COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 84, nº 1 (1992), p. 43-52

<http://www.numdam.org/item?id=CM_1992__84_1_43_0>

Compositio Mathematica **84**: 43–52, 1992.

Varieties of small Kodaira dimension whose cotangent bundles are semiample

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Received 15 November 1990; accepted 12 June 1991

We work in the category of complex projective algebraic varieties, and study the fundamental structures of nonsingular varieties of Kodaira dimension 0 and 1 whose cotangent bundles are semiample. Our results are summarized as follows.

A nonsingular variety X is called a *para-abelian variety* if it admits a finite unramified Galois covering $A \to X$ with an abelian variety A. It is clear that a para-abelian variety X is attended with semiample cotangent bundle and of Kodaira dimension $\kappa(X) = 0$. Conversely, we obtain the following:

THEOREM I. *Let X be a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 0$. Then X is a para-abelian variety.*

To simplify our statement of the next result, we introduce a special type of variety.

DEFINITION. Let $V = F \times C$ be the product of a para-abelian variety F and a nonsingular curve C of genus g, and let $X = V/G$ be the quotient of V by a finite group G which acts effectively both on V and on C so that:

(1) $\varphi \circ \sigma = \sigma \circ \varphi$ for every $\sigma \in G$ and for the projection $\varphi: V \to C$;
(2) If $\sigma \in G$ has a fixed point $v \in V$, then $\sigma(v') = v'$ for every point $v' \in \varphi^{-1}(\varphi(v))$.

For each point $c \in C$ put $G_c = \{\sigma \in G \mid \sigma(v') = v' \text{ for every point } v' \in \varphi^{-1}(c)\}$, and set

$$R = \sum_{c \in C} (|G_c| - 1),$$

where $|G_c|$ is the order of the subgroup G_c. Then, in case $R < 2g - 2$, we call X a *variety of type Q_+*.

We shall show that a variety X of type Q_+ is a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$. Such a variety X may seem too typical for the converse to be verified. Nevertheless we obtain the following:

THEOREM II. *Let X be a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$. Then X is a variety of type Q_+.*

The author would like to express his thanks to Prof. T. Fujita, Prof. S. Iitaka, Prof. F. Sakai and Prof. S. Kawai for many helpful suggestions.

Notation and Terminology

$\mathscr{L}^{\otimes m}$	the mth tensor power of a line bundle $\mathscr{L}$
$S^m\mathscr{E}$	the mth symmetric tensor power of a vector bundle $\mathscr{E}$
$\det\mathscr{E}$	the determinant bundle of a vector bundle $\mathscr{E}$
$\mathscr{E}^*$	the dual bundle of a vector bundle $\mathscr{E}$
$\mathbf{P}(\mathscr{E})$	the projective space bundle $\mathbf{Proj}(\oplus_{m\geqslant 0}S^m\mathscr{E})$ associated to a vector bundle $\mathscr{E}$
$\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$	the tautological line bundle of $\mathbf{P}(\mathscr{E})$
$\mathscr{O}_{\mathbf{P}(\mathscr{E})}(m)$	the mth tensor power of $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$
$c_1(\mathscr{E})$	the first Chern class of a vector bundle $\mathscr{E}$
$\mathscr{O}_X$	the structure sheaf of a variety X
$\mathscr{T}_X$	the tangent sheaf of a variety X
Ω_X^1	the sheaf of regular 1-forms on a variety X (the cotangent bundle of a variety X)
ω_X	the canonical sheaf of a variety X
$\Omega_{X/Y}$	the sheaf of relative differentials of a variety X over a variety Y

A *vector bundle* means a locally free sheaf of finite rank. A line bundle is said to be *spanned* if it is generated by its global sections. A vector bundle $\mathscr{E}$ is defined to be *semiample* if for some positive integer m the line bundle $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(m)$ is spanned. We say that a surjective homomorphism h of vector bundles is *splitting* if the short exact sequence derived from h splits.

Given a line bundle $\mathscr{L}$ on a nonsingular variety X, we let $N(\mathscr{L})$ be the set of all positive integers m such that $H^0(X, \mathscr{L}^{\otimes m}) \neq 0$, and for each $m \in N(\mathscr{L})$ let $\Phi_m: X \to \mathbf{P}(H^0(X, \mathscr{L}^{\otimes m}))$ be the canonical rational map. Then we put

$$\kappa(\mathscr{L}, X) = \begin{cases} \max\{\dim \Phi_m(X) \mid m \in N(\mathscr{L})\} & \text{if } N(\mathscr{L}) \neq \phi, \\ -\infty & \text{if } N(\mathscr{L}) = \phi. \end{cases}$$

This is the $\mathscr{L}$-dimension of X introduced by Iitaka [5]. For the canonical sheaf ω_X of X, we put $\kappa(X) = \kappa(\omega_X, X)$ and call it the *Kodaira dimension* of X.

A *fibration* is a dominating morphism of normal varieties with connected fibres. A *fibre bundle* is an analytically locally trivial fibration.

1. Semiample vector bundles

In this section, we study some fundamental properties of semiample vector bundles. We use frequently the following lemmata:

LEMMA 1 (Fujita [2]). *Let $f: X \to Y$ be a dominating morphism of nonsingular varieties and let $\mathscr{E}$ be a vector bundle on Y. Then $\mathscr{E}$ is semiample if and only if the pull-back $f^*\mathscr{E}$ is semiample.*

LEMMA 2 (Fujita). *Let $\mathscr{E}, \mathscr{F}$ be vector bundles on a nonsingular variety X. Then the direct sum $\mathscr{E} \oplus \mathscr{F}$ is semiample if and only if both $\mathscr{E}$ and $\mathscr{F}$ are semiample.*

Proof. Put $\mathscr{G} = \mathscr{E} \oplus \mathscr{F}$. The natural surjective homomorphisms $\mathscr{G} \to \mathscr{E}$, $\mathscr{G} \to \mathscr{F}$ define embeddings $i_1: \mathbf{P}(\mathscr{E}) \to \mathbf{P}(\mathscr{G})$, $i_2: \mathbf{P}(\mathscr{F}) \to \mathbf{P}(\mathscr{G})$ such that $i_1^* \mathscr{O}_{\mathbf{P}(\mathscr{G})}(1) \cong \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$, $i_2^* \mathscr{O}_{\mathbf{P}(\mathscr{G})}(1) \cong \mathscr{O}_{\mathbf{P}(\mathscr{F})}(1)$ respectively. Hence $\mathscr{E}$ and $\mathscr{F}$ are semiample if so is $\mathscr{G}$. Put $Y_1 = i_1(\mathbf{P}(\mathscr{E}))$, $Y_2 = i_2(\mathbf{P}(\mathscr{F}))$. Then the natural injective homomorphisms $\mathscr{E} \to \mathscr{G}$, $\mathscr{F} \to \mathscr{G}$ define morphisms $j_1: \mathbf{P}(\mathscr{G}) \setminus Y_2 \to \mathbf{P}(\mathscr{E})$, $j_2: \mathbf{P}(\mathscr{G}) \setminus Y_1 \to \mathbf{P}(\mathscr{F})$ such that $j_1^* \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1) \cong \mathscr{O}_{\mathbf{P}(\mathscr{G})}(1)|_{\mathbf{P}(\mathscr{G}) \setminus Y_2}$, $j_2^* \mathscr{O}_{\mathbf{P}(\mathscr{F})}(1) \cong \mathscr{O}_{\mathbf{P}(\mathscr{G})}(1)|_{\mathbf{P}(\mathscr{G}) \setminus Y_1}$ respectively. We have $Y_1 \cap Y_2 = \phi$. Therefore, if both $\mathscr{E}$ and $\mathscr{F}$ are semiample, then so is $\mathscr{G}$. Q.E.D.

LEMMA 3 (Iitaka [5]). *Let $f: X \to Y$ be a dominating morphism of nonsingular varieties and let $\mathscr{L}$ be a line bundle on Y. Then $\kappa(f^*\mathscr{L}, X) = \kappa(\mathscr{L}, Y)$.*

LEMMA 4 (cf. Proposition 4.1 in [1]). *Let $h: \mathscr{E} \to \mathscr{L}$ be a surjective homomorphism from a vector bundle $\mathscr{E}$ to a line bundle $\mathscr{L}$. If there exists a positive integer m for which the derived homomorphism $S^m h: S^m \mathscr{E} \to \mathscr{L}^{\otimes m}$ is splitting, then h is splitting.*

Proof. If $m \geqslant 2$, consider the derived homomorphism $S^{m-1}h: S^{m-1}\mathscr{E} \to \mathscr{L}^{\otimes m-1}$. Tensoring with $\mathscr{L}$, we obtain a homomorphism $\alpha: S^{m-1}\mathscr{E} \otimes \mathscr{L} \to \mathscr{L}^{\otimes m}$. On the other hand, the dual homomorphism $h^*: \mathscr{L}^* \to \mathscr{E}^*$ gives rise to the Koszul type exact sequence $0 \to S^{m-1}\mathscr{E}^* \otimes \mathscr{L}^* \to S^m \mathscr{E}^* \to S^m \mathscr{F} \to 0$, where $\mathscr{F}$ is the cokernel of h^*. Hence we obtain a homomorphism $\beta: S^m \mathscr{E} \to S^{m-1}\mathscr{E} \otimes \mathscr{L}$, and then we have $S^m h = \alpha \circ \beta$. This implies that α is splitting and hence so is $S^{m-1}h$. Then by induction we can find that h is splitting.

Q.E.D.

PROPOSITION 1. *Let $\mathscr{E}$ be a semiample vector bundle on a nonsingular variety X, and let $h: \mathscr{E} \to \mathscr{O}_X$ be a nonzero homomorphism. Then h is surjective and splitting.*

Proof. We let m be a positive integer for which the line bundle $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(m)$ is spanned. The homomorphism h is surjective on some open subset $U \subseteq X$. Then there exists a morphism $\rho: U \to \mathbf{P}(\mathscr{E})$ such that $\rho^* \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1) \cong \mathscr{O}_U$. Therefore the natural homomorphism $H^0(X, S^m \mathscr{E}) \otimes_{\mathbf{C}} \mathscr{O}_X \to \mathscr{O}_X$ is surjective on U, and hence on X. This implies that the derived homomorphism $S^m h: S^m \mathscr{E} \to \mathscr{O}_X$ is surjective and splitting, and hence that h is surjective. Then by Lemma 4 we obtain the result.

Q.E.D.

PROPOSITION 2. *Let $\mathscr{E}$ be a semiample vector bundle on a nonsingular variety X. Then the determinant bundle* $\det \mathscr{E}$ *is semiample.*

Proof. We let $\pi: \mathbf{P}(\mathscr{E}) \to X$ be the projective space bundle associated to $\mathscr{E}$, and put $r = \operatorname{rank} \mathscr{E}$. Let m be a positive integer for which the line bundle $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(m)$ is spanned. For any point $x \in X$, choosing suitable r global sections of $\mathcal{O}_{\mathbf{P}(\mathscr{E})}(m)$ and taking the intersection of the divisors defined by them, we can find a nonnegative cycle ξ on $\mathbf{P}(\mathscr{E})$ which represents the class $m^r c_1(\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1))^r$ and which does not meet $\pi^{-1}(x)$. Then the projection $\pi_*(\xi)$ is a nonnegative cycle on X which represents the class $m^r c_1(\mathscr{E})$ (cf. [3]) and which does not contain the point x. This implies that the m^rth tensor power $(\det \mathscr{E})^{\otimes m^r}$ of $\det \mathscr{E}$ has a global section which does not vanish at x. Thus we see that the line bundle $(\det \mathscr{E})^{\otimes m^r}$ is spanned. Q.E.D.

COROLLARY 1. *Let $\mathscr{E}$ be a semiample vector bundle on a nonsingular variety X. Then the vector bundle $\mathscr{E}^* \otimes \det \mathscr{E}$ is semiample.*

Proof. Let $\pi: \mathbf{P}(\mathscr{E}^* \otimes \det \mathscr{E}) \to X$ be the projective space bundle associated to $\mathscr{E}^* \otimes \det \mathscr{E}$, and let $\mathscr{K}$ be the cokernel of the natural homomorphism $\mathcal{O}_{\mathbf{P}(\mathscr{E}^* \otimes \det \mathscr{E})}(-1) \otimes \pi^* \det \mathscr{E} \to \pi^* \mathscr{E}$. Since $\mathscr{E}$ is semiample, so is $\mathscr{K}$. Hence the line bundle $\mathcal{O}_{\mathbf{P}(\mathscr{E}^* \otimes \det \mathscr{E})}(1) \cong \det \mathscr{K}$ is semiample. Q.E.D.

REMARK. If $\mathscr{E}$ is a semiample vector bundle on a nonsingular variety X, then it follows from Proposition 2 that $\kappa(\det \mathscr{E}, X) \geqslant 0$, where the equality holds if and only if $c_1(\mathscr{E}) = 0$ modulo torsion.

PROPOSITION 3. *Let $\mathscr{E}$ be a semiample vector bundle on a nonsingular variety X such that $\kappa(\det \mathscr{E}, X) = 0$. Then there exists a finite unramified covering $f: \tilde{X} \to X$ such that the pull-back $f^*\mathscr{E}$ is a trivial bundle.*

Proof. Let π, r, m be the same as in Proof of Proposition 2, and let $\Phi: \mathbf{P}(\mathscr{E}) \to \mathbf{P}(H^0(\mathbf{P}(\mathscr{E}), \mathcal{O}_{\mathbf{P}(\mathscr{E})}(m)))$ be the canonical morphism. Clearly $\dim \Phi(\mathbf{P}(\mathscr{E})) \geqslant r - 1$. If $\dim \Phi(\mathbf{P}(\mathscr{E})) \geqslant r$, then we can find a positive cycle which represents the class $m^r c_1(\mathscr{E})$. However this contradicts the fact that $\kappa(\det \mathscr{E}, X) = 0$. Thus we have $\dim \Phi(\mathbf{P}(\mathscr{E})) = r - 1$. Therefore if we let W be an irreducible component of a smooth fibre of Φ and let $\lambda_1: W \to X$ be the restriction of the projection π to W, then λ_1 is a finite covering. Furthermore, it follows that

$$\mathcal{O}_{\mathbf{P}(\mathscr{E})}(m) \otimes \mathcal{O}_W \cong \mathcal{O}_W, \tag{1.1}$$

$$\omega_W \cong \omega_{\mathbf{P}(\mathscr{E})} \otimes \mathcal{O}_W. \tag{1.2}$$

On the other hand, we have

$$\omega_{\mathbf{P}(\mathscr{E})} \cong \mathcal{O}_{\mathbf{P}(\mathscr{E})}(-4) \otimes \pi^*(\omega_X \otimes \det \mathscr{E}) \tag{1.3}$$

(cf. Proposition 8.4 in [4]). Recall that there exists a positive integer k for which $(\det \mathscr{E})^{\otimes k} \cong \mathcal{O}_X$. Then from (1.1)–(1.3) we obtain $\omega_W^{\otimes km} \cong \lambda_1^* \omega_X^{\otimes km}$. However the finite covering λ_1 induces a nonzero homomorphism $\lambda_1^* \omega_X \to \omega_W$. Therefore this implies that $\omega_W \cong \lambda_1^* \omega_X$, and hence that λ_1 is a finite unramified covering. By virtue of (1.1), we can find a finite unramified covering $\lambda_2: V \to W$ for which $\lambda_2^*(\mathcal{O}_{\mathbf{P}(\mathscr{E})}(1) \otimes \mathcal{O}_W) \cong \mathcal{O}_V$. Put $\lambda = \lambda_1 \circ \lambda_2: V \to X$. Then λ is a finite unramified covering. The universal quotient $\pi^*\mathscr{E} \to \mathcal{O}_{\mathbf{P}(\mathscr{E})}(1)$ of $\mathbf{P}(\mathscr{E})$ induces a surjective homomorphism $\lambda^*\mathscr{E} \to \mathcal{O}_V$ on V. Then it follows from Proposition 1 that $\lambda^*\mathscr{E} \cong \mathcal{O}_V \oplus \mathscr{F}$ with a vector bundle $\mathscr{F}$ of rank $r-1$. By Lemma 2, $\mathscr{F}$ is semiample, and by Lemma 3, $\kappa(\det \mathscr{F}, V) = 0$. Hence, using induction, we obtain the result. Q.E.D.

COROLLARY 2. *Let $\mathscr{E}$ be a semiample vector bundle on a nonsingular variety X such that $\kappa(\det \mathscr{E}, X) = 0$. Then the dual bundle $\mathscr{E}^*$ is semiample, and $\kappa(\det \mathscr{E}^*, X) = 0$.*

Proof. The result follows immediately from Proposition 3 and Lemma 1, 3. Q.E.D.

COROLLARY 3. *Let $\mathscr{E}$ be a semiample vector bundle of rank r on a nonsingular variety X such that $\kappa(\det \mathscr{E}, X) = 0$ and $\dim H^0(X, \mathscr{E}) = k$. Then $\mathscr{E} \cong \mathscr{F}_1 \oplus \mathscr{F}_2$ with a trivial bundle $\mathscr{F}_1$ of rank k and a semiample vector bundle $\mathscr{F}_2$ of rank $r-k$ such that $\kappa(\det \mathscr{F}_2, X) = 0$ and $H^0(X, \mathscr{F}_2) = 0$.*

Proof. Put $\mathscr{F}_1 = H^0(X, \mathscr{E}) \otimes_{\mathbf{C}} \mathcal{O}_X \cong \bigoplus_{1 \leqslant i \leqslant k} \mathscr{L}_i$ with $\mathscr{L}_i \cong \mathcal{O}_X$ $(i = 1, 2, \ldots, k)$. Then the natural homomorphism $h: \mathscr{E}^* \to \mathscr{F}_1^*$ induces a nonzero homomorphism $h_i: \mathscr{E}^* \to \mathscr{L}_i^*$ for every i. By Corollary 2 the dual bundle $\mathscr{E}^*$ is semiample. Hence, by Proposition 1 we obtain the result. Q.E.D.

COROLLARY 4. *Let $h: \mathscr{E} \to \mathscr{F}$ be a generically surjective homomorphism of vector bundles on a nonsingular variety X. If $\mathscr{E}$ and $\mathscr{F}$ are semiample and if $\kappa(\det \mathscr{F}, X) = 0$, then h is surjective and splitting.*

Proof. By Proposition 3 there exists a finite unramified covering $f: \tilde{X} \to X$ such that $f^*\mathscr{F}$ is a trivial bundle. The homomorphism h is surjective and splitting if so is the pull-back $f^*h: f^*\mathscr{E} \to f^*\mathscr{F}$. Therefore we may assume that $\mathscr{F}$ is a trivial bundle. Then the result follows from Proposition 1. Q.E.D.

2. Varieties with semiample cotangent bundle

Let X be a para-abelian variety. Then X admits a finite unramified covering $f: A \to X$ with an abelian variety A. Since $f^*\Omega_X^1 \cong \Omega_A^1$ is a trivial bundle, by Lemma 1 the cotangent bundle Ω_X^1 is semiample, and $\kappa(X) = 0$ by Lemma 3. Conversely we have Theorem I, which follows immediately from Proposition 3.

Proof of Theorem I. By Proposition 3, there exists a finite unramified Galois covering $f: A \to X$ such that $f^*\Omega_X^1$ is a trivial bundle. Since $\Omega_A^1 \cong f^*\Omega_X^1$, the covering space A is an abelian variety (cf. [6]). Q.E.D.

Before proving the second theorem, we have to study varieties of type Q_+.

PROPOSITION 4. *A variety X of type Q_+ is a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$.*

Proof. We use the same notation as in Introduction. Put $\Gamma = \{\sigma \in G \mid \sigma(v) = v$ for some point $v \in V\}$, and let $H \subseteq G$ be the subgroup generated by Γ. Then $V/H \to X$ is a finite unramified covering. Hence by Lemma 1 and 3, we may assume that $G = H$.

Let v be an arbitrary point in V and put $c = \varphi(v)$, $w = \psi(v)$ where $\psi\colon V \to F$ is the projection. Let s be a regular element of $\mathcal{O}_{C,c}$, and let $\{t_1, t_2, \dots, t_n\}$ be a regular system of parameters of $\mathcal{O}_{F,w}$, where $n = \dim F$. Then we can regard the set $\{s, t_1, t_2, \dots, t_n\}$ as a regular system of parameters of $\mathcal{O}_{V,v}$. For each $\sigma_i \in G_c$, the restriction of the action σ_i to the fibre $\varphi^{-1}(c)$ is the identity. Therefore σ_i gives rise to an automorphism $\check{\sigma}_i$ of the local ring $\mathcal{O}_{V,v}$ such that $\check{\sigma}_i(t_j) = t_j + \varepsilon_{ij}s$ with some $\varepsilon_{ij} \in \mathcal{O}_{V,v}$ for every j. Put $T_j = |G_c|^{-1} \sum_{\sigma_i \in G_c} \check{\sigma}_i(t_j)$. Then for every j we have

$$T_j = t_j + \varepsilon_j s \quad \text{with some } \varepsilon_j \in \mathcal{O}_{V,v}, \tag{2.1}$$

$$\check{\sigma}_i(T_j) = T_j \quad \text{for every } i. \tag{2.2}$$

The group G_c acts also on $\mathcal{O}_{C,c}$. Hence for each i we have $\check{\sigma}_i(s) = \zeta_i s + \eta_i s^2$ with some $\zeta_i \in \mathbb{C}^*$ and some $\eta_i \in \mathcal{O}_{C,c}$. Put $S = |G_c|^{-1} \sum_{\sigma_i \in G_c} \zeta_i^{-1} \check{\sigma}_i(s)$. Then we have

$$S = s + \eta s^2 \quad \text{with some } \eta \in \mathcal{O}_{C,c}, \tag{2.3}$$

$$\check{\sigma}_i(S) = \zeta_i S \quad \text{for every } i. \tag{2.4}$$

Since G_c acts effectively on C, $\zeta_{i_1} \neq \zeta_{i_2}$ if $i_1 \neq i_2$. Let $\hat{\mathcal{O}}_{V,v}$ be the completion of the local ring $\mathcal{O}_{V,v}$, and let $\mathcal{S}_v$ be the subring of all invariant elements in $\hat{\mathcal{O}}_{V,v}$ with respect to the action of G_c. Then from (2.1)–(2.4) we obtain $\hat{\mathcal{O}}_{V,v} \cong \mathbb{C}[[S, T_1, T_2, \dots, T_n]]$ and $\mathcal{S}_v \cong \mathbb{C}[[S^d, T_1, T_2, \dots, T_n]]$ with $d = |G_c|$. Note that $\mathcal{S}_v$ is a regular local ring. Let $f\colon V \to X$ be the quotient morphism and put $x = f(v)$. Then the completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{S}_v$. Thus we see that the quotient space X is nonsingular, and obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*\omega & \longrightarrow & f^*\Omega_X^1 & \longrightarrow & \psi^*\Omega_F^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varphi^*\omega_C & \longrightarrow & \Omega_V^1 & \longrightarrow & \psi^*\Omega_F^1 & \longrightarrow & 0,
\end{array}$$

where $\omega = \omega_C \otimes \mathcal{O}_C(-\sum_{c \in C}(|G_c| - 1)\cdot c)$.

We claim that $f^*\Omega_X^1 \cong \psi^*\Omega_F^1 \oplus \varphi^*\omega$. Let $\{e_1, e_2, \dots, e_k\}$ be a basis of the vector space $H^0(F, \Omega_F^1)$. Each $\sigma \in G$ gives rise to an automorphism σ^* of $H^0(V, \Omega_V^1)$. If $\sigma \in \Gamma$, then for some point $c \in C$ the restriction of the action σ to the fibre $\varphi^{-1}(c)$ is the identity, and therefore the image of $\sigma^*(\psi^* e_j)$ in $H^0(V, \psi^*\Omega_F^1)$ is $\psi^* e_j$. However, since $G = H$, this is true for every $\sigma \in G$. If we put $E_j = |G|^{-1} \sum_{\sigma \in G} \sigma^*(\psi^* e_j)$, then for all $\sigma \in G$ we have $\sigma^*(E_j) = E_j$. Hence every E_j is a section of the vector bundle $f^*\Omega_X^1$, whose image in $H^0(V, \psi^*\Omega_F^1)$ is $\psi^* e_j$. By Corollary 3 we have $\Omega_F^1 \cong \Omega_0 \oplus \Omega$, where $\Omega_0 = \oplus_{1 \leqslant j \leqslant k} \mathcal{O}_F \cdot e_j$ and Ω is a semiample vector bundle of rank $n - k$ such that $\kappa(\det \Omega, F) = 0$ and $H^0(F, \Omega) = 0$. Put $\mathscr{F}_0 = \psi^*\Omega_0$, $\mathscr{F} = \psi^*\Omega$ and put $\mathscr{E} = \oplus_{1 \leqslant j \leqslant k} \mathcal{O}_V \cdot E_j$. Then we have $\psi^*\Omega_F^1 \cong \mathscr{F}_0 \oplus \mathscr{F}$ and $\mathscr{E} \cong \mathscr{F}_0$. Thus we obtain $f^*\Omega_X^1 \cong \mathscr{E} \oplus \mathscr{G}_1$, $\Omega_V^1 \cong \mathscr{E} \oplus \mathscr{G}_2$ with some vector bundles $\mathscr{G}_1$, $\mathscr{G}_2$ for which we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varphi^*\omega \arrow[r] \arrow[d] & \mathscr{G}_1 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varphi^*\omega_C \arrow[r] & \mathscr{G}_2 \arrow[r] & \mathscr{F} \arrow[r] & 0.
\end{tikzcd}$$

Let

$$\delta_1 \colon H^0(V, \mathscr{F} \otimes \mathscr{F}^*) \to H^1(V, \varphi^*\omega \otimes \mathscr{F}^*),$$

$$\delta_2 \colon H^0(V, \mathscr{F} \otimes \mathscr{F}^*) \to H^1(V, \varphi^*\omega_C \otimes \mathscr{F}^*)$$

be the canonical homomorphisms and let $1_{\mathscr{F}} \in H^0(V, \mathscr{F} \otimes \mathscr{F}^*)$ be the identity of $\mathscr{F}$. Since the bottom exact sequence splits, we see that $\delta_2(1_{\mathscr{F}}) = 0$. Consider the following exact sequence:

$$H^0(V, \mathscr{H} \otimes \mathscr{F}^*) \to H^1(V, \varphi^*\omega \otimes \mathscr{F}^*) \to H^1(V, \varphi^*\omega_C \otimes \mathscr{F}^*),$$

where $\mathscr{H} = \varphi^*\omega_C / \varphi^*\omega$. By Corollary 2 and 3, we have $H^0(F, \Omega^*) = 0$. Hence it can be easily checked that $H^0(V, \mathscr{H} \otimes \mathscr{F}^*) = 0$. Thus we find that $\delta_1(1_{\mathscr{F}}) = 0$, and hence obtain

$$f^*\Omega_X^1 \cong \mathscr{E} \oplus \mathscr{G}_1 \cong \mathscr{E} \oplus \mathscr{F} \oplus \varphi^*\omega \cong \mathscr{F}_0 \oplus \mathscr{F} \oplus \varphi^*\omega \cong \psi^*\Omega_F^1 \oplus \varphi^*\omega.$$

Since $R < 2g - 2$, the line bundle ω is ample. Hence by Lemma 1 and 2, we see that the cotangent bundle Ω_X^1 is semiample. Furthermore we have $\kappa(X) = \kappa(f^*\omega_X, V) = \kappa(\psi^*\omega_F \otimes \varphi^*\omega, V) = \kappa(\varphi^*\omega, V) = \kappa(\omega, C) = 1$ by Lemma 3. Q.E.D.

REMARK. In the above proof, it can be easily seen that the condition $R < 2g - 2$ is not only sufficient but also necessary for the quotient X to be attended with semiample cotangent bundle and of Kodaira dimension $\kappa(X) = 1$.

Proof of Theorem II. By Proposition 2, the line bundle ω_X is semiample. Hence we have a fibration $\Phi: X \to B$ with a nonsingular curve B such that $\omega_X^{\otimes k} \cong \Phi^*\mathscr{L}_0$ for some positive integer k and some line bundle $\mathscr{L}_0$ on B. Any smooth fibre of Φ is a para-abelian variety by Theorem I. Let $\mathscr{L}$ be the full subbundle of Ω_X^1 associated to the pull-back $\Omega^*\omega_B$ of the canonical sheaf ω_B. For each point $b \in B$, decompose the fibre $\Phi^{-1}(b) = \sum a_i D_i$ as a sum of irreducible components and set $D(\Phi)_b = \sum (a_i - 1)D_i$. Put $D(\Phi) = \sum_{b\in B} D(\Phi)_b$. Then we have $\mathscr{L} \cong \Phi^*\omega_B \otimes \mathscr{O}_X(D(\Phi))$ (cf. [10]). Consider the natural homomorphism $h: \mathscr{F}_X \to \mathscr{L}^*$. There exists a closed subset Y of codimension 2 such that h is surjective at every point in $X \backslash Y$. Tensoring with ω_X, we obtain a homomorphism $h_1: \mathscr{T}_X \otimes \omega_X \to \mathscr{L}^* \otimes \omega_X$. By Corollary 1 the vector bundle $\mathscr{T}_X \otimes \omega_X$ is semiample. Hence for some positive integer m, the homomorphism

$$h_2: H^0(X, S^m(\mathscr{T}_X \otimes \omega_X)) \otimes_{\mathbb{C}} \mathscr{O}_X \to (\mathscr{L}^* \otimes \omega_X)^{\otimes m}$$

derived from h_1 is surjective at every point in $X \backslash Y$. Write $\mathscr{M} = (\mathscr{L}^* \otimes \omega_X)^{\otimes m}$. Since the direct image $\Phi_*\mathscr{M}$ is a line bundle and since $H^0(B, \Phi_*\mathscr{M}) = H^0(X, \mathscr{M})$, one has $\Phi^*\Phi_*\mathscr{M} \cong \mathscr{M}$ and the zero set of each global section consists of fibres of Φ. Therefore h_2 must be surjective at every point in X. This implies that h is surjective, and hence that any fibre $\Phi^{-1}(b)$ is a multiple of a smooth irreducible component.

Choosing a suitable finite covering $\gamma: C \to B$ with a nonsingular curve C and taking the normalization V of the product $X \times_B C$, we obtain a smooth fibration $\varphi: V \to C$, a finite covering $f: V \to X$,

$$\begin{array}{ccc} V & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ C & \xrightarrow[\gamma]{} & B \end{array}$$

and the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\mathscr{L} & \longrightarrow & f^*\Omega_X^1 & \longrightarrow & \Omega_{V/C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varphi^*\omega_C & \longrightarrow & \Omega_V^1 & \longrightarrow & \Omega_{V/C} & \longrightarrow & 0 \end{array} \tag{2.5}$$

(cf. Theorem 6.3 in [8]). From the upper exact sequence, we obtain the following one:

$$0 \to \Omega_{V/C}^* \otimes f^*\omega_X \to f^*(\mathscr{T}_X \otimes \omega_X) \to f^*(\mathscr{L}^* \otimes \omega_X) \to 0. \tag{2.6}$$

Since the homomorphism h_2 is surjective and since $(\mathscr{L}^* \otimes \omega_X) \otimes m$ is the pull-back of the line bundle $\Phi_* \mathscr{M}$ on B, we can find an open covering $\{U_i\}$ of B such that on each open subset $\Phi^{-1}(U_i)$ the restricted homomorphism $S^m(\mathscr{T}_X \otimes \omega_X)|_{\Phi^{-1}(U_i)} \to (\mathscr{L}^* \otimes \omega_X)^{\otimes m}|_{\Phi^{-1}(U_i)}$ is splitting, and hence so is the homomorphism $\mathscr{T}_X \otimes \omega_X|_{\Phi^{-1}(U_i)} \to \mathscr{L}^* \otimes \omega_X|_{\Phi^{-1}(U_i)}$ by Lemma 4. Then, restricted on each open subset $\varphi^{-1}(\gamma^{-1}(U_i))$, the exact sequence (2.6) splits, and therefore so do both of the exact sequences in (2.5). This implies that the canonical homomorphism $\mathscr{T}_C \to R^1\varphi_*(\Omega^*_{V/C})$ vanishes at every point in C, and hence that φ is a fibre bundle (cf. Theorem 5.1 in [7]).

Since the fibre of φ is a para-abelian variety, we see that $\kappa(\det \Omega_{V/C}, V) = 0$ (cf. [9]). Therefore by Proposition 3, we have a finite unramified Galois covering $\mu: \tilde{V} \to V$ such that $\mu^*\Omega_{V/C}$ is a trivial bundle. Clearly we may assume that the projection $\tilde{\varphi}: \tilde{V} \to C$ is a fibration. Then, since $\Omega_{\tilde{V}/C}$ is a trivial bundle, $\tilde{\varphi}$ is a fibre bundle whose fibre A is an abelian variety. Furthermore we may assume that $\tilde{\varphi}$ has a section $\rho: C \to \tilde{V}$. Choose and fix a basis of $H^0(\tilde{V}, \Omega_{\tilde{V}/C})$. Then the basis and the section $\rho(C)$ determine isomorphisms $\tilde{\varphi}^{-1}(c) \cong A$ for all $c \in C$, which define an isomorphism $\tilde{V} \cong A \times C$. Each element χ in the Galois group $\mathrm{Gal}(\tilde{V}/V)$ gives rise to automorphisms of the fibres $\tilde{\varphi}^{-1}(c)$, and hence defines a continuous mapping $\chi_{\#}: C \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of automorphisms of A. However the order of the element χ is finite. Therefore $\chi_{\#}$ must be constant. Hence we see that φ is a trivial fibre bundle whose fibre is a para-abelian variety.

We may assume that γ is a Galois covering. Then, since the variety V is the normalization of the product $X \times_B C$, it follows that f is a finite Galois covering whose Galois group $G = \mathrm{Gal}(V/X)$ acts effectively both on V and on C so that $\varphi \circ \sigma = \sigma \circ \varphi$ for every $\sigma \in G$. Let F be an arbitrary fibre of φ. Then by Corollary 4 the natural homomorphism $f^*\Omega^1_X \otimes \mathcal{O}_F \to \Omega^1_F$ is surjective, and hence the restriction of f to F is unramified. This implies that if $\sigma \in G$ has a fixed point $v \in V$, then $\sigma(v') = v'$ for every point $v' \in \varphi^{-1}(\varphi(v))$. Finally from Remark to Proposition 4, we infer that the condition $R < 2g - 2$ holds. Thus we find that X is a variety of type Q_+. Q.E.D.

REMARK. Let X be the same as in Definition of a variety of type Q_+, and assume that the condition $R > 2g - 2$ holds in place of $R < 2g - 2$. Then we call X a *variety of type* Q_-. In Proof of Proposition 4, we can easily see that a variety X of type Q_- is a nonsingular variety with semiample tangent bundle such that $\kappa^{-1}(X) = 1$, where $\kappa^{-1}(X) = \kappa(\omega_X^*, X)$ is the *anti-Kodaira dimension* of X. Conversely, in the same manner as in Proof of Theorem II, we obtain the following:

THEOREM II$'$. *Let X be a nonsingular variety with semiample tangent bundle such that $\kappa^{-1}(X) = 1$. Then X is a variety of type Q_-.*

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