p_{ij} = p + \dim \mathscr{F} - \dim T$;
(ii) *For any $t \in T$, $(j_t \times \mathrm{id}_V)^*(X_{ij})$ is $\mathbb{Q}$-homologous to zero on $\mathscr{F}_t \times V$, where $j_t: \mathscr{F}_t \to \mathscr{F}$ is the inclusion.*
(iii) $(e \times \mathrm{id}_V)^*(Z) = \sum_j (\pi \times \mathrm{id}_V)_*(X_{ij} \cdot \cdots \cdot X_{lj})$ *in* $\mathrm{CH}(T \times V)$.

Let $\pi_\alpha: \mathscr{F}_\alpha \to T_\alpha$ $(\alpha \in A)$ be countable families of smooth projective morphisms such that T_α are affine algebraic schemes over k and that for any smooth projective variety W, there exist an $\alpha \in A$ and $t \in T_\alpha$ with $W \simeq (\mathscr{F}_\alpha)_t$.

For a smooth projective morphism $q: X \to T$, integers $p_1, \ldots, p_l \geqslant 0$, let

$$\mathscr{U} = \{(Z_1, \ldots, Z_l) \in C^{p_1}_{X/T} \times \cdots \times C^{p_l}_{X/T};$$
$$(Z_1)_t, \ldots, (Z_l)_t \text{ intersect properly for } t \in T\}.$$

$\mathscr{U}$ is an open subscheme of $\prod_{i/T} C^{p_i}_{X/T}$ and we have a morphism ([1], 8.1).

$$\begin{array}{ccc} \prod_{i/T} C^{p_i}_{X/T} \supset \mathscr{U} & \longrightarrow & C^{\bar{p}}_{X/T}, \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (Z_1, \ldots, Z_l) & \longrightarrow & Z_1 \underset{T}{\cdot} \cdots \underset{T}{\cdot} Z_l \end{array} \qquad \bar{p} = \sum_i p_i.$$

By [1], 7.1.6, there is a morphism

$$\mathrm{Hilb}^{\times 2, H\mathbb{Q}}_{X/T, r_i} \subset \mathrm{Hilb}^{p_i}_{X/T} \times \mathrm{Hilb}^{p_i}_{X/T} \to C^{p_i}_{X/T} \times C^{p_i}_{X/T}$$

where $\mathrm{Hilb}^{\times 2, H\mathbb{Q}}_{X/T, r_i}$ are defined in 2.4.3 (cf. also, 2.4; note that $\mathbb{Q}$-homological and $\mathbb{Q}_e$-homological equivalences coincide since we are in characteristic zero), $r_i = \mathrm{rel.dim}\, X/T - p_i$, hence their product

$$\iota\colon \prod\nolimits_{i/T} \mathrm{Hilb}^{\times 2, H\mathbb{Q}}_{X/T, r_i} \to \prod\nolimits_{i/T} (C^{p_i}_{X/T} \times C^{p_i}_{X/T}).$$

Let $2^{[1,l]}$ be the set of maps from the interval $[1, l]$ of integers to the set $\{0, 1\}$ and for $\sigma \in 2^{[1,l]}$, let

$$\mathrm{pr}_\sigma\colon \prod\nolimits_{i/T} (C^{p_i}_{X/T} \times C^{p_i}_{X/T}) \to \prod\nolimits_{i/T} C^{p_i}_{X/T}$$

be the product of projections $\mathrm{pr}_{\sigma(i)}$ where $\mathrm{pr}_{\sigma(i)}$ is the projection to the first factor if $\sigma(i) = 0$, and to the second factor if $\sigma(i) = 1$. Then we have

$$\bigcap_\sigma \mathrm{pr}_\sigma^{-1}(\mathscr{U}) \subset \prod\nolimits_{i/T} (C^{p_i}_{X/T} \times C^{p_i}_{X/T}),$$

and,

$$\begin{array}{ccc} \bigcap_\sigma \mathrm{pr}_\sigma^{-1}(\mathscr{U}) & \longrightarrow & C^{\bar{p}}_{X/T} \times C^{\bar{p}}_{X/T} \\ \Downarrow & & \Downarrow \\ (Z_i^{(0)}, Z_i^{(1)}) & \longrightarrow & \left(\sum_{|\sigma| \equiv 0} \prod_{i/T} Z_i^{(\sigma(i))}, \sum_{|\sigma| \equiv 1} \prod_{i/T} Z_i^{(\sigma(i))} \right), \end{array}$$

where $|\sigma| = \sum_i \sigma(i)$, and $|\sigma| \equiv 0$ means that the summation is over all σ with even $|\sigma|$, and $|\sigma| \equiv 1$ means the summation over σ with odd $|\sigma|$.

Let $\mathscr{H}^{p_1, \ldots, p_l}_{X/T}$ be the pull-back of $\bigcap_\sigma \mathrm{pr}_\sigma^{-1}(\mathscr{U})$ by the morphism ι. Thus we get a morphism

$$\mathscr{H}^{p_1, \ldots, p_l}_{X/T} \to C^{\bar{p}}_{X/T} \times_T C^{\bar{p}}_{X/T}.$$

Consider the morphisms

$$\begin{aligned} \mathscr{H}^{\mathbf{p}}_{V \times \mathscr{F}_\alpha/S_\alpha} &= \mathscr{H}^{p_1, \ldots p_l}_{V \times \mathscr{F}_\alpha/S_\alpha} \to C^{|\mathbf{p}|}_{V \times \mathscr{F}_\alpha/S_\alpha} \times_{S_\alpha} C^{|\mathbf{p}|}_{V \times \mathscr{F}_\alpha/S_\alpha} \\ &\to \bar{C}^{p}_{V \times S_\alpha/S_\alpha} \times \bar{C}^{p}_{V \times S_\alpha/S_\alpha} \\ &= \bar{C}^{p}_V \times \bar{C}^{p}_V \times S_\alpha \to \bar{C}^{p}_V \times \bar{C}^{p}_V, \end{aligned}$$

where $\mathfrak{p} = (p_1, \ldots, p_l)$, $|\mathfrak{p}| = \sum p_i$, $p = |\mathfrak{p}| - \mathrm{rel.dim}\,\mathscr{F}_\alpha/S_\alpha$, and $\bar{C}^p_{X/S} = C^p_{X/S} \coprod o(S)$ (cf. 3.4) and the second arrow is induced by the morphism $V \times \mathscr{F}_\alpha \to V \times S_\alpha$.

For an integer $n \geqslant 1$ and a sequence of l-tuples $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ with $|\mathfrak{p}_j| = p + \mathrm{rel.dim}\,\mathscr{F}_\alpha/S_\alpha$, putting

$$\mathscr{Q}_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n} = \prod\nolimits_{j/S_\alpha} \mathscr{H}^{\mathfrak{p}_j}_{V \times \mathscr{F}_\alpha/S_\alpha},$$

we get a morphism

$$\psi^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n}_{V \times \mathscr{F}_\alpha/S_\alpha} = \psi_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n} \colon \mathscr{Q}_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n} \to (\bar{C}^p_V \times \bar{C}^p_V)^{\times n} \to \bar{C}^p_V \times \bar{C}^p_V,$$

the second arrow being the sum given by

$$((Y_1, Y'_1), \ldots, (Y_n, Y'_n)) \mapsto \left(\sum_i Y_i, \sum_i Y'_i\right),$$

(cf. 3.8). For a k-rational point x of the left hand side $\mathscr{Q}_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n}$, the image in $\bar{C}^p_V \times \bar{C}^p_V$ is given as follows:

Let s be the image of x in S_α. Then x consists of subschemes $(Z^{(0)}_{i,j}, Z^{(1)}_{i,j})$ of $V \times (\mathscr{F}_\alpha)_s$ of codimension $p_{i,j}$ (where $\mathfrak{p}_j = (p_{1,j}, \ldots, p_{l,j})$) such that the associated cycles to $Z^{(0)}_{i,j}$ and $Z^{(1)}_{i,j}$ are $\mathbb{Q}$-homologically equivalent on $V \times (\mathscr{F}_\alpha)_s$. The image of x in $\bar{C}^p_V \times \bar{C}^p_V$ corresponds to the pairs of cycles

$$\left(\sum_j \sum_{|\sigma| \equiv 0} (\pi_\alpha)_*(Z^{(\sigma(1))}_{1,j} \cdot \ldots \cdot Z^{(\sigma(l))}_{l,j}),\ \sum_j \sum_{|\sigma| \equiv 1} (\pi_\alpha)_*(Z^{(\sigma(1))}_{1,j} \cdot \ldots \cdot Z^{(\sigma(l))}_{l,j})\right),$$

where for simplicity, we denote by $Z^{(\sigma(i))}_{i,j}$ the associated cycles on $V \times (\mathscr{F}_\alpha)_s$ to the subschemes $Z^{(\sigma(i))}_{i,j}$, and by π_α the morphism $\pi_\alpha \colon V \times (\mathscr{F}_\alpha)_s \to V$. Since $\mathrm{Hilb}_{V \times \mathscr{F}_\alpha/S_\alpha, r} \to C^p_{V \times \mathscr{F}_\alpha/S_\alpha}$ is surjective, any r-cycles on V which are $H^{*l}_{\mathbb{Q}}$-equivalent to zero can be written as the differences $z - z'$ of pairs (Z, Z') in this form for some α (cf. 1.6.2). We have a morphism defined by

$$\begin{array}{ccccc} C^p_V \times C^p_V \times \mathscr{Q}_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n} & \xrightarrow{\mathrm{id} \times \psi_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n}} & C^p_V \times C^p_V \times \bar{C}^p_V \times \bar{C}^p & \longrightarrow & \bar{C}^p_V \times \bar{C}^p_V. \\ & & \in & & \in \\ & & (Z_1, Z_2, Z_3, Z_4) & \longrightarrow & (Z_2 + Z_3, Z_1 + Z_4) \end{array}$$

Denote by $\mathscr{R}_\alpha^{\mathfrak{p}_1, \ldots, \mathfrak{p}_n}$ the pull-back of the diagonal of $\bar{C}^p_V \times \bar{C}^p_V$, and consider the projection $\pi_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}: \mathscr{R}_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n} \to C_V^{\mathfrak{p}} \times C_V^{\mathfrak{p}}$. The union of the images for all n, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ and α

$$\bigcup \operatorname{Im} \pi_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n} = C_V^{\mathfrak{p}} \times C_V^{\mathfrak{p}}$$

is the set of the pairs (Z, Z') of effective r-cycles which are $H_{\mathbb{Q}}^{*l}$-equivalent.

Since the set of possible n, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, α is countable and the number of irreducible components of $\mathscr{R}_{V\times\mathscr{F}_\alpha/S_\alpha}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}$ is countable, the above union is a countable union of irreducible subsets. Now, shrinking S if necessary, write Z as a difference of effective cycles which are non-degenerate on S: $Z = Z^+ - Z^-$. It defines a morphism

$$\varphi: S \to C_V^{\mathfrak{p}} \times C_V^{\mathfrak{p}}.$$

By hypotheses,

$$\operatorname{Im} \varphi \subset \bigcup \operatorname{Im} \pi_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}$$

as k-rational point. Since the ground field k is uncountable, we can find n, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, and α such that there exists a locally closed subvariety of $\mathscr{R}_{V\times\mathscr{F}_\alpha/S_\alpha}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}$ such that the image of the restriction of $\pi_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}$ to the subvariety contains the generic point of $\overline{\operatorname{Im} \varphi}$. Hence we have a diagram

$$\begin{tikzcd}
S_{C_V^{\mathfrak{p}} \times C_V^{\mathfrak{p}}} \times \mathscr{R}_{V\times\mathscr{F}_\alpha/S_\alpha}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n} \arrow[r] \arrow[d] & \mathscr{R}_{V\times\mathscr{F}_\alpha/S_\alpha}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n} \arrow[d, "\pi_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}"] \\
S \arrow[r, "\varphi"'] & C_V^{\mathfrak{p}} \times C_V^{\mathfrak{p}}
\end{tikzcd}$$

and the left vertical arrow is dominant. There exist, therefore, a smooth affine variety T, and a dominant morphism $e: T \to S$ which sit in the diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "e"] & \mathscr{R}_{V\times\mathscr{F}_\alpha/S_\alpha}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n} \arrow[d, "\pi_\alpha^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}"] \\
S \arrow[r, "\varphi"'] & C_V^{\mathfrak{p}} \times C_V^{\mathfrak{p}}.
\end{tikzcd}$$

We have the morphism $T \to S_\alpha$ and let

$$\mathscr{F} = \mathscr{F}_\alpha \times_{S_\alpha} T.$$

By base-change, we get an element ξ of $\mathscr{R}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}_{V\times\mathscr{F}/T}(T)$ whose image by $\pi^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}_{V\times\mathscr{F}/T}$ is $\varphi\circ e\in C^{\mathfrak{p}}_V\times C^{\mathfrak{p}}_V(T)$. Let the image of ξ under the morphism induced by ι

$$\mathscr{R}^{\mathfrak{p}_1,\dots,\mathfrak{p}_n}_{\alpha}\to\prod\nolimits_{i/T}\prod\nolimits_{j/T}(C^{p_{i,j}}_{V\times\mathscr{F}/T}\times_T C^{p_{i,j}}_{V\times\mathscr{F}/T})$$

be $((Z^{(0)}_{i,j}, Z^{(1)}_{i,j}))$. If we denote the generic point of T by τ, the pull-backs $(Z^{(0)}_{i,j})_\tau$ and $(Z^{(1)}_{i,j})_\tau$ are the cycles on $V\times\mathscr{F}_\tau/\kappa(\tau)$. Let $\bar{Z}^{(0)}_{i,j}$ and $\bar{Z}^{(1)}_{i,j}$ be the closures of them in $V\times\mathscr{F}$, and put

$$X_{ij}=\bar{Z}^{(0)}_{i,j}-\bar{Z}^{(1)}_{i,j}.$$

Then X_{ij} and $e\colon T\to S$ satisfy the conditions of the theorem.

5. Definition of the functor

In the sequel, *the ground field is assumed to be the field of complex numbers.*

5.1. Recall the definition of coniveau filtration (cf. [12]):

For a smooth variety, let

$$\begin{aligned}N^pH^n(V,\mathbb{Q}) &:= \bigcup \operatorname{Im}(H^n_F(V,\mathbb{Q})\to H^n(V,\mathbb{Q})) \\ &= \bigcup \operatorname{Ker}(H^n(V,\mathbb{Q})\to H^n(V\backslash F,\mathbb{Q})),\end{aligned} \tag{5.1.1}$$

where F runs over the set of Zariski closed subsets of V of codimension $\geqslant p$. $N^pH^n(V,\mathbb{Q})$ define a decreasing filtration of $H^n(V,\mathbb{Q})$ and we denote by $\mathrm{gr}^pH^n(V,\mathbb{Q})$ the associated graded module:

$$\mathrm{gr}^pH^n(V,\mathbb{Q})=N^pH^n(V,\mathbb{Q})/N^{p+1}H^n(V,\mathbb{Q}).$$

We have $H^n(V,\mathbb{Q})=N^0H^n(V,\mathbb{Q})$ and $N^pH^n(V,\mathbb{Q})=0$ if $n<2p$. Note that $H^n(V,\mathbb{Q})$ has a mixed $\mathbb{Q}$-Hodge structure. In view of 5.1.1, $N^pH^n(V,\mathbb{Q})$ is a mixed Hodge sub-structure of $H^n(V,\mathbb{Q})$, and hence, $\mathrm{gr}^pH^n(V,\mathbb{Q})$ has also a mixed $\mathbb{Q}$-Hodge structure. If V is projective, it is pure of weight n.

The coniveau filtration has the following functorial properties:

(i) For a morphism $f\colon V\to W$, $N^pH^n(W,\mathbb{Q})\subset H^n(W,\mathbb{Q})$ is mapped into $N^pH^n(V,\mathbb{Q})$ by the pull-back $f^*\colon N^n(W,\mathbb{Q})\to H^n(V,\mathbb{Q})$; hence f^* induces the map

$$f^*\colon \mathrm{gr}^pH^n(W,\mathbb{Q})\to\mathrm{gr}^pH^n(V,\mathbb{Q}).$$

(ii) For a proper morphism $f\colon V\to W$, $N^pH^n(V,\mathbb{Q})\subset H^n(V,\mathbb{Q})$ is mapped into

$N^{p-d}H^{n-2d}(W, \mathbb{Q})(-d)$ by the push-forward $f^*: H^n(V, \mathbb{Q}) \to H^{n-2d}(W, \mathbb{Q})(-d)$, where $d = \dim W - \dim V$. Hence f^* induces the map

$$f_*: \mathrm{gr}^p H^N(V, \mathbb{Q}) \to \mathrm{gr}^{p-d} H^{n-2d}(W, \mathbb{Q})(-d).$$

(iii) The cup-product $\cup: H^n(V, \mathbb{Q}) \times H^{n'}(V, \mathbb{Q}) \to H^{n+n'}(V, \mathbb{Q})$ maps

$$N^p H^n(V, \mathbb{Q}) \times N^{p'} H^{n'}(V, \mathbb{Q})$$

into

$$N^{p+p'} H^{n+n'}(V, \mathbb{Q});$$

hence we get

$$\cup: \mathrm{gr}^p H^n(V, \mathbb{Q}) \times \mathrm{gr}^{p'} H^{n'}(V, \mathbb{Q}) \to \mathrm{gr}^{p+p'} H^{n+n'}(V, \mathbb{Q}).$$

The fundamental class of an algebraic cycle z of codimension p on V will be denoted by $\{z\} \in \mathrm{gr}^p H^{2p}(V, \mathbb{Q})(p) = N^p H^{2p}(V, \mathbb{Q})(p) \subset H^{2p}(V, \mathbb{Q})(p)$.

For smooth varieties T, V, with V projective, $\dim V = m$, and for $z \in \mathrm{CH}^p(T \times V)$, l an integer, $r = m - p$, we define a morphism of mixed Hodge structure

$$\{{}^t z\}: \mathrm{gr}^r H^{2r+l}(V, \mathbb{Q})(r) \to \mathrm{gr}^0 H^l(T, \mathbb{Q})$$

as the composite

$$\mathrm{gr}^r H^{2r+l}(V, \mathbb{Q})(r) \xrightarrow{\mathrm{pr}_V^*} \mathrm{gr}^r H^{2r+l}(T \times V, \mathbb{Q})(r)$$
$$\longrightarrow \mathrm{gr}^m H^{2m+l}(T \times V, \mathbb{Q})(m) \xrightarrow{\mathrm{pr}_{T*}} \mathrm{gr}^0 H^l(T, \mathbb{Q}),$$

where the second map is defined by the cup-product with

$$\{{}^t z\} \in \mathrm{gr}^p H^{2p}(T \times V, \mathbb{Q})(p).$$

THEOREM 5.2. *Let V be a smooth projective variety of dimension m, S a smooth variety, $z \in \mathrm{CH}^p(S \times V)$, $r = m - p$, and l an integer. If $z(s) \in H_{\mathbb{Q}}^{*(l+1)}\mathrm{CH}^p(V)$ for all $s \in S$, then the map*

$$\{{}^t z\}: \mathrm{gr}^r H^{2r+l}(V, \mathbb{Q})(r) \to \mathrm{gr}^0 H^l(S, \mathbb{Q})$$

is zero.

Let Z be a cycle on $S \times V$ representing $z \in \mathrm{CH}^p(S \times V)$. By shrinking S, if necessary, we may assume that $Z(s)$ are defined for all $s \in S$. Then $Z(s)$ are $H^{*(l+1)}_{\mathbb{Q}}$-equivalent to zero. By theorem 4.2, there exist a smooth variety T, a dominant morphism $e: T \to S$, a smooth projective morphism $\pi: \mathscr{F} \to T$, and cycles X_{ij} of codimension p_{ij} on $\mathscr{F} \times V$ $(0 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant n)$ such that

$$\sum_j p_{ij} = \dim \mathscr{F} - \dim T + p;$$

For any $t \in T$, $X_{ij} | \mathscr{F}_t \times V$ are $\mathbb{Q}$-homologous to zero;

$$(e \times \mathrm{id}_V)^*(Z) = \sum_j \pi_*(X_{0j} \cdot \cdots \cdot X_{lj}) \text{ in } \mathrm{CH}(T \times V).$$

We have a factorization

$$\{{}^t(e \times \mathrm{id}_V)^*(Z)\} : \mathrm{gr}^r H^{2r+l}(V, \mathbb{Q}(r)) \xrightarrow{\{{}^t z\}} \mathrm{gr}^0 H^l(S, \mathbb{Q}) \xrightarrow{e^*} \mathrm{gr}^0 H^l(T, \mathbb{Q}),$$

and e^* is injective (cf. [12], 1.7). The following lemma will complete the proof of the theorem:

LEMMA 5.2.1. *Let T, X, V be smooth varieties, $g: X \to V$ be a morphism and $f: X \to T$ a smooth proper morphism of relative dimension m, Z_i $(0 \leqslant i \leqslant l)$ be cycles on X of codimension p_i such that the restriction of Z_i to a fiber X_t is $\mathbb{Q}$-homologically equivalent to zero. Put $p = p_0 + \cdots + p_l$, $r = m - p$, and*

$$z = \{Z_0\} \cup \cdots \cup \{Z_l\} \in H^{2p}(X, \mathbb{Q}(p)).$$

Then the map

$$H^{2r+l}(V, \mathbb{Q}(r)) \xrightarrow{g^*} H^{2r+l}(X, \mathbb{Q}(r)) \xrightarrow{\cup z} H^{2m+l}(X, \mathbb{Q}(m)) \xrightarrow{f_*} H^l(T, \mathbb{Q})$$

is zero.

We have the Leray spectral sequence

$$E_2^{p,n-p}(f) = H^p(T, R^{n-p} f_* \mathbb{Q}(k)) \Rightarrow F^p H^n(X, \mathbb{Q}(k)).$$

(i) By intersection, we get a pairing of spectral sequence

$$\begin{array}{ccccc} H^p(T, R^{n-p} f_* \mathbb{Q}(k)) & \times & H^{p'}(T, R^{n'-p'} f_* \mathbb{Q}(k')) & \xrightarrow{\cup} & H^{p+p'}(T, R^{n+n-p-p'} f_* \mathbb{Q}(k+k')) \\ \Downarrow & & \Downarrow & & \Downarrow \\ H^n(X, \mathbb{Q}(k)) & \times & H^{n'}(X, \mathbb{Q}(k')) & \xrightarrow{\cup} & H^{n+n'}(X, \mathbb{Q}(k+k')) \end{array}$$

in particular, we have

$$F^pH^n(X, \mathbb{Q}(k)) \cup F^{p'}H^{n'}(X, \mathbb{Q}(k')) \subset F^{p+p'}H^{n+n'}(X, \mathbb{Q}(k+k')).$$

(ii) If $f': X' \to T$ is smooth of relative dimension m' and $h: X \to X'$ is a proper T-morphism, and if $d = m - m'$, we have a morphism of spectral sequence

$$\begin{array}{ccccc} E_2^{p,n-p}(f) & = & H^p(T, R^{n-p}f_*\mathbb{Q}(k)) & \Rightarrow & F^pH^n(X, \mathbb{Q}(k)) \\ & & \Big\downarrow{\scriptstyle h_*} & & \Big\downarrow{\scriptstyle h_*} \\ e_2^{p,n-2d-p}(f') & = & H^p(T, R^{n-2d-p}f'_*\mathbb{Q}(k-d)) & \Rightarrow & F^pH^{n-2d}(X', \mathbb{Q}(k-d)), \end{array}$$

in particular, $h_* F^pH^n(X', \mathbb{Q}(k)) \subset F^pH^{n-2d}(X', \mathbb{Q}(k-d))$.

By Lemma 2.4.1 (see also Remark 2.4.3.),

$$\{Z_i\} \in F^1H^{2p_i}(X, \mathbb{Q}(p_i)).$$

For

$$\alpha \in H^{2r+l}(V, \mathbb{Q}(r)),$$
$$g^*(\alpha) \in H^{2r+l}(X, \mathbb{Q}(r)) = F^0H^{2r+l}(X, \mathbb{Q}(r)),$$

and by (ii) and iterated use of (i), we obtain

$$f_*(z \cup g^*(\alpha)) = f_*(\{Z_0\} \cup \cdots \cup \{Z_l\} \cup g^*(\alpha)) \in F^{l+1}H^l(T, \mathbb{Q}) = 0,$$

hence, the lemma is proven.

5.3. For a smooth projective variety W and integers q, l, we consider the condition:

$H(W, q, l)$: There exist smooth projective varieties T_j, and cycles u_j on $T_j \times W$ of codimension $\dim W - q$ such that

(i) the map

$$\mathrm{gr}^q H^{2q+l}(W, \mathbb{Q}(q)) \to \coprod_j \mathrm{gr}^0 H^l(T_j, \mathbb{Q})$$

induced by u_j is injective;

(ii) The following condition $H(T_j, l)$ holds for l and all T_j. $H(T, l)$: There exist smooth varieties S, $\mathscr{F}$, morphisms $\mathscr{F} \to S$, and cycles $x_{1k}, \ldots, x_{lk}$ ($1 \leqslant k \leqslant n$, $n \geqslant 1$) on $\mathscr{F} \times T$ such that

(i) $\mathscr{F} \to S$ is smooth projective;

(ii) $\sum_i \operatorname{codim} x_{ik} = \operatorname{rel.dim} \mathscr{F}/S + \dim T$, for all k;
(iii) $x_{ik} \mid \mathscr{F}_s \times T$ are homologous to zero for all i, k and $s \in S$;
(iv) The map

$$\mathrm{gr}^0 H^l(T, \mathbb{Q}) \to \mathrm{gr}^0 H^l(S, \mathbb{Q})$$

induced by the cycle $\sum_k x_{1k} \cdot \cdots \cdot x_{lk}$ is injective.

(For $l = 0$, $\sum_k x_{1k} \cdot \cdots \cdot x_{lk} = n \cdot 1_{\mathscr{F} \times T} \neq 0$, and by (ii), T must be a point; in that case, $H(T, 0)$ always holds).

The reason we introduce the condition is this:

COROLLARY 5.4. *Let V and W be smooth projective varieties of dimension m and n respectively, $z \in \mathrm{CH}^{p+q}(W \times V)$, $r = m - p$, and l an integer, and suppose that the condition $H(W, q, l)$ holds. If the map*

$$[z]: \mathrm{Gr}^l \mathrm{CH}_q(W) \to \mathrm{Gr}^l \mathrm{CH}_r(V)$$

(cf. 1.8) *is zero, then the map*

$$\{{}^t z\}: \mathrm{gr}^r H^{2r+l}(V, \mathbb{Q}(r)) \to \mathrm{gr}^q H^{2q+l}(W, \mathbb{Q}(q))$$

is also zero.

With the notations of 5.3, we have

$$0 = [z \circ u_j]: \mathrm{Gr}^l \mathrm{CH}_0(T_j) \xrightarrow{[u_j]} \mathrm{Gr}^l \mathrm{CH}_q(W) \xrightarrow{[z]} \mathrm{Gr}^l \mathrm{CH}_r(V)$$

and

$$\sum_j \{{}^t(z \circ u_j)\}: \mathrm{gr}^r H^{2r}(V, \mathbb{Q}(r)) \xrightarrow{\{{}^t z\}} \mathrm{gr}^q H^{2q+l}(W, \mathbb{Q}(q))$$
$$\xrightarrow{\sum \{{}^t u_j\}} \coprod_j \mathrm{gr}^0 H^l(T_j, \mathbb{Q}),$$

and since $\sum_j \{{}^t u\}$ is injective, we may assume that $H(W, l)$ holds and $q = 0$. The notation being as in the definition of $H(T, l)$ with $T = W$, let $x = \sum_k x_{1k} \cdot \cdots \cdot x_{lk}$. If $\pi: \mathscr{F} \to S$ is the morphism, then we set $y = (\pi \times \mathrm{id}_T)_*(x) \in \mathrm{CH}(S \times T)$. For $s \in S$,

$$y(s) \in \langle H^{*l} \rangle_0 \mathrm{CH}_0(T),$$

and $z \circ y(s) \in H^{*(l+1)} \mathrm{CH}_r(V)$. By the theorem, we have that

$$\{{}^t(z \circ y)\}: \mathrm{gr}^r H^{2r+l}(V, \mathbb{Q}(r)) \xrightarrow{\{{}^t z\}} \mathrm{gr}^0 H^l(W, \mathbb{Q}(q)) \xrightarrow{\{{}^t y\}} \mathrm{gr}^0 H^l(S, \mathbb{Q})$$

is zero. But by 5.3, (iv), $\{{}^t y\}$ is injective, so that $\{{}^t z\}$ is zero.

5.5. We shall reformulate the corollary 5.4. To do so, we introduce a pseudo-abelian category $\mathscr{C}(l)$. First, we define an additive category $\mathscr{C}^*(l)_Z$ as follows:

Objects: formal sum $\coprod_i \mathrm{Gr}^l\mathrm{CH}_{r_i}(V_i)$, where the condition $H(V_i, r_i, l)$ holds for each smooth projective variety V_i.

Morphisms:

$$\begin{aligned}&\mathrm{Hom}(\mathrm{Gr}^l\mathrm{CH}_q(W), \mathrm{Gr}^l\mathrm{CH}_r(V))\\ &\quad = \{[z]: \mathrm{Gr}^l\mathrm{CH}_q(W) \to \mathrm{Gr}^l\mathrm{CH}_r(V);\ z \in \mathrm{CH}^{p+q}(W \times V),\ p+r = \dim V\}\end{aligned}$$

and for general objects, we define

$$\mathrm{Hom}\left(\coprod_j \mathrm{Gr}^l\mathrm{CH}_{q_j}(W_j), \coprod_i \mathrm{Gr}^l\mathrm{CH}_{r_i}(V_i)\right) = \coprod_{i,j} \mathrm{Hom}(\mathrm{Gr}^l\mathrm{CH}_{q_j}(W_j), \mathrm{Gr}^l\mathrm{CH}_{r_i}(V_i)).$$

It is clear that $\mathscr{C}^*(l)_Z$ is an additive category, and we define a $\mathbb{Q}$-additive category $\mathscr{C}^*(l)$ having the same objects as $\mathscr{C}^*(l)_Z$ and

$$\mathrm{Hom}_{\mathscr{C}^*(l)}(M, N) = \mathrm{Hom}_{\mathscr{C}^*(l)_\mathbb{Z}}(M, N) \otimes \mathbb{Q}.$$

Then the pseudo-abelian category $\mathscr{C}(l)$ is obtained as the pseudo-abelian envelope of $\mathscr{C}^*(l)$.

Let Hdg be the category of polarizable $\mathbb{Q}$-Hodge structures and Hdg(l) be the full subcategory of Hdg whose objects are effective of weight l. As noted above, $\mathrm{gr}^r H^{2r+l}(V, \mathbb{Q}(r)) \in \mathrm{Hdg}(l)$. By Corollary 5.4, we have

COROLLARY 5.6. *We have an additive contravariant functor*

$$n: \mathscr{C}(l) \to \mathrm{Hdg}(l),\ \mathrm{Gr}^l\mathrm{CH}_r(V) \mapsto \mathrm{gr}^r H^{2r+l}(V, \mathbb{Q}(r)).$$

LEMMA 5.7. (i) *If z is a cycle of codimension $q + \dim W' - q'$ on the product $W \times W'$ of smooth projective varieties such that the induced map*

$$\{{}^t z\}: \mathrm{gr}^{q'} H^{2q'+l}(W', \mathbb{Q}(q)) \to \mathrm{gr}^q H^{2q+l}(W, \mathbb{Q}(q))$$

is injective and if the condition $H(W, q, l)$ holds, then the condition $H(W', q', l)$ also holds.

(ii) *If $H(T, l)$ holds. then $H(T, 0, l)$ also holds.*

For (i), let T_j's and u_j's be as in 5.3 (for $H(W, q, l)$). Then, the map

$$\mathrm{gr}^{q'} H^{2q'+l}(W', \mathbb{Q}(q')) \xrightarrow{\{{}^t z\}} \mathrm{gr}^q H^{2q+l}(W, \mathbb{Q}(q)) \to \coprod_j \mathrm{gr}^0 H^l(T_j, \mathbb{Q})$$

induced by $z \circ u_j$ is injective, and $H(T_j, l)$ hold for all T_j.

(ii) is trivial by taking the diagonal as $u = u_1$ in the definition of $H(T, 0, l)$.

PROPOSITION 5.8. *For a smooth projective variety V, the condition $H(V, 0, 2)$ holds.*

Let $i: V' \hookrightarrow V$ be a smooth hyperplane section. Then

$$i^*: H^2(V, \mathbb{Q}) \to H^2(V', \mathbb{Q})$$

is injective if $\dim V' \geqslant 2$. Note that

$$N^1 H^2(V, \mathbb{Q}) = H^2(V, \mathbb{Q}) \cap H^{1,1}(V) = \mathbb{Q}(-1)^{\oplus \rho}.$$

Since $\mathrm{Hom}_{\mathrm{Hdg}}(\mathrm{gr}^0 H^2(V, \mathbb{Q}), \mathbb{Q}(-1)) = 0 = \mathrm{Hom}_{\mathrm{Hdg}}(\mathbb{Q}(-1), \mathrm{gr}^0 H^2(V, \mathbb{Q}))$, we have the canonical decomposition

$$H^2(V, \mathbb{Q}) = \mathrm{gr}^0 H^2(V, \mathbb{Q}) \oplus \mathrm{gr}^1 H^2(V, \mathbb{Q})$$

and $i^*: \mathrm{gr}^0 H^2(V, \mathbb{Q}) \to \mathrm{gr}^0 H^2(V', \mathbb{Q})$ is also injective. Therefore, there exist a surface S, and $j: S \to V$ such that the map

$$j^*: \mathrm{gr}^0 H^2(V, \mathbb{Q}) \to \mathrm{gr}^0 H^2(S, \mathbb{Q})$$

is injective. Then, by lemma 5.7, it suffices to show $H(S, 0, 2)$.

If $b: S' \to S$ is surjective, $b^*: \mathrm{gr}^0 H^2(S, \mathbb{Q}) \to \mathrm{gr}^0 H^2(S', \mathbb{Q})$ is injective. In view of 5.7, considering a Lefschetz pencil and its base change over $\mathbb{P}^1$, for example, we can suppose S has a fibration $\pi: S \to C$ over a curve C with smooth generic fibre, and a section $\sigma: C \to S$.

LEMMA 5.8.1. *Let S be a smooth projective surface. Then there exists a 2-cycle Z on $S \times S$ with $\mathbb{Q}$-coefficients inducing the projector $H^{\cdot}(s \cdot \mathbb{Q}) \to \mathrm{gr}^0 H^2(S, \mathbb{Q})$, i.e., the induced map $H^n(S, \mathbb{Q}) \to H^n(S, \mathbb{Q})$ are zero for $n \neq 2$, $N^1 H^2(S, \mathbb{Q}) \to N^1 H^2(S, \mathbb{Q})$ is also zero, and the map $\mathrm{gr}^0 H^2(S, \mathbb{Q}) \to \mathrm{gr}^0 H^2(S, \mathbb{Q})$ is the identity.*

Sketch of proof. Let $\mathscr{V}$ be the full subcategory of smooth projective schemes consisting of schemes whose components V satisfy the condition $B(V)$ of [7]. Note that the condition B is stable under product, that the Künneth components of the class of the diagonal of V are algebraic (loc. cit., 2.5, 2.9), that the condition $I(V, L)$ (loc. cit.) holds for those schemes by the Hodge theory, and that all the curves and all the surfaces (and all the abelian varieties) belong to $\mathscr{V}$. Starting from $\mathscr{V}$, employing algebraic cycles modulo numerical equivalence as morphisms, we can construct the category $\mathscr{M}$ of motives as in [10]. The category $\mathscr{M}$ is semi-simple, and we have a faithful functor

$$H:\mathscr{M} \to \mathrm{Hdg}$$

with $H(h^n(V)) = H^n(V, \mathbb{Q})$, the Betti realization. By [6], there exist a finite number of curves $C_1, \dots, C_k$ and morphisms $\varphi_i: C_i \to S$ such that the image of

$$\coprod_i H^0(C_i, \mathbb{Q}) \xrightarrow{\sum \varphi_{i*}} H^2(S, \mathbb{Q})(1)$$

is $N^1H^2(S, \mathbb{Q})(1)$. Since C_i, $S \in \mathrm{Ob}\,\mathscr{V}$, we have as well

$$\coprod_i h^0(C_i) \xrightarrow{\sum \varphi_{i*}} h^2(S)(1).$$

in $\mathscr{M}$. Denote the image by I. Since the category $\mathscr{M}$ is semi-simple, we have the projector $p: h^2(S)(1) \to I \subset h^2(S)(1)$. The composite of the morphism $h(S)(1) \to h^2(S)(1)$with $\mathrm{id} - p$ is represented by a 2-cycle with $\mathbb{Q}$-coefficient on $S \times S$ which has the required properties, by considering the Betti realization.

LEMMA 5.8.2. *For a surface S which has a fibration $\pi: S \to T$ over a curve with smooth generic fiber and a section $\sigma: C \to S$, the condition $H(S, 2)$ holds.*

Let C_0 be an open subset of C such that $\pi_0: S_0 := \pi^{-1}(C_0) \to C_0$ is smooth, and set $\mathscr{F} = S_0 \times_C S$. We have the projections $\pi_1: \mathscr{F} \to S_0$ and $\pi_2: \mathscr{F} \to S$, and put

$$\bar{\pi}_1 = \pi_1 \times \mathrm{id}: \mathscr{F} \times S \to S_0 \times S,$$

and

$$\bar{\pi}_2 = \pi_2 \times \mathrm{id}: \mathscr{F} \times S \to S \times S.$$

Note that π_1 is smooth projective, so that $\mathscr{F}$ is smooth. Let Z be the cycle with $\mathbb{Q}$-coefficients as in 5.8.1, and let N be a sufficiently large integer > 0 such that $Z_1 = N \cdot {}^tZ$ has $\mathbb{Z}$-coefficients, and put $X_1 = \bar{\pi}_2^*(Z_1)$, a 3-cycle on $\mathscr{F} \times S$. To the C-morphisms $\psi_1: S_0 \to S$, the inclusion, and $\psi_2 = \sigma \circ \pi: S_0 \to S$, there correspond the sections τ_1, $\tau_2: S_0 \to \mathscr{F}$ of π_1, and $\bar{\tau}_1$, $\bar{\tau}_2: S_0 \times S \to \mathscr{F} \times S$, the base-changes. Finally, we set

$$X_2 = \bar{\tau}_{1*}(1_{S_0 \times S}) - \bar{\tau}_{2*}(1_{S_0 \times S}).$$

For $s \in S_0$, putting $c = \pi(s)$, $\mathscr{F}_s = s \times S_c = S_c$, and we have

$$j^*(X_2) = \psi_1(s) \times S - \psi_2(s) \times S,$$

where $j: S_c \times S = \mathscr{F}_s \times S \to \mathscr{F} \times S$, and $j^*(X_2)$ is homologous to zero on $\mathscr{F}_s \times S$. Denoting the natural inclusion $S_c \times S \to S \times S$ by j', we have

$$j^*(X_1) = (\bar{\pi}_2 \circ j)^*(Z_1) = j'^*(Z_1) = Z_1 \mid S_c \times S.$$

In the Künneth decomposition

$$H^4(S \times S, \mathbb{Q}(2)) = \coprod_{0 \leqslant i \leqslant 4} H^{4-i}(S, \mathbb{Q}) \otimes H^i(S, \mathbb{Q})(2),$$

Z_1 has no other than $H^2(S) \otimes H^2(S)$-component, and further, in the decomposition

$$\begin{aligned} &H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})(2) \\ &\quad = \mathrm{gr}^0 H^2(S, \mathbb{Q}) \otimes \mathrm{gr}^0 H^2(S, \mathbb{Q})(2) \oplus \mathrm{gr}^0 H^2(S, \mathbb{Q}) \otimes \mathrm{gr}^1 H^2(S, \mathbb{Q})(2) \\ &\qquad \oplus \mathrm{gr}^1 H^2(S, \mathbb{Q}) \otimes \mathrm{gr}^0 H^2(S, \mathbb{Q})(2) \oplus \mathrm{gr}^1 H^2(S, \mathbb{Q}) \otimes \mathrm{gr}^1 H^2(S, \mathbb{Q})(2), \end{aligned}$$

Z_1 has only $\mathrm{gr}^0 H^2 \otimes \mathrm{gr}^0 H^2$-component. Hence $Z_1 \mid S_c \times S$ is $\mathbb{Q}$-homologous to zero, by $\mathrm{gr}^0 H^2(S_c, \mathbb{Q}) = 0$. Taking N larger if necessary, we may assume that it is $\mathbb{Z}$-homologous to zero. We claim that

$$\{{}^t(\bar{\pi}_{1*}(X_1 \cdot X_2))\} = N \cdot \psi_1^* : \mathrm{gr}^0 H^2(S, \mathbb{Q}) \to \mathrm{gr}^0 H^2(S_0, \mathbb{Q}),$$

hence, injective. We have $X_1 \cdot X_2 = X_1 \cdot \bar{\tau}_{1*}(1_{S_0 \times S}) - X_1 \cdot \bar{\tau}_{2*}(1_{S_0 \times S})$, and

$$\bar{\pi}_{1*}(X_1 \cdot \bar{\tau}_{i*}(1_{S_0 \times S})) = \bar{\pi}_{1*}(\bar{\tau}_{i*}(\bar{\tau}_i^* \bar{\pi}_2^*(Z_1))) = (\psi_i \times \mathrm{id}_S)^*(Z_1),$$

for $i = 1, 2$. Therefore,

$$\begin{aligned} &\{{}^t(\pi_{1*}(X_1 \cdot X_2))\} : \mathrm{Gr}^0 H^2(S, \mathbb{Q}) \xrightarrow{\{{}^t Z_1\}} \mathrm{gr}^0 H^2(S, \mathbb{Q}) \\ &\quad \xrightarrow{\psi_1^* - \psi_2^*} \mathrm{gr}^0 H^2(S_0, \mathbb{Q}). \end{aligned}$$

On $\mathrm{gr}^0 H^2(S, \mathbb{Q})$, $\{{}^t Z_1\} = N \cdot \mathrm{id}$, $\psi_2^* = 0$ because

$$\psi_2^* : \mathrm{gr}^0 H^2(S, \mathbb{Q}) \xrightarrow{\sigma^*} \mathrm{gr}^0 H^2(C, \mathbb{Q}) = 0 \xrightarrow{\pi^*} \mathrm{gr}^0 H^2(S_0, \mathbb{Q}).$$

This completes the proof of 5.8.2 and hence that of 5.8.

REMARK 5.9. We can prove similarly that for a smooth projective variety V, the condition $H(V, 0, 1)$ holds.

6. Generalization of Abel's theorem

THEOREM 6.1. *Let V and T be a smooth projective varieties, $z \in \mathrm{CH}^{p+q}(T \times V)$, $q' = \dim T - q$. If the map*

$$\{z\} : \mathrm{gr}^{q'-1} H^{2q'-1}(T, \mathbb{Q})(q') \to \mathrm{gr}^{p-1} H^{2p-1}(V, \mathbb{Q})(p).$$

is zero, then the map

$$[z] : \mathrm{Gr}^1 \mathrm{CH}_q(T) \to \mathrm{Gr}^1 \mathrm{CH}^p(V)$$

is also zero.

To see $[z] = 0$, it suffices to show that for any curve C and for any $u \in \mathrm{CH}^{q'}(C \times T)$, the composite

$$[z \circ u] : \mathrm{Gr}^1 \mathrm{CH}_0(C) \xrightarrow{[u]} \mathrm{Gr}^1 \mathrm{CH}_q(T) \xrightarrow{[z]} \mathrm{Gr}^1 \mathrm{CH}_r(V)$$

vanishes because $\langle H \rangle_0 \mathrm{CH}_q(T) = \mathrm{ACH}_q(T)$ is generated by $u(\mathrm{ACH}_0(C))$ for all C and u. By hypothesis,

$$\{z \circ u\} : H^1(C, \mathbb{Q}) = \mathrm{gr}^0 H^1(C) \xrightarrow{\{u\}} \mathrm{gr}^{q'-1} H^{2q'-1}(T)$$
$$\xrightarrow{\{z\}} \mathrm{gr}^{p-1} H^{2p-1}(V) \subset H^{2p-1}(V)$$

is zero. Hence, in the Künneth decomposition,

$$H^{2p}(C \times V) = H^0(C) \otimes H^{2p}(V) \oplus H^1(C) \otimes H^{2p-1}(V) \oplus H^2(C) \otimes H^{2p-2}(V),$$

the $H^1 \otimes H^{2p-1}$-component of the class $\{z \circ u\}$ in $H^{2p}(C \times V, \mathbb{Q})$ vanishes. Some multiple of $z \circ u$ is, therefore, homologically equivalent to the sum of cycles of the form $C \times$(cycle on V of codimension p) and of the form point $\times$(cycle on V of codimension $p-1$). Since these cycles induce the zero map $\mathrm{Gr}^1 \mathrm{CH}_0(C) \to \mathrm{Gr}^1 \mathrm{CH}^p(V)$, the multiple of $z \circ u$ induces the zero map $\mathrm{Gr}^1 CH_0(C) \to \mathrm{Gr}^1 \mathrm{CH}^p(V)$ by 1.8. We have $[z \circ u] = 0$ by divisibility of $\mathrm{Gr}^1 \mathrm{CH}_0(C)$.

THEOREM 6.2. *Let V be a smooth projective variety, and p be an integer. Then,*

(i) $\mathrm{Gr}^1\mathrm{CH}^p(V)$ *has a structure of abelian variety, and the canonical map* $\mathrm{ACH}^p(V) \to \mathrm{Gr}^1\mathrm{CH}^p(V)$ *is regular:* i.e., *for an arbitrary smooth projective variety T, a cycle $z \in \mathrm{CH}^p(T \times V)$, and $t_0 \in T$, the map*

$$T \to \mathrm{Gr}^1\mathrm{CH}^p(V), \quad t \mapsto z((t) - (t_0))$$

is a morphism of varieties.

(ii) *The canonical mapping* (*cf.* 1.11)

$$\gamma^p: \mathrm{Gr}^1\mathrm{CH}^p(V) \to J_a^p(V)$$

is surjective and the kernel is finite.

(iii) *If* $\langle H \rangle_0 \mathrm{CH}^p(V)_{\mathrm{tors}} \to J_a^p(V)$ *is injective, then γ^p is bijective, where* $\langle H \rangle_0 \mathrm{CH}^p(V)_{\mathrm{tors}}$ *denotes the torsion part.*

LEMMA 6.2.1. *There exist an abelian variety A of dimension a and $u \in \mathrm{CH}^p(A \times V)$ such that the induced mapping*

$$\{u\}: \mathrm{gr}^{a-1}H^{2a-1}(A, \mathbb{Q}) \to \mathrm{gr}^{p-1}H^{2p-1}(V, \mathbb{Q})$$

is bijective. Moreover, putting $'\mathrm{Gr}^1\mathrm{CH}^p(V) = \mathrm{ACH}^p(V)/A * \mathrm{HCH}^p(V)$, *the mapping*

$$[u]: {}'\mathrm{Gr}^1\mathrm{CH}_0(A) \to {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$$

is surjective.

We have a surjective map $H_1(P) \to N^{p-1}H^{2p-1}(V)$ induced by an algebraic cycle, where P is an abelian variety. In fact, by [6], $N^{p-1}H^{2p-1}(V)$ is the sum of the images of $H^1(T)$ by f_*, where $f: T \to V$ is projective with $\mathrm{codim} f(T) = p - 1$. Let P_T be the Picard variety of T; then, the Poincaré divisor induces the bijection $H_1(P_T) \to H^1(T)$. Therefore, $N^{p-1}H^{2p-1}(V)$ is the sum of the images of $H_1(P_T) \to H^{2p-1}(V)$ induced by algebraic cycles u_T. Since $N^{p-1}H^{2p-1}(V)$ is finite dimensional, we can find a finite number of T_i such that the sum of the images of $H_1(P_{T_i}) \to H^{2p-1}(V)$ is $N^{p-1}H^{2p-1}(V)$. Let P be the product of P_{T_i}'s, and u' be the sum of pull-backs of u_{T_i} to $P \times V$. Then $H_1(P) = \coprod H_1(P_{T_i})$, and the image of $H_1(P)$ in $H^{2p-1}(V)$ induced by u' is $N^p H^{2p-1}(V)$.

Since the kernel of $H_1(P) \to N^{p-1}H^{2p-1}(V)$ is a sub-Hodge structure of weight -1 of $H_1(P)$, there exists an abelian variety K_1 of P such that $0 \to H_1(K_1) \to H_1(P) \to N^{p-1}H^{2p-1}(V)$ is exact. Let A be an abelian subvariety of P such that $A + K_1 = P$ and $A \cap K_1$ is finite. Then the map $H_1(A) \to N^{p-1}H^{2p-1}(V)$ induced by the restriction $u \in \mathrm{CH}^p(A \times V)$ of u' to $A \times V$ is an isomorphism. Replacing u by $u - 0 \times u(0)$, we may assume $u(0) = 0$.

We shall show that $[u]\colon {}'\mathrm{Gr}^1\mathrm{CH}_0(A) \to {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$ is surjective. It suffices to show that $u\colon A \to {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$, $x \mapsto u(x)$ is surjective.

Let B be an abelian variety and $z \in \mathrm{CH}^p(B \times V)$, $z(0) = 0$. Put $w = 1_B \times u + 1_A \times z \in \mathrm{CH}^p(B \times A \times V)$. We have

$$u\colon A \simeq 0 \times A \hookrightarrow B \times A \xrightarrow{w} {}'\mathrm{Gr}^1\mathrm{CH}^p(V),$$

$$z\colon B \simeq B \times 0 \hookrightarrow B \times A \xrightarrow{w} {}'\mathrm{Gr}^1\mathrm{CH}^p(V).$$

Let $K \subset B \times A$ be an abelian subvariety such that

$$H_1(K) = \mathrm{Ker}(H_1(B \times A) \xrightarrow{w} H^{2p-1}(V)).$$

By 6.1, $K \subset B \times A \xrightarrow{w} {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$ vanishes. Therefore, we obtain

$$\begin{tikzcd}
 & & \mathrm{A} \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] & B \times A \arrow[r] \arrow[drr, "w"'] & (B \times A)/K \arrow[r] \arrow[dr, dotted] & 0, \\
 & & & & {}'\mathrm{Gr}^1\mathrm{CH}^p(V)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & H_1(A) \arrow[d] & & \\
0 \arrow[r] & H_1(K) \arrow[r] & H_1(B \times A) \arrow[r] \arrow[drr, "\{w\}"'] & H_1((B \times A)/K) \arrow[r] \arrow[dr, dotted] & 0, \\
 & & & & N^{p-1}H^{2p-1}(V)
\end{tikzcd}$$

where the dotted maps are not known to be algebraic. Since $\{u\}\colon H_1(A) \to H_1((B \times A)/K) \hookrightarrow N^{p-1}H^{2p-1}(V)$ is bijective, so is the map $H_1(A) \to H_1((B \times A)/K)$. Hence $A \to (B \times A)/K$ is an isogeny, in particular, is surjective. It therefore follows that

$$\mathrm{Im}(B \to \mathrm{Gr}^1\mathrm{CH}^p(V)) \subset \mathrm{Im}(B \times A \xrightarrow{w} {}'\mathrm{Gr}^1\mathrm{CH}^p(V))$$

$$\subset \mathrm{Im}((B \times A)/K \to {}'\mathrm{Gr}^1\mathrm{CH}^p(V)) = \mathrm{Im}(A \xrightarrow{u} {}'\mathrm{Gr}^1\mathrm{CH}^p(V))$$

Since, for any element of $'\mathrm{Gr}^1\mathrm{CH}^p(V)$, we can find an abelian variety B and $z \in \mathrm{CH}^p(B \times V)$ as above such that the element is contained in the image of $B \xrightarrow{z} {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$, we see that $A \xrightarrow{u} {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$ is surjective.

We shall prove the theorem 6.2. Note that we have

$$'\gamma^p : {}'\mathrm{Gr}^1\mathrm{CH}^p(V) \to \mathrm{Gr}^1\mathrm{CH}^p(V) \xrightarrow{\gamma^p} J_a^p(V).$$

Let $'N$ be the kernel of $A \xrightarrow{u} {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$. The map $A \xrightarrow{u} {}'\mathrm{Gr}^1\mathrm{CH}^p(V) \xrightarrow{'\gamma^p} J_a^p(V)$ is an isogeny, since its H_1 is identified with the bijection $H_1(A) \xrightarrow{u} N^{p-1}H^{2p-1}(V)$. Hence $'N$ is contained in the kernel, and finite. By the surjectivity of $A \xrightarrow{u} {}'\mathrm{Gr}^1\mathrm{CH}^p(V)$, and of the maps in the factorization of $'\gamma^p$, the kernel of each of these maps is finite.

Suppose $\mathrm{ACH}^p(V)_{\mathrm{tors}} \to J_a^p(V)_{\mathrm{tors}}$ is injective, and put

$$\bar{K} = \mathrm{Ker}(\mathrm{ACH}^p(V) \to J_a^p(V)).$$

For $k \in \mathbb{Z}$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bar{K} & \to & \mathrm{ACH}^p(V) & \to & J_a^p(V) & \to & 0 \\ & & \downarrow \times k & & \downarrow \times k & & \downarrow \times k & & \\ 0 & \to & \bar{K} & \to & \mathrm{ACH}^p(V) & \to & J_a^p(V) & \to & 0 \end{array}$$

and we see that $\bar{K}$ is torsion-free. From $A * \mathrm{HCH}^p(V) \subset \bar{K}$ follows that $A * \mathrm{HCH}^p(V)$ is torsion-free and divisible (cf. 1.10). Hence $\mathrm{Ker}('\gamma^p) = \bar{K}/A * \mathrm{HCH}^p(V)$ is torsion-free. Since it is finite, $\mathrm{Ker}('\gamma^p) = 0$. As its quotient, $\mathrm{Ker}(\gamma^p) = 0$, too. In particular, for $p = \dim V$, the maps $'\gamma^p$ and γ^p are bijective by [9].

We shall prove (i). Putting $N = \mathrm{Ker}(A \xrightarrow{u} \mathrm{Gr}^1\mathrm{CH}^p(V))$, a finite group, we have

$$A/N \simeq \mathrm{Gr}^1\mathrm{CH}^p(V).$$

The left hand side has a structure of abelian variety, and we endow the right hand side with the structure of abelian variety via the isomorphism above. We shall show that the natural homomorphism

$$\mathrm{ACH}^p(V) \to \mathrm{Gr}^1\mathrm{CH}^p(V)$$

is regular. Let T and z be as in (i), and B be the Albanese variety of $T: \beta : T \to B$, with $\beta(t_0) = 0$. Assume $z = (\beta \times \mathrm{id}_V)^*(z')$, $z' \in \mathrm{CH}^p(B \times V)$. Then,

$$z : T \xrightarrow{\beta} B \xrightarrow{z'} \mathrm{Gr}^1\mathrm{CH}^p(V).$$

With the notations of the proof of lemma 6.2.1, z replaced by z', we have

$$\begin{tikzcd} & B \arrow[d] & \\ A \arrow[r] \arrow[drr, "u"'] & B \times A \arrow[r] & (B \times A)/K. \arrow[d] \\ & & \mathrm{Gr}^1\mathrm{CH}^p(V) \end{tikzcd}$$

Let $N' = \mathrm{Ker}(A \to (B \times A)/K)$. We get $(B \times A)/K = A/N'$ and $N' \subset N$. Then,

$$A/N = (A/N')/(N/N') = ((B \times A)/K)/(N/N'),$$

and the map

$$B \to (B \times A)/K \to ((B \times A)/K)/(N/N') = A/N$$

is a morphism. (Notice we are in characteristic 0.) Therefore,

$$z: T \xrightarrow{\beta} B \xrightarrow{z'} \mathrm{Gr}^1\mathrm{CH}^p(V)$$

is also a morphism.

Next we shall assume $\dim T = 1$. Let J be the jacobian of T and $\mathfrak{P}$ be the Poincaré divisor on $J \times T$. The map

$$\{\mathfrak{P}\}: H_1(J) \to H^1(T) = H_1(T)$$

is the inverse of $\beta: H_1(T) \to H_1(J)$. For $z \in \mathrm{CH}^p(T \times V)$, let

$$z' = z \circ \mathfrak{P} \in \mathrm{CH}^p(J \times V).$$

We have

$$\{z\} = \{(\beta \times \mathrm{id}_V)^*(z')\}: H_1(T) \xrightarrow{\beta_*} H_1(J) \xrightarrow{\{\mathfrak{P}\}} H_1(T) \xrightarrow{\{z\}} H^{2p-1}(V),$$

hence

$$z = (\beta \times \mathrm{id}_V)^*(z'): T \to \mathrm{Gr}^1\mathrm{CH}^p(V)$$

and

$$J \simeq \mathrm{Gr}^1\mathrm{CH}_0(T) \to \mathrm{Gr}^1\mathrm{CH}^p(V)$$

is a morphism.

Consider the general case. Let C be a general curve of T. $i: C \hookrightarrow T$. Then $i_*: \mathrm{Alb}(C) \to \mathrm{Alb}(T)$ is surjective, and we have

$$\begin{array}{ccccc} C & \xrightarrow{i} & T & \xrightarrow{z} & \mathrm{ACH}^p(V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Alb}(C) & \xrightarrow{i_*} & \mathrm{Alb}(T) & \xrightarrow{z} & \mathrm{Gr}^1\mathrm{CH}^p(V) \end{array}.$$

As shown above, $z \circ i_*: \mathrm{Alb}(C) \to \mathrm{Gr}^1\mathrm{CH}^p(V)$ is a morphism, and so is the map $\mathrm{Alb}(T) \to \mathrm{Gr}^1\mathrm{CH}^p(V)$ by the surjectivity of i_*. It follows that $z: T \to \mathrm{Gr}^1\mathrm{CH}^p(V)$ is a morphism.

COROLLARY 6.3. *For* $p = 0, 1, 2, \dim V$, *the canonical map*

$$\gamma^p: \mathrm{Gr}^1\mathrm{CH}^p(V) \to J_a^p(V)$$

is bijective.

We may assume $p = 2$. By virtue of [8], for any prime e, we have an isomorphism $\mathrm{CH}^2(V)(e) \simeq N^1H^3(V, \mathbb{Q}_e/\mathbb{Z}_e(2))$, where $\mathrm{CH}^p(V)(e)$ is the e-torsion subgroup of $\mathrm{CH}^p(V)$, and the map is induced by Bloch's map [4]. Summing up over all primes, we get

$$\begin{aligned} \mathrm{ACH}^2(V)_{\mathrm{tors}} &\subset \mathrm{CH}^2(V)_{\mathrm{tors}} \simeq N^1H^3(V, \mathbb{Q}/\mathbb{Z}) \\ &\subset H^3(V, \mathbb{Q}/\mathbb{Z}) \simeq T^2(V)_{\mathrm{tors}}, \end{aligned}$$

which is induced from the Abel-Jacobi map

$$\mathrm{ACH}^2(V) \to J_a^2(V) \subset T^2(V).$$

REMARK. 6.4. In the course of the proof of 6.2, we have proven that the subgroups

$$\begin{aligned} A * \mathrm{HCH}^p(V) &\subset \langle H\rangle_0\mathrm{CH}^p(V) \cap H^{*2}\mathrm{CH}^p(V) \\ &\subset \mathrm{Ker}(\langle H\rangle_0\mathrm{CH}^p(V) \to J_a^p(V)) \end{aligned}$$

coincide up to finite groups, and if the assumption 6.2, (iii) is satisfied, then they coincide precisely.

7. The equivalence of categories

THEOREM 7.1. *Let T, V be smooth projective varieties, $m = \dim V$, $z \in \mathrm{CH}^{p+q}(T \times W)$ and assume the condition $D(V, r, 2)$:*

$$N^r H^{2r+2}(V, \mathbb{Q})(r) \otimes N^{p-2} H^{2p-2}(V, \mathbb{Q})(p) \to H^{2m}(V, \mathbb{Q})(m) = \mathbb{Q}$$

is a perfect pairing, where $p = m - r$. If

$$0 = \{{}^t z\} : \mathrm{gr}^r H^{2r+2}(V, \mathbb{Q})(r) \to \mathrm{gr}^q H^{2q+2}(T, \mathbb{Q})(q),$$

then, we have

$$0 = [z] : \mathrm{Gr}^2 \mathrm{CH}_q(T) \to \mathrm{Gr}^2 \mathrm{CH}_r(V).$$

LEMMA 7.1.1. *The adequate equivalence relation $\langle H^{*2} \rangle_0$ is generated by $\langle H^{*2} \rangle_0 \mathrm{CH}_0$ of surfaces. More precisely, for an arbitrary smooth projective variety V, we have*

$$\langle H^{*2} \rangle_0 \mathrm{CH}(V) = \sum z(\langle H^{*2} \rangle_0 \mathrm{CH}_0(S)),$$

where S ranges over all surfaces, and z ranges over all of elements of $\mathrm{CH}(S \times V)$.

We denote the right hand side by $\mathrm{ECH}(V)$. It is clear that E gives an adequate equivalence relation, and that $\langle H^{*2} \rangle_0 \supset E$. Note that $\langle H^{*2} \rangle_0$ is generated by $H^{*2}\mathrm{CH}_0$, and, by 6.3, and 6.4, we have $\langle H^{*2} \rangle_0 \mathrm{CH}_0 = A * \mathrm{HCH}_0$, hence that

$$\mathrm{ECH}(V) = \sum z(A * \mathrm{HCH}_0(S)),$$

where S runs over all surfaces and $z \in \mathrm{CH}(S \times V)$. We may assume $\dim V > 2$ and it is sufficient to show that $A * \mathrm{HCH}_0(V) = \mathrm{ECH}_0(V)$, i.e., that for a smooth projective variety T, $x \in \mathrm{HCH}^p(T \times V)$, $y \in \mathrm{ACH}^q(T \times V)$, with $p + q = \dim T + \dim V$, we have

$$\mathrm{pr}_{V*}(x \cdot y) \in \mathrm{ECH}_0(V).$$

By definition, there exist a curve C, $u \in \mathrm{CH}^q(C \times T \times V)$, and points a and b of C such that $y = u(\gamma)$, where $\gamma = (a) - (b)$. Since

$$\mathrm{pr}_{V*}(x \cdot y) = \mathrm{pr}_{V*}(1_C \times x \cdot u \cdot \gamma \times 1_T \times 1_V),$$

it suffices to show that $1_C \times x \cdot u \cdot \gamma \times 1_T \times 1_V \in \mathrm{ECH}_0(C \times T \times V)$. We are thus reduced to show the following assertion:

Let V be a smooth projective variety of dimension >2 with a morphism $\pi: V \to C$ to a curve, $x \in \mathrm{HCH}_1(V)$, $\gamma \in \mathrm{ACH}_0(C)$. Then $x \cdot \pi^*(\gamma) \in \mathrm{ECH}(V)$.

Let X be a 1-cycle representing x, and let Supp(X) denote the support of X with reduced scheme structure. Blowing-up V at singular points of Supp(X), we get $b: \tilde{V} \to V$ such that the proper transform of Supp(X) is smooth. Then the proper transform $\tilde{X}$ of X is a 1-cycle whose support is smooth and $b_*(\tilde{X}) = X$. By the following sublemma, we can find a smooth hyperplane section $V' \subset V$, with respect to some embedding into a projective space, containing the support of $\tilde{X}$, if $\dim V > 2$.

SUBLEMMA 7.1.2. *Let X be an r-dimensional smooth subscheme of a smooth projective variety V, I_X the ideal sheaf of X in V, L an ample line bundle. If $2r < \dim V$, a general member of $|I_X \otimes L^{\otimes n}|$ is a smooth variety containing the scheme X for sufficiently large n.*

For sufficiently large n, the map

$$H^0(V, I_X \otimes L^{\otimes n}) \otimes O_V \to I_X \otimes L^{\otimes n}$$

is surjective. Then $(I_X/I_X^2) \otimes L^{\otimes n}$ is generated by the global sections of $H^0(V, I_X \otimes L^{\otimes n})$. Since the rank of the vector bundle I_X/I_X^2 on X is $\dim V - r > r$, the image of a general member s of $|I_X \otimes L^{\otimes n}|$ by the canonical map $I_X \otimes L^{\otimes n} \to (I_X/I_X^2) \otimes L^{\otimes n}$ vanishes nowhere. Then, $V' = (s) \subset V$ is smooth at the points of X. By Bertini's theorem, it is smooth off X, whence the sublemma 7.1.2.

We return to the proof of 7.1.1. Taking hyperplane sections repeatedly, we obtain a smooth surface $S \overset{i}{\hookrightarrow} \tilde{V}$ containing the support of $\tilde{X}$. Let $b' = b \circ i$. Denoting by X' the 1-cycle $\tilde{X}$ regarded as a cycle on S, we have $b'_*(X') = X$. In the commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}^0 V = \mathrm{Gr}^1\mathrm{CH}^1(V) & \xrightarrow[b'^*]{} & \mathrm{Gr}^1\mathrm{CH}^1(S) \\
\big\downarrow \cdot X & & \big\downarrow \cdot X' \\
\mathrm{Alb}\, V = \mathrm{Gr}^1\mathrm{CH}_0(V) & \xleftarrow[b'_*]{} & \mathrm{Gr}^1\mathrm{CH}_0(S),
\end{array}$$

the horizontal map below is an isogeny, since

$$b'^*: H^1(V) \xrightarrow{b^*} H^1(\tilde{V}) \xrightarrow{i^*} H^1(S)$$

is an isomorphism. The cycle X is homologous to zero, hence the left vertical arrow vanishes, which means that $b'^*(\alpha) \cdot X' = 0$ in $\mathrm{Gr}^1\mathrm{CH}_0(S)$, for any

$\alpha \in \mathrm{Gr}^1\mathrm{CH}^1(V) = \mathrm{ACH}^1(V)$. In other words, $b'^*(\alpha)\cdot X' \in A * \mathrm{HCH}_0(S)$. It follows that

$$\alpha \cdot X = b'_*(b'^*(\alpha)\cdot X') \in \mathrm{ECH}_0(V).$$

It is now enough to take $\alpha = \pi^*(\gamma)$.

We shall prove the theorem 7.1. By means of 7.1.1, we are reduced to the case where $q = 0$ and T is a surface, as in the proof of theorem 6.1. We shall show that there exists an integer $N \neq 0$ with

$$0 = N\cdot[z]: \mathrm{Gr}^2\mathrm{CH}_0(T) \to \mathrm{Gr}^2\mathrm{CH}_r(V).$$

Then, since $A * \mathrm{HCH}_0(T) = \langle H^{*2}\rangle_0 \mathrm{CH}_0(T)$ as noted above, $\mathrm{Gr}^2\mathrm{CH}_0(T)$ is divisible by 1.10, so that $[z] = 0$.

Since T is a surface, the Künneth components $\Delta_i \in H^i(T) \otimes H^{4-i}(T)$ of the diagonal $\{\Delta_T\} \in H^4(T \times T, \mathbb{Q})$ are algebraic; moreover, for a hyperplane section $h \in H^2(T, \mathbb{Q})$, the inverses of the bijective maps $h^i\cup: H^{2-i}(T, \mathbb{Q}) \to H^{2+i}(T, \mathbb{Q})$ are algebraic (cf. [7]). Put

$$z_i = \{z\} \circ \Delta_i \in H^i(T, \mathbb{Q}) \otimes H^{2p-i}(V, \mathbb{Q}).$$

They are the Künneth components of $\{z\} \in H^{2p}(T \times V, \mathbb{Q})$ and are algebraic. For each i, there exists some integer N such that $N\cdot z_i$ is integral (and algebraic) and induces the zero map

$$\mathrm{Gr}^2\mathrm{CH}_0(T) \to \mathrm{Gr}^2\mathrm{CH}_r(V),$$

which follows from the following two lemmae.

LEMMA 7.1.3. *For $i \neq 2$, there exists an integer N such that $N\cdot\Delta_i$ is integral algebraic and induces the zero map*

$$\mathrm{Gr}^2\mathrm{CH}_0(T) \to \mathrm{Gr}^2\mathrm{CH}_0(T).$$

LEMMA 7.1.4. *Under the hypothesis of* 7.1, *there exists an integer N such that $N\cdot z_2$ is integral algebraic and induces the zero map*

$$\mathrm{Gr}^2\mathrm{CH}_0(T) \to \mathrm{Gr}^2\mathrm{CH}_r(V).$$

Proof of 7.1.3. $L^j = h^j\cup: H^{2-j}(T, \mathbb{Q}) \to H^{2+j}(T, \mathbb{Q})$ has the algebraic inverse $L^{-j} \in H^{4-2j}(T \times T)$. We distinguish two cases:

CASE $i < 2$. Put $\Delta_i' = L^{i-2} \circ \Delta_i \in H^{2i}(T \times T)$. Then $(1 \times h^{2-i} \cdot \Delta_T) \circ \Delta_i' = \Delta_i$. In fact, the left hand side induces the map

$$H^j(T, \mathbb{Q}) \xrightarrow{\Delta_i} H^j(T, \mathbb{Q}) \xrightarrow{L^{i-2}} H^{j+2i-4}(T, \mathbb{Q}) \xrightarrow{h^{2-i} \cup} H^j(T, \mathbb{Q}),$$

which vanishes unless $j = 4 - i$, and in that case, which is *id*. Take integers N_1, N_2 so that $N_1 \cdot \Delta_i$ and $N_1 \Delta_i'$ are integral and that $N_2 N_1 (1 \times h^{2-i} \cdot \Delta_T) \circ \Delta_i'$ and $N_2(N_1 \Delta_i)$ are $\mathbb{Z}$-homologically equivalent. Set $N = N_1 N_2$. By 1.8,

$$N\Delta \colon \mathrm{Gr}^2\mathrm{CH}_0(T) \xrightarrow{N_1 \Delta_i'} \mathrm{Gr}^2\mathrm{CH}^i(T) \xrightarrow{N_2 h^{2-i}} \mathrm{Gr}^2\mathrm{CH}_0(T),$$

which is zero because $\mathrm{Gr}^2\mathrm{CH}^i(T) = 0$ by 1.11.

CASE $i > 2$. Let $\Delta_i' = \Delta_i \circ L^{2-i} \in H^{8-2i}(T \times T, \mathbb{Q})$. Then, $\Delta_i' \circ (1 \times h^{i-2} \cdot \Delta_T) = \Delta_i$:

$$H^j(T, \mathbb{Q}) \xrightarrow{h^{i-2}} H^{j+2i-4}(T, \mathbb{Q}) \xrightarrow{L^{2-i}} H^j(T, \mathbb{Q}) \xrightarrow{\Delta_i} H^j(T, \mathbb{Q}),$$

which is zero unless $j = 4 - i$, and is *id* in that case. Take an integer N similarly as above. Then symbolically,

$$\Delta_i \colon \mathrm{Gr}^2\mathrm{CH}_0(T) \xrightarrow{h^{i-2}} \mathrm{Gr}^2\mathrm{CH}_{2-i}(T) \xrightarrow{\Delta_i'} \mathrm{Gr}^2\mathrm{CH}_0(T).,$$

and $\mathrm{Gr}^2\mathrm{CH}_{2-i}(T) = 0$ because $2 - i < 0$.

Proof of 7.1.4. First, we prove

$$z_2 \in H^2(T) \otimes N^{p-2} H^{2p-2}(V, \mathbb{Q}). \qquad 7.1.5$$

Let $e_1, \ldots, e_b$ be a basis for $H^2(T, \mathbb{Q})$, where $b = \dim H^2(T, \mathbb{Q})$; $e_1^*, \ldots, e_b^*$ be the dual basis for $H^2(T, \mathbb{Q})$ via the intersection product $H^2(T, \mathbb{Q}) \otimes H^2(T, \mathbb{Q}) \to H^4(T, \mathbb{Q}) = \mathbb{Q}$, and write

$$z_2 = \sum e_i \otimes x_i, \quad x_i \in H^{2p-2}(V, \mathbb{Q}).$$

Then,

$$x_i = \mathrm{pr}_{V*}(\{z\} \cdot e_i^* \otimes 1_V).$$

Since $\{z\} \in N^p H^{2p}(V, \mathbb{Q})$, we have $\{z\} \cdot e_i^* \otimes 1_V \in N^p H^{2p+2}(T \times V, \mathbb{Q})$, hence, $x_i \in N^{p-2} H^{2p-2}(V, \mathbb{Q})$, i.e., 7.1.5.

The hypothesis

$$0 = \{{}^t z\} : \mathrm{gr}^r H^{2r+2}(V, \mathbb{Q}) \to \mathrm{gr}^0 H^2(T, \mathbb{Q})$$

means $\{{}^t z\}(N^r H^{2r+2}(V, \mathbb{Q})) \subset N^1 H^2(T, \mathbb{Q})$, or ${}^t z_2(N^r H^{2r+2}(V, \mathbb{Q})) \subset N^1 H^2(T, \mathbb{Q})$. The condition $D(V, r, 2)$ together with 7.1.5 implies $z_2 \in N^1 H^2(T, \mathbb{Q}) \otimes N^{p-2} H^{2p-2}(V, \mathbb{Q})$. Choose the basis $e_1, \ldots, e_b$ so that $e_1, \ldots, e_\rho$ form a basis for $N^1 H^2(T, \mathbb{Q})$, where $\rho = \dim N^1 H^2(T, \mathbb{Q})$, the Picard number of T. Then $e_1^*, \ldots, e_\rho^*$ are the dual basis for $N^1 H^2(T, \mathbb{Q})$, since the restriction to $N^1 H^2(T, \mathbb{Q})$ of the intersection product is perfect. We can write

$$z_2 = \sum_{i \leqslant \rho} e_i \otimes x_i, \quad x_i \in N^{p-2} H^{2p-2}(V, \mathbb{Q}).$$

For $i \leqslant \rho$, we have $e_i^* \otimes 1_V \in N^1 H^2(T \times V, \mathbb{Q})$, and hence, $\{z\} \cdot e_i^* \otimes 1_V \in N^{p+1} H^{2p+2}(T \times V, \mathbb{Q})$, from which we get

$$x_i \in N^{p-1} H^{2p-2}(V, \mathbb{Q}),$$

i.e., x_i is algebraic. Let N_1, N_2 be the non-zero integers such that $N_1 \cdot e_i$'s are represented by integral divisors E_i on T, and that $N_2 \cdot x_i$'s are represented by integral algebraic cycles X_i on V. Put $N = N_1 N_2$. Then $N \cdot z_2$ is represented by the cycle $\sum E_i \times X_i$ which induces the zero map

$$\mathrm{Gr}^2 \mathrm{CH}_0(T) \to \mathrm{Gr}^2 \mathrm{CH}_r(V).$$

7.2. We shall define the pseudo-abelian category $\mathscr{C}'(2)$, as in 5.5, starting from $\mathrm{Gr}^2 \mathrm{CH}_r(V)$ with $H(V, r, 2)$ and $D(V, r, 2)$. Then, $\mathscr{C}'(2)$ is a full subcategory of $\mathscr{C}(2)$ and we have the composite

$$\mathscr{C}'(2) \subset \mathscr{C}(2) \xrightarrow{\eta} \mathrm{Hdg}(2),$$

which we shall also denote by η. Note that $\mathrm{Gr}^2 \mathrm{CH}_0(V)$ are objects of $\mathscr{C}'(2)$ for all smooth projective varieties V, since the condition $D(V, 0, 2)$ holds trivially for $r = 0$, and the condition $H(V, 0, 2)$ holds by 5.8.

COROLLARY 7.3. *The contravariant functor*

$$\eta : \mathscr{C}'(2) \to \mathrm{Hdg}(2)$$

is faithful.

7.4. Let $\mathscr{C}(2)_{\mathrm{surf}}$ be the full pseudo-abelian subcategory of $\mathscr{C}(2)$ obtained from $\mathrm{Gr}^2 \mathrm{CH}_0(S)$ with surfaces S, and let $\mathscr{M}_2$ be the full subcategory of motives defined in 5.8.1, consisting of the subobjects of sums of $\mathrm{gr}^0 h^2(S)$, where S is a surface, $\mathrm{gr}^0 h^2(S) = h^2(S)/N^1 h^2(S)$, and $N^1 h^2(S)$ is the submotive of $h^2(S)$ whose Betti realization is $N^1 H^2(S, \mathbb{Q})$ (cf. 5.7.1). Then $\mathcal{M}_2$ is a semi-simple abelian subcategory of $\mathcal{M}$. Note that by Betti realization, we have a faithful functor

$$H: \mathcal{M}_2 \to \mathrm{Hdg}(2).$$

$\mathscr{C}(2)_{\mathrm{surf}}$ is a full subcategory of $\mathscr{C}'(2)$ and, we have the restriction $\eta: \mathscr{C}(2)_{\mathrm{surf}} \to \mathrm{Hdg}(2)$, which is factorized as

$$\eta: \mathscr{C}(2)_{\mathrm{surf}} \xrightarrow{\eta'} \mathcal{M}_2 \xrightarrow{H} \mathrm{Hdg}(2),$$

where $\eta': \mathscr{C}(2)_{\mathrm{surf}} \to \mathcal{M}_2$ is given by $\mathrm{Gr}^2\mathrm{CH}_0(S) \mapsto \mathrm{gr}^0 h^2(S)$.

COROLLARY 7.5. *The functor η' gives an anti-equivalence of categories:*

$$\eta': \mathscr{C}(2)_{\mathrm{surf}} \xrightarrow{\sim} \mathcal{M}_2.$$

In particular, the category $\mathscr{C}(2)_{\mathrm{surf}}$ is a semi-simple $\mathbb{Q}$-abelian category.

We have shown that η' is faithful. By definition, the morphisms from $\mathrm{gr}^0 h^2(S)$ to $\mathrm{gr}^0 h^2(S')$ are induced by algebraic cycles of codimension 2 on $S' \times S$. Hence it is clear that η' is fully-faithful, and its essential image is $\mathcal{M}_2$, because $\eta'(\mathrm{Gr}^2\mathrm{CH}_0(S)) = \mathrm{gr}^0 h^2(S)$.

REMARKS 7.6.1. Since $\mathrm{gr}^0 H^2(S, \mathbb{Q})$ and $\mathrm{gr}^0 H^2(S, \mathbb{Q})(2)$ are dual via intersection, we could formulate the corollary 7.5 as

$$\mathscr{C}(2)_{\mathrm{surf}} \to \mathcal{M}_2,\ \mathrm{Gr}^2\mathrm{CH}_0(S) \mapsto \mathrm{gr}^0 h^2(S),\ [z] \mapsto \{z\}$$

is an equivalence of categories.

7.6.2. By 7.1.1, for any smooth projective variety V, $\mathrm{Gr}^2\mathrm{CH}_0(V)$ is generated by those of surfaces as abelian group. We do not know, however, whether the inclusion from $\mathscr{C}(2)_{\mathrm{surf}}$ into the category generated by all $\mathrm{Gr}^2\mathrm{CH}_0(V)$ is an equivalence of categories, or more generally, whether $\mathscr{C}(2)_{\mathrm{surf}} \hookrightarrow \mathscr{C}(2)$ is an equivalence of categories. Assume, however, that the standard conjecture $B(V)$ holds universally. Then the conditions $H(V, r, 2)$ and $D(V, r, 2)$ are true and $\mathrm{Gr}^2\mathrm{CH}_r(V)$ is an object of $\mathscr{C}'(2)$ for arbitrary V and r. Hence, $\mathscr{C}'(2) = \mathscr{C}(2)$, and $\mathscr{C}(2)_{\mathrm{surf}} \hookrightarrow \mathscr{C}(2)$ is an equivalence of categories, and they are equivalent to the category $\mathcal{M}_2$ via the functor η.

REMARK 7.7. So far, we have assumed that the ground field k is the complex numbers. Some statements remain true even if k is algebraically closed of characteristic zero. For example, theorems 6.1 and 7.1 are those ones when the

Betti cohomology is replaced by etale cohomology or De Rham cohomology, the proof being reduced to the case of complex numbers by the comparison theorem. However, theorem 4.1 (hence 5.1) makes essential use of the hypothesis that the ground field is uncountable, and it is plausible that it is false if k is the algebraic closure of the field of rational numbers. Hence it might be a right formulation to define first a functor of the form

$$\mathscr{M}_l \to \mathscr{C}(l)$$

and to show it is (fully) faithful when k is, for example, the field of complex numbers.

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