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On the average of inversion of Dirichlet's L-functions*

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Abstract. The main purpose of this paper is to use the estimates of the character sums and the elementary method to give a sharper asymptotic formula for the first power mean of the inversion of Dirichlet's L-functions.

1. Introduction

For real number $Q > 2$, let integer $q \leqslant Q$, χ denote a typical Dirichlet character mod q, and $L(s, \chi)$ be the corresponding Dirichlet L-function. The main purpose of this paper is to study the asymptotic property of the first power mean

$$\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \frac{1}{|L(1, \chi)|} \tag{1}$$

where $\sum_{\chi \bmod q}$ denotes the summation over all characters mod q and the function

$$A(q) = \prod_{p/q} \left(1 + \frac{1}{4p^2} + \cdots + \frac{\binom{2n}{n}^2}{4^{2n}(2n-1)^2 p^{2n}} + \cdots \right).$$

For mean value (1) or a similar problem, it seems to have not been studied before. This paper, using the elementary method and R. C. Vaughan's work, studies the asymptotic property of (1) and proves the following theorem:

THEOREM. *Let real number $Q > 2$, then we have the asymptotic formula*

$$\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \frac{1}{|L(1, \chi)|} = Q \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + 0(Q^{1/2} \ln^2 Q)$$

where $r(n)$ is a multiplicative function defined as follows.

2. Some lemmas

In this section, we shall give several basic lemmas which are necessary in the

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course of proving the theorem. First we define the multiplicative function $r(n)$ as follows:

For any prime p and any non-negative integer α, we define

$$r(p^{\alpha}) = -\binom{2\alpha}{\alpha}\Big/ 4^{\alpha}(2\alpha - 1),$$

where

$$\binom{2n}{n} = (2n)!/(n!)^2, \quad 0! = 1.$$

For any positive integer n, let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, define $r(n) = r(p_1^{\alpha_1}) \cdot r(p_2^{\alpha_2}) \cdots r(p_k^{\alpha_k})$. It is clear that the function $r(n)$ is a multiplicative function. Using this function we can deduce the following.

LEMMA 1. *If integer $n > 0$, $\mu(n)$ is a Möbius function, then we have identity*

$$\mu(n) = \sum_{d/n} r(d) r(n/d).$$

Proof. Since $\mu(n)$ and $r(n)$ are multiplicative functions of n, we prove that lemma 1 holds only for prime powers $n = p^{\alpha}$. Considering the power series expansion of function $(1 - x)^{1/2}$:

$$\begin{aligned}(1 - x)^{1/2} &= 1 - \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n(2n - 1)} x^n \\ &= \sum_{n=0}^{\infty} r(p^n) x^n, \quad |x| < 1. \qquad (2)\end{aligned}$$

Squaring (2) and comparing coefficients we may get

$$1 - x = \sum_{n=0}^{\infty} \left(\sum_{t+s=n} r(p^s) r(p^t) \right) x^n$$

and

$$\begin{aligned}\sum_{k=0}^{\alpha} r(p^k) r(p^{\alpha-k}) &= \sum_{d/p\alpha} r(d) r(p^{\alpha}/d) \\ &= \begin{cases} 1, & \alpha = 0, \\ -1, & \alpha = 1, \\ 0, & \alpha \geqslant 2, \end{cases} = \mu(p^{\alpha}). \qquad \square\end{aligned}$$

LEMMA 2. *If real number $Q > 2$, then we have*

$$\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \left| \sum_{n \leqslant Q^2} \frac{\chi(n)\mu(n)}{n} \right|$$
$$= Q \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + 0(Q^{1/2} \ln Q).$$

Proof. From lemma 1 we may immediately get

$$\sum_{n \leqslant Q^2} \frac{\chi(n)\mu(n)}{n} = \sum_{mn \leqslant Q^2} \frac{r(m)r(n)\chi(m)\chi(n)}{mn}$$
$$= \left(\sum_{n \leqslant Q} \frac{r(n)\chi(n)}{n} \right)^2 + 2 \sum_{\substack{mn \leqslant Q^2 \\ n > Q}} \frac{r(m)r(n)\chi(m)\chi(n)}{mn}.$$

From the orthogonality of the character, Schwarz's inequality, and the above we may get

$$\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \left| \sum_{n \leqslant Q^2} \frac{\chi(n)\mu(n)}{n} \right|$$
$$= \sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \left| \sum_{n \leqslant Q} \frac{r(n)\chi(n)}{n} \right|^2$$
$$+ 0 \left(\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \left| \sum_{\substack{mn \leqslant Q^2 \\ n > Q}} \frac{r(m)r(n)\chi(mn)}{n} \right| \right)$$
$$= \sum_{q \leqslant Q} A(q) \left[\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{r^2(n)}{n^2} + 0 \left(\sum_{n \geqslant Q} \frac{r^2(n)}{n^2} \right) \right]$$
$$+ 0 \left(\sum_{q \leqslant Q} A(q) \sum_{\substack{n < m < Q \\ m \equiv n(q)}} \frac{r(m)r(n)}{mn} \right)$$
$$+ 0 \left[\sum_{q \leqslant Q} \frac{1}{\sqrt{\phi(q)}} \left(\sum_{\chi \bmod q} \left| \sum_{\substack{mn \leqslant Q^2 \\ n > Q}} \frac{r(m)r(n)\chi(mn)}{mn} \right|^2 \right)^{1/2} \right]$$
$$= Q \cdot \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + 0(1) + 0(\ln^3 Q)$$
$$+ 0 \left[\sum_{q \leqslant Q} \left(\sum_{n > Q}^{\infty} \frac{1}{n^2} + \ln^2 Q/q \right)^{1/2} \right]$$
$$= Q \cdot \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + 0(Q^{1/2} \ln Q).$$

This completes the proof of lemma 2. □

LEMMA 3. *Let real number $Q > 1$, $y > 2$. Then we have*

$$\sum_{q\leqslant Q} \sup_{\substack{a,z\\ z\leqslant y}} \left| \sum_{\substack{n\leqslant z\\ n\equiv a(q)}} \mu(n) \right| \ll_A y\ln^{-A} y + y^{1/2} Q \ln^4 (Qy).$$

Proof. (See [1] Theorem 4) □

LEMMA 4. *Let $y > Q^2 > 1$, $A(y, \chi) = \sum_{Q^2 < n \leqslant y} \chi(n)\mu(n)$, then we have estimate*

$$\sum_{q\leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} |A(y, x)| \ll_A Q^{1/2} y \ln^{-A} y + y^{3/4} Q \ln^2 (Qy).$$

Proof. From Schwarz's inequality and lemma 3, notice that if $A(q) \ll 1$ we may get

$$\begin{aligned}
\sum_{q\leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} |A(y, \chi)| &\ll \sum_{q\leqslant Q} \left(\frac{1}{\phi(q)} \sum_{\chi \bmod q} |A(y, \chi)|^2 \right)^{1/2} \\
&\ll Q^{1/2} \left(\sum_{q\leqslant Q} \frac{1}{\phi(q)} \sum_{\chi \bmod q} |A(y, \chi)|^2 \right)^{1/2} \\
&\ll Q^{1/2} \left(\sum_{q\leqslant Q} \sum_{\substack{Q^2 < n\leqslant y\\ (n,q)=1}} \sum_{\substack{Q^2 < m\leqslant y\\ m\equiv n(q)}} \mu(m)\mu(n) \right)^{1/2} \\
&\ll Q^{1/2} \left(Qy + y \sum_{q\leqslant Q} \sup_{\substack{a,z\\ z\leqslant y}} \left| \sum_{\substack{n\leqslant z\\ n\equiv a(z)}} \mu(n) \right| \right)^{1/2} \\
&\ll Q^{1/2} y \ln^{-A} y + Q y^{3/4} \ln^2 (Qy).
\end{aligned}$$

□

3. Proof of the theorem

In this section, we shall carry out the proof of the theorem, first for $\mathrm{Re}(s) > 1$. We can easily deduce that

$$\frac{1}{L(s, \chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^s}$$

From the above and the Abel's summation we may get

$$\frac{1}{L(s, \chi)} = \sum_{n\leqslant Q^2} \frac{\chi(n)\mu(n)}{n^s} + s \int_{Q^2}^{\infty} \frac{A(y, \chi)}{y^{s+1}}\, \mathrm{d}y \tag{3}$$

Since $L(1,\chi) \neq 0$, thus (3) also holds for $s = 1$. Taking $A = 4$, $s = 1$, from (3), lemma 2 and lemma 4 we may get

$$
\begin{aligned}
&\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \frac{1}{|L(1,\chi)|} \\
&= \sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} \left| \sum_{n \leqslant Q^2} \frac{\chi(n)\mu(n)}{n} \right| \\
&\quad + 0\left[\int_{Q^2}^{+\infty} y^{-2} \left(\sum_{q \leqslant Q} \frac{A(q)}{\phi(q)} \sum_{\chi \bmod q} |A(y,\chi)| \right) \mathrm{d}y \right] \\
&= Q \cdot \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + 0(Q^{1/2} \ln Q) \\
&\quad + 0\left[Q^{1/2} \int_{Q^2}^{+\infty} \frac{1}{y \ln^4 y} \,\mathrm{d}y + Q \int_{Q^2}^{+\infty} \frac{\ln^2(yQ)}{y^{5/4}} \,\mathrm{d}y \right] \\
&= Q \cdot \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} + 0(Q^{1/2} \ln^2 Q).
\end{aligned}
$$

This completes the proof of the theorem. □

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