

COMPOSITIO MATHEMATICA

TSUYOSHI FUJIWARA

Varieties of small Kodaira dimension whose cotangent bundles are semiample

Compositio Mathematica, tome 84, n° 1 (1992), p. 43-52

http://www.numdam.org/item?id=CM_1992__84_1_43_0

© Foundation Compositio Mathematica, 1992, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Varieties of small Kodaira dimension whose cotangent bundles are semiample

TSUYOSHI FUJIWARA

Nagasaki Institute of Applied Science, 536 Aba-machi, Nagasaki 851-01, Japan

Received 15 November 1990; accepted 12 June 1991

We work in the category of complex projective algebraic varieties, and study the fundamental structures of nonsingular varieties of Kodaira dimension 0 and 1 whose cotangent bundles are semiample. Our results are summarized as follows.

A nonsingular variety X is called a *para-abelian variety* if it admits a finite unramified Galois covering $A \rightarrow X$ with an abelian variety A . It is clear that a para-abelian variety X is attended with semiample cotangent bundle and of Kodaira dimension $\kappa(X) = 0$. Conversely, we obtain the following:

THEOREM I. *Let X be a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 0$. Then X is a para-abelian variety.*

To simplify our statement of the next result, we introduce a special type of variety.

DEFINITION. Let $V = F \times C$ be the product of a para-abelian variety F and a nonsingular curve C of genus g , and let $X = V/G$ be the quotient of V by a finite group G which acts effectively both on V and on C so that:

- (1) $\varphi \circ \sigma = \sigma \circ \varphi$ for every $\sigma \in G$ and for the projection $\varphi: V \rightarrow C$;
- (2) If $\sigma \in G$ has a fixed point $v \in V$, then $\sigma(v') = v'$ for every point $v' \in \varphi^{-1}(\varphi(v))$.

For each point $c \in C$ put $G_c = \{\sigma \in G \mid \sigma(v') = v' \text{ for every point } v' \in \varphi^{-1}(c)\}$, and set

$$R = \sum_{c \in C} (|G_c| - 1),$$

where $|G_c|$ is the order of the subgroup G_c . Then, in case $R < 2g - 2$, we call X a *variety of type Q_+* .

We shall show that a variety X of type Q_+ is a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$. Such a variety X may seem too typical for the converse to be verified. Nevertheless we obtain the following:

THEOREM II. *Let X be a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$. Then X is a variety of type Q_+ .*

The author would like to express his thanks to Prof. T. Fujita, Prof. S. Iitaka, Prof. F. Sakai and Prof. S. Kawai for many helpful suggestions.

Notation and Terminology

$\mathcal{L}^{\otimes m}$	the m th tensor power of a line bundle \mathcal{L}
$S^m \mathcal{E}$	the m th symmetric tensor power of a vector bundle \mathcal{E}
$\det \mathcal{E}$	the determinant bundle of a vector bundle \mathcal{E}
\mathcal{E}^*	the dual bundle of a vector bundle \mathcal{E}
$\mathbf{P}(\mathcal{E})$	the projective space bundle $\mathbf{Proj}(\bigoplus_{m \geq 0} S^m \mathcal{E})$ associated to a vector bundle \mathcal{E}
$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$	the tautological line bundle of $\mathbf{P}(\mathcal{E})$
$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$	the m th tensor power of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$
$c_1(\mathcal{E})$	the first Chern class of a vector bundle \mathcal{E}
\mathcal{O}_X	the structure sheaf of a variety X
\mathcal{T}_X	the tangent sheaf of a variety X
Ω_X^1	the sheaf of regular 1-forms on a variety X (the cotangent bundle of a variety X)
ω_X	the canonical sheaf of a variety X
$\Omega_{X/Y}$	the sheaf of relative differentials of a variety X over a variety Y

A *vector bundle* means a locally free sheaf of finite rank. A line bundle is said to be *spanned* if it is generated by its global sections. A vector bundle \mathcal{E} is defined to be *semiample* if for some positive integer m the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$ is spanned. We say that a surjective homomorphism h of vector bundles is *splitting* if the short exact sequence derived from h splits.

Given a line bundle \mathcal{L} on a nonsingular variety X , we let $N(\mathcal{L})$ be the set of all positive integers m such that $H^0(X, \mathcal{L}^{\otimes m}) \neq 0$, and for each $m \in N(\mathcal{L})$ let $\Phi_m: X \rightarrow \mathbf{P}(H^0(X, \mathcal{L}^{\otimes m}))$ be the canonical rational map. Then we put

$$\kappa(\mathcal{L}, X) = \begin{cases} \max\{\dim \Phi_m(X) \mid m \in N(\mathcal{L})\} & \text{if } N(\mathcal{L}) \neq \emptyset, \\ -\infty & \text{if } N(\mathcal{L}) = \emptyset. \end{cases}$$

This is the \mathcal{L} -dimension of X introduced by Iitaka [5]. For the canonical sheaf ω_X of X , we put $\kappa(X) = \kappa(\omega_X, X)$ and call it the *Kodaira dimension* of X .

A *fibration* is a dominating morphism of normal varieties with connected fibres. A *fibre bundle* is an analytically locally trivial fibration.

1. Semiample vector bundles

In this section, we study some fundamental properties of semiample vector bundles. We use frequently the following lemmata:

LEMMA 1 (Fujita [2]). *Let $f: X \rightarrow Y$ be a dominating morphism of nonsingular varieties and let \mathcal{E} be a vector bundle on Y . Then \mathcal{E} is semiample if and only if the pull-back $f^*\mathcal{E}$ is semiample.*

LEMMA 2 (Fujita). *Let \mathcal{E}, \mathcal{F} be vector bundles on a nonsingular variety X . Then the direct sum $\mathcal{E} \oplus \mathcal{F}$ is semiample if and only if both \mathcal{E} and \mathcal{F} are semiample.*

Proof. Put $\mathcal{G} = \mathcal{E} \oplus \mathcal{F}$. The natural surjective homomorphisms $\mathcal{G} \rightarrow \mathcal{E}$, $\mathcal{G} \rightarrow \mathcal{F}$ define embeddings $i_1: \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}(\mathcal{G})$, $i_2: \mathbf{P}(\mathcal{F}) \rightarrow \mathbf{P}(\mathcal{G})$ such that $i_1^* \mathcal{O}_{\mathbf{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$, $i_2^* \mathcal{O}_{\mathbf{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ respectively. Hence \mathcal{E} and \mathcal{F} are semiample if so is \mathcal{G} . Put $Y_1 = i_1(\mathbf{P}(\mathcal{E}))$, $Y_2 = i_2(\mathbf{P}(\mathcal{F}))$. Then the natural injective homomorphisms $\mathcal{E} \rightarrow \mathcal{G}$, $\mathcal{F} \rightarrow \mathcal{G}$ define morphisms $j_1: \mathbf{P}(\mathcal{G}) \setminus Y_2 \rightarrow \mathbf{P}(\mathcal{E})$, $j_2: \mathbf{P}(\mathcal{G}) \setminus Y_1 \rightarrow \mathbf{P}(\mathcal{F})$ such that $j_1^* \mathcal{O}_{\mathbf{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_{\mathbf{P}(\mathcal{G}) \setminus Y_2}$, $j_2^* \mathcal{O}_{\mathbf{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)|_{\mathbf{P}(\mathcal{G}) \setminus Y_1}$ respectively. We have $Y_1 \cap Y_2 = \emptyset$. Therefore, if both \mathcal{E} and \mathcal{F} are semiample, then so is \mathcal{G} . Q.E.D.

LEMMA 3 (Iitaka [5]). *Let $f: X \rightarrow Y$ be a dominating morphism of nonsingular varieties and let \mathcal{L} be a line bundle on Y . Then $\kappa(f^*\mathcal{L}, X) = \kappa(\mathcal{L}, Y)$.*

LEMMA 4 (cf. Proposition 4.1 in [1]). *Let $h: \mathcal{E} \rightarrow \mathcal{L}$ be a surjective homomorphism from a vector bundle \mathcal{E} to a line bundle \mathcal{L} . If there exists a positive integer m for which the derived homomorphism $S^m h: S^m \mathcal{E} \rightarrow \mathcal{L}^{\otimes m}$ is splitting, then h is splitting.*

Proof. If $m \geq 2$, consider the derived homomorphism $S^{m-1} h: S^{m-1} \mathcal{E} \rightarrow \mathcal{L}^{\otimes m-1}$. Tensoring with \mathcal{L} , we obtain a homomorphism $\alpha: S^{m-1} \mathcal{E} \otimes \mathcal{L} \rightarrow \mathcal{L}^{\otimes m}$. On the other hand, the dual homomorphism $h^*: \mathcal{L}^* \rightarrow \mathcal{E}^*$ gives rise to the Koszul type exact sequence $0 \rightarrow S^{m-1} \mathcal{E}^* \otimes \mathcal{L}^* \rightarrow S^m \mathcal{E}^* \rightarrow S^m \mathcal{F} \rightarrow 0$, where \mathcal{F} is the cokernel of h^* . Hence we obtain a homomorphism $\beta: S^m \mathcal{E} \rightarrow S^{m-1} \mathcal{E} \otimes \mathcal{L}$, and then we have $S^m h = \alpha \circ \beta$. This implies that α is splitting and hence so is $S^{m-1} h$. Then by induction we can find that h is splitting. Q.E.D.

PROPOSITION 1. *Let \mathcal{E} be a semiample vector bundle on a nonsingular variety X , and let $h: \mathcal{E} \rightarrow \mathcal{O}_X$ be a nonzero homomorphism. Then h is surjective and splitting.*

Proof. We let m be a positive integer for which the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$ is spanned. The homomorphism h is surjective on some open subset $U \subseteq X$. Then there exists a morphism $\rho: U \rightarrow \mathbf{P}(\mathcal{E})$ such that $\rho^* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \cong \mathcal{O}_U$. Therefore the natural homomorphism $H^0(X, S^m \mathcal{E}) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow \mathcal{O}_X$ is surjective on U , and hence on X . This implies that the derived homomorphism $S^m h: S^m \mathcal{E} \rightarrow \mathcal{O}_X$ is surjective and splitting, and hence that h is surjective. Then by Lemma 4 we obtain the result. Q.E.D.

PROPOSITION 2. *Let \mathcal{E} be a semiample vector bundle on a nonsingular variety X . Then the determinant bundle $\det \mathcal{E}$ is semiample.*

Proof. We let $\pi: \mathbf{P}(\mathcal{E}) \rightarrow X$ be the projective space bundle associated to \mathcal{E} , and put $r = \text{rank } \mathcal{E}$. Let m be a positive integer for which the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$ is

spanned. For any point $x \in X$, choosing suitable r global sections of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$ and taking the intersection of the divisors defined by them, we can find a nonnegative cycle ξ on $\mathbf{P}(\mathcal{E})$ which represents the class $m^r c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))^r$ and which does not meet $\pi^{-1}(x)$. Then the projection $\pi_*(\xi)$ is a nonnegative cycle on X which represents the class $m^r c_1(\mathcal{E})$ (cf. [3]) and which does not contain the point x . This implies that the m^r th tensor power $(\det \mathcal{E})^{\otimes m^r}$ of $\det \mathcal{E}$ has a global section which does not vanish at x . Thus we see that the line bundle $(\det \mathcal{E})^{\otimes m^r}$ is spanned. Q.E.D.

COROLLARY 1. *Let \mathcal{E} be a semiample vector bundle on a nonsingular variety X . Then the vector bundle $\mathcal{E}^* \otimes \det \mathcal{E}$ is semiample.*

Proof. Let $\pi: \mathbf{P}(\mathcal{E}^* \otimes \det \mathcal{E}) \rightarrow X$ be the projective space bundle associated to $\mathcal{E}^* \otimes \det \mathcal{E}$, and let \mathcal{X} be the cokernel of the natural homomorphism $\mathcal{O}_{\mathbf{P}(\mathcal{E}^* \otimes \det \mathcal{E})}(-1) \otimes \pi^* \det \mathcal{E} \rightarrow \pi^* \mathcal{E}$. Since \mathcal{E} is semiample, so is \mathcal{X} . Hence the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E}^* \otimes \det \mathcal{E})}(1) \cong \det \mathcal{X}$ is semiample. Q.E.D.

REMARK. If \mathcal{E} is a semiample vector bundle on a nonsingular variety X , then it follows from Proposition 2 that $\kappa(\det \mathcal{E}, X) \geq 0$, where the equality holds if and only if $c_1(\mathcal{E}) = 0$ modulo torsion.

PROPOSITION 3. *Let \mathcal{E} be a semiample vector bundle on a nonsingular variety X such that $\kappa(\det \mathcal{E}, X) = 0$. Then there exists a finite unramified covering $f: \tilde{X} \rightarrow X$ such that the pull-back $f^* \mathcal{E}$ is a trivial bundle.*

Proof. Let π, r, m be the same as in Proof of Proposition 2, and let $\Phi: \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}(H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)))$ be the canonical morphism. Clearly $\dim \Phi(\mathbf{P}(\mathcal{E})) \geq r - 1$. If $\dim \Phi(\mathbf{P}(\mathcal{E})) \geq r$, then we can find a positive cycle which represents the class $m^r c_1(\mathcal{E})$. However this contradicts the fact that $\kappa(\det \mathcal{E}, X) = 0$. Thus we have $\dim \Phi(\mathbf{P}(\mathcal{E})) = r - 1$. Therefore if we let W be an irreducible component of a smooth fibre of Φ and let $\lambda_1: W \rightarrow X$ be the restriction of the projection π to W , then λ_1 is a finite covering. Furthermore, it follows that

$$\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m) \otimes \mathcal{O}_W \cong \mathcal{O}_W, \tag{1.1}$$

$$\omega_W \cong \omega_{\mathbf{P}(\mathcal{E})} \otimes \mathcal{O}_W. \tag{1.2}$$

On the other hand, we have

$$\omega_{\mathbf{P}(\mathcal{E})} \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-4) \otimes \pi^*(\omega_X \otimes \det \mathcal{E}) \tag{1.3}$$

(cf. Proposition 8.4 in [4]). Recall that there exists a positive integer k for which $(\det \mathcal{E})^{\otimes k} \cong \mathcal{O}_X$. Then from (1.1)–(1.3) we obtain $\omega_W^{\otimes km} \cong \lambda_1^* \omega_X^{\otimes km}$. However the finite covering λ_1 induces a nonzero homomorphism $\lambda_1^* \omega_X \rightarrow \omega_W$. Therefore this implies that $\omega_W \cong \lambda_1^* \omega_X$, and hence that λ_1 is a finite unramified covering. By

virtue of (1.1), we can find a finite unramified covering $\lambda_2: V \rightarrow W$ for which $\lambda_2^*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{O}_W) \cong \mathcal{O}_V$. Put $\lambda = \lambda_1 \circ \lambda_2: V \rightarrow X$. Then λ is a finite unramified covering. The universal quotient $\pi^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ of $\mathbf{P}(\mathcal{E})$ induces a surjective homomorphism $\lambda^*\mathcal{E} \rightarrow \mathcal{O}_V$ on V . Then it follows from Proposition 1 that $\lambda^*\mathcal{E} \cong \mathcal{O}_V \oplus \mathcal{F}$ with a vector bundle \mathcal{F} of rank $r - 1$. By Lemma 2, \mathcal{F} is semiample, and by Lemma 3, $\kappa(\det \mathcal{F}, V) = 0$. Hence, using induction, we obtain the result. Q.E.D.

COROLLARY 2. *Let \mathcal{E} be a semiample vector bundle on a nonsingular variety X such that $\kappa(\det \mathcal{E}, X) = 0$. Then the dual bundle \mathcal{E}^* is semiample, and $\kappa(\det \mathcal{E}^*, X) = 0$.*

Proof. The result follows immediately from Proposition 3 and Lemma 1, 3. Q.E.D.

COROLLARY 3. *Let \mathcal{E} be a semiample vector bundle of rank r on a nonsingular variety X such that $\kappa(\det \mathcal{E}, X) = 0$ and $\dim H^0(X, \mathcal{E}) = k$. Then $\mathcal{E} \cong \mathcal{F}_1 \oplus \mathcal{F}_2$ with a trivial bundle \mathcal{F}_1 of rank k and a semiample vector bundle \mathcal{F}_2 of rank $r - k$ such that $\kappa(\det \mathcal{F}_2, X) = 0$ and $H^0(X, \mathcal{F}_2) = 0$.*

Proof. Put $\mathcal{F}_1 = H^0(X, \mathcal{E}) \otimes_{\mathbb{C}} \mathcal{O}_X \cong \bigoplus_{1 \leq i \leq k} \mathcal{L}_i$ with $\mathcal{L}_i \cong \mathcal{O}_X$ ($i = 1, 2, \dots, k$). Then the natural homomorphism $h: \mathcal{E}^* \rightarrow \mathcal{F}_1^*$ induces a nonzero homomorphism $h_i: \mathcal{E}^* \rightarrow \mathcal{L}_i^*$ for every i . By Corollary 2 the dual bundle \mathcal{E}^* is semiample. Hence, by Proposition 1 we obtain the result. Q.E.D.

COROLLARY 4. *Let $h: \mathcal{E} \rightarrow \mathcal{F}$ be a generically surjective homomorphism of vector bundles on a nonsingular variety X . If \mathcal{E} and \mathcal{F} are semiample and if $\kappa(\det \mathcal{F}, X) = 0$, then h is surjective and splitting.*

Proof. By Proposition 3 there exists a finite unramified covering $f: \tilde{X} \rightarrow X$ such that $f^*\mathcal{F}$ is a trivial bundle. The homomorphism h is surjective and splitting if so is the pull-back $f^*h: f^*\mathcal{E} \rightarrow f^*\mathcal{F}$. Therefore we may assume that \mathcal{F} is a trivial bundle. Then the result follows from Proposition 1. Q.E.D.

2. Varieties with semiample cotangent bundle

Let X be a para-abelian variety. Then X admits a finite unramified covering $f: A \rightarrow X$ with an abelian variety A . Since $f^*\Omega_X^1 \cong \Omega_A^1$ is a trivial bundle, by Lemma 1 the cotangent bundle Ω_X^1 is semiample, and $\kappa(X) = 0$ by Lemma 3. Conversely we have Theorem I, which follows immediately from Proposition 3.

Proof of Theorem I. By Proposition 3, there exists a finite unramified Galois covering $f: A \rightarrow X$ such that $f^*\Omega_X^1$ is a trivial bundle. Since $\Omega_A^1 \cong f^*\Omega_X^1$, the covering space A is an abelian variety (cf. [6]). Q.E.D.

Before proving the second theorem, we have to study varieties of type Q_+ .

PROPOSITION 4. *A variety X of type Q_+ is a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$.*

Proof. We use the same notation as in Introduction. Put $\Gamma = \{\sigma \in G \mid \sigma(v) = v \text{ for some point } v \in V\}$, and let $H \subseteq G$ be the subgroup generated by Γ . Then $V/H \rightarrow X$ is a finite unramified covering. Hence by Lemma 1 and 3, we may assume that $G = H$.

Let v be an arbitrary point in V and put $c = \varphi(v)$, $w = \psi(v)$ where $\psi: V \rightarrow F$ is the projection. Let s be a regular element of $\mathcal{O}_{C,c}$, and let $\{t_1, t_2, \dots, t_n\}$ be a regular system of parameters of $\mathcal{O}_{F,w}$, where $n = \dim F$. Then we can regard the set $\{s, t_1, t_2, \dots, t_n\}$ as a regular system of parameters of $\mathcal{O}_{V,v}$. For each $\sigma_i \in G_c$, the restriction of the action σ_i to the fibre $\varphi^{-1}(c)$ is the identity. Therefore σ_i gives rise to an automorphism $\check{\sigma}_i$ of the local ring $\mathcal{O}_{V,v}$ such that $\check{\sigma}_i(t_j) = t_j + \varepsilon_{ij}s$ with some $\varepsilon_{ij} \in \mathcal{O}_{V,v}$ for every j . Put $T_j = |G_c|^{-1} \sum_{\sigma_i \in G_c} \check{\sigma}_i(t_j)$. Then for every j we have

$$T_j = t_j + \varepsilon_j s \quad \text{with some } \varepsilon_j \in \mathcal{O}_{V,v}, \quad (2.1)$$

$$\check{\sigma}_i(T_j) = T_j \quad \text{for every } i. \quad (2.2)$$

The group G_c acts also on $\mathcal{O}_{C,c}$. Hence for each i we have $\check{\sigma}_i(s) = \zeta_i s + \eta_i s^2$ with some $\zeta_i \in \mathbb{C}^*$ and some $\eta_i \in \mathcal{O}_{C,c}$. Put $S = |G_c|^{-1} \sum_{\sigma_i \in G_c} \zeta_i^{-1} \check{\sigma}_i(s)$. Then we have

$$S = s + \eta s^2 \quad \text{with some } \eta \in \mathcal{O}_{C,c}, \quad (2.3)$$

$$\check{\sigma}_i(S) = \zeta_i S \quad \text{for every } i. \quad (2.4)$$

Since G_c acts effectively on C , $\zeta_{i_1} \neq \zeta_{i_2}$ if $i_1 \neq i_2$. Let $\widehat{\mathcal{O}}_{V,v}$ be the completion of the local ring $\mathcal{O}_{V,v}$, and let \mathcal{L}_v be the subring of all invariant elements in $\widehat{\mathcal{O}}_{V,v}$ with respect to the action of G_c . Then from (2.1)–(2.4) we obtain $\widehat{\mathcal{O}}_{V,v} \cong_{\mathbb{C}}[[S, T_1, T_2, \dots, T_n]]$ and $\mathcal{L}_v \cong_{\mathbb{C}}[[S^d, T_1, T_2, \dots, T_n]]$ with $d = |G_c|$. Note that \mathcal{L}_v is a regular local ring. Let $f: V \rightarrow X$ be the quotient morphism and put $x = f(v)$. Then the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is isomorphic to \mathcal{L}_v . Thus we see that the quotient space X is nonsingular, and obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varphi^* \omega & \longrightarrow & f^* \Omega_X^1 & \longrightarrow & \psi^* \Omega_F^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varphi^* \omega_c & \longrightarrow & \Omega_V^1 & \longrightarrow & \psi^* \Omega_F^1 \longrightarrow 0, \end{array}$$

where $\omega = \omega_c \otimes \mathcal{O}_C(-\sum_{c \in C} (|G_c| - 1) \cdot c)$.

We claim that $f^* \Omega_X^1 \cong \psi^* \Omega_F^1 \oplus \varphi^* \omega$. Let $\{e_1, e_2, \dots, e_k\}$ be a basis of the vector space $H^0(F, \Omega_F^1)$. Each $\sigma \in G$ gives rise to an automorphism σ^* of $H^0(V, \Omega_V^1)$. If $\sigma \in \Gamma$, then for some point $c \in C$ the restriction of the action σ to the

fibre $\varphi^{-1}(c)$ is the identity, and therefore the image of $\sigma^*(\psi^*e_j)$ in $H^0(V, \psi^*\Omega_F^1)$ is ψ^*e_j . However, since $G = H$, this is true for every $\sigma \in G$. If we put $E_j = |G|^{-1} \sum_{\sigma \in G} \sigma^*(\psi^*e_j)$, then for all $\sigma \in G$ we have $\sigma^*(E_j) = E_j$. Hence every E_j is a section of the vector bundle $f^*\Omega_X^1$, whose image in $H^0(V, \psi^*\Omega_F^1)$ is ψ^*e_j . By Corollary 3 we have $\Omega_F^1 \cong \Omega_0 \oplus \Omega$, where $\Omega_0 = \bigoplus_{1 \leq j \leq k} \mathcal{O}_F \cdot e_j$ and Ω is a semiample vector bundle of rank $n - k$ such that $\kappa(\det \Omega, F) = 0$ and $H^0(F, \Omega) = 0$. Put $\mathcal{F}_0 = \psi^*\Omega_0$, $\mathcal{F} = \psi^*\Omega$ and put $\mathcal{E} = \bigoplus_{1 \leq j \leq k} \mathcal{O}_V \cdot E_j$. Then we have $\psi^*\Omega_F^1 \cong \mathcal{F}_0 \oplus \mathcal{F}$ and $\mathcal{E} \cong \mathcal{F}_0$. Thus we obtain $f^*\Omega_X^1 \cong \mathcal{E} \oplus \mathcal{G}_1$, $\Omega_V^1 \cong \mathcal{E} \oplus \mathcal{G}_2$ with some vector bundles $\mathcal{G}_1, \mathcal{G}_2$ for which we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \varphi^*\omega & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{F} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \varphi^*\omega_C & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{F} \longrightarrow 0.
 \end{array}$$

Let

$$\delta_1: H^0(V, \mathcal{F} \otimes \mathcal{F}^*) \rightarrow H^1(V, \varphi^*\omega \otimes \mathcal{F}^*),$$

$$\delta_2: H^0(V, \mathcal{F} \otimes \mathcal{F}^*) \rightarrow H^1(V, \varphi^*\omega_C \otimes \mathcal{F}^*)$$

be the canonical homomorphisms and let $1_{\mathcal{F}} \in H^0(V, \mathcal{F} \otimes \mathcal{F}^*)$ be the identity of \mathcal{F} . Since the bottom exact sequence splits, we see that $\delta_2(1_{\mathcal{F}}) = 0$. Consider the following exact sequence:

$$H^0(V, \mathcal{H} \otimes \mathcal{F}^*) \rightarrow H^1(V, \varphi^*\omega \otimes \mathcal{F}^*) \rightarrow H^1(V, \varphi^*\omega_C \otimes \mathcal{F}^*),$$

where $\mathcal{H} = \varphi^*\omega_C / \varphi^*\omega$. By Corollary 2 and 3, we have $H^0(F, \Omega^*) = 0$. Hence it can be easily checked that $H^0(V, \mathcal{H} \otimes \mathcal{F}^*) = 0$. Thus we find that $\delta_1(1_{\mathcal{F}}) = 0$, and hence obtain

$$f^*\Omega_X^1 \cong \mathcal{E} \oplus \mathcal{G}_1 \cong \mathcal{E} \oplus \mathcal{F} \oplus \varphi^*\omega \cong \mathcal{F}_0 \oplus \mathcal{F} \oplus \varphi^*\omega \cong \psi^*\Omega_F^1 \oplus \varphi^*\omega.$$

Since $R < 2g - 2$, the line bundle ω is ample. Hence by Lemma 1 and 2, we see that the cotangent bundle Ω_X^1 is semiample. Furthermore we have $\kappa(X) = \kappa(f^*\omega_X, V) = \kappa(\psi^*\omega_F \otimes \varphi^*\omega, V) = \kappa(\varphi^*\omega, V) = \kappa(\omega, C) = 1$ by Lemma 3. Q.E.D.

REMARK. In the above proof, it can be easily seen that the condition $R < 2g - 2$ is not only sufficient but also necessary for the quotient X to be attended with semiample cotangent bundle and of Kodaira dimension $\kappa(X) = 1$.

Proof of Theorem II. By Proposition 2, the line bundle ω_X is semiample. Hence we have a fibration $\Phi: X \rightarrow B$ with a nonsingular curve B such that $\omega_X^{\otimes k} \cong \Phi^* \mathcal{L}_0$ for some positive integer k and some line bundle \mathcal{L}_0 on B . Any smooth fibre of Φ is a para-abelian variety by Theorem I. Let \mathcal{L} be the full subbundle of Ω_X^1 associated to the pull-back $\Omega^* \omega_B$ of the canonical sheaf ω_B . For each point $b \in B$, decompose the fibre $\Phi^{-1}(b) = \sum a_i D_i$ as a sum of irreducible components and set $D(\Phi)_b = \sum (a_i - 1) D_i$. Put $D(\Phi) = \sum_{b \in B} D(\Phi)_b$. Then we have $\mathcal{L} \cong \Phi^* \omega_B \otimes \mathcal{O}_X(D(\Phi))$ (cf. [10]). Consider the natural homomorphism $h: \mathcal{F}_X \rightarrow \mathcal{L}^*$. There exists a closed subset Y of codimension 2 such that h is surjective at every point in $X \setminus Y$. Tensoring with ω_X , we obtain a homomorphism $h_1: \mathcal{F}_X \otimes \omega_X \rightarrow \mathcal{L}^* \otimes \omega_X$. By Corollary 1 the vector bundle $\mathcal{F}_X \otimes \omega_X$ is semiample. Hence for some positive integer m , the homomorphism

$$h_2: H^0(X, S^m(\mathcal{F}_X \otimes \omega_X)) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow (\mathcal{L}^* \otimes \omega_X)^{\otimes m}$$

derived from h_1 is surjective at every point in $X \setminus Y$. Write $\mathcal{M} = (\mathcal{L}^* \otimes \omega_X)^{\otimes m}$. Since the direct image $\Phi_* \mathcal{M}$ is a line bundle and since $H^0(B, \Phi_* \mathcal{M}) = H^0(X, \mathcal{M})$, one has $\Phi^* \Phi_* \mathcal{M} \cong \mathcal{M}$ and the zero set of each global section consists of fibres of Φ . Therefore h_2 must be surjective at every point in X . This implies that h is surjective, and hence that any fibre $\Phi^{-1}(b)$ is a multiple of a smooth irreducible component.

Choosing a suitable finite covering $\gamma: C \rightarrow B$ with a nonsingular curve C and taking the normalization V of the product $X \times_B C$, we obtain a smooth fibration $\varphi: V \rightarrow C$, a finite covering $f: V \rightarrow X$,

$$\begin{array}{ccc} V & \xrightarrow{f} & X \\ \varphi \downarrow & & \downarrow \Phi \\ C & \xrightarrow{\gamma} & B \end{array}$$

and the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^* \mathcal{L} & \longrightarrow & f^* \Omega_X^1 & \longrightarrow & \Omega_{V/C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varphi^* \omega_C & \longrightarrow & \Omega_V^1 & \longrightarrow & \Omega_{V/C} \longrightarrow 0 \end{array} \quad (2.5)$$

(cf. Theorem 6.3 in [8]). From the upper exact sequence, we obtain the following one:

$$0 \rightarrow \Omega_{V/C}^* \otimes f^* \omega_X \rightarrow f^*(\mathcal{F}_X \otimes \omega_X) \rightarrow f^*(\mathcal{L}^* \otimes \omega_X) \rightarrow 0. \quad (2.6)$$

Since the homomorphism h_2 is surjective and since $(\mathcal{L}^* \otimes \omega_X) \otimes m$ is the pull-back of the line bundle $\Phi_* \mathcal{M}$ on B , we can find an open covering $\{U_i\}$ of B such that on each open subset $\Phi^{-1}(U_i)$ the restricted homomorphism $S^m(\mathcal{T}_X \otimes \omega_X)|_{\Phi^{-1}(U_i)} \rightarrow (\mathcal{L}^* \otimes \omega_X)^{\otimes m}|_{\Phi^{-1}(U_i)}$ is splitting, and hence so is the homomorphism $\mathcal{T}_X \otimes \omega_X|_{\Phi^{-1}(U_i)} \rightarrow \mathcal{L}^* \otimes \omega_X|_{\Phi^{-1}(U_i)}$ by Lemma 4. Then, restricted on each open subset $\varphi^{-1}(\gamma^{-1}(U_i))$, the exact sequence (2.6) splits, and therefore so do both of the exact sequences in (2.5). This implies that the canonical homomorphism $\mathcal{T}_C \rightarrow R^1 \varphi_* (\Omega_{V/C}^*)$ vanishes at every point in C , and hence that φ is a fibre bundle (cf. Theorem 5.1 in [7]).

Since the fibre of φ is a para-abelian variety, we see that $\kappa(\det \Omega_{V/C}, V) = 0$ (cf. [9]). Therefore by Proposition 3, we have a finite unramified Galois covering $\mu: \tilde{V} \rightarrow V$ such that $\mu^* \Omega_{V/C}$ is a trivial bundle. Clearly we may assume that the projection $\tilde{\varphi}: \tilde{V} \rightarrow C$ is a fibration. Then, since $\Omega_{\tilde{V}/C}$ is a trivial bundle, $\tilde{\varphi}$ is a fibre bundle whose fibre A is an abelian variety. Furthermore we may assume that $\tilde{\varphi}$ has a section $\rho: C \rightarrow \tilde{V}$. Choose and fix a basis of $H^0(\tilde{V}, \Omega_{\tilde{V}/C})$. Then the basis and the section $\rho(C)$ determine isomorphisms $\tilde{\varphi}^{-1}(c) \cong A$ for all $c \in C$, which define an isomorphism $\tilde{V} \cong A \times C$. Each element χ in the Galois group $\text{Gal}(\tilde{V}/V)$ gives rise to automorphisms of the fibres $\tilde{\varphi}^{-1}(c)$, and hence defines a continuous mapping $\chi_\#: C \rightarrow \text{Aut}(A)$, where $\text{Aut}(A)$ is the group of automorphisms of A . However the order of the element χ is finite. Therefore $\chi_\#$ must be constant. Hence we see that φ is a trivial fibre bundle whose fibre is a para-abelian variety.

We may assume that γ is a Galois covering. Then, since the variety V is the normalization of the product $X \times_B C$, it follows that f is a finite Galois covering whose Galois group $G = \text{Gal}(V/X)$ acts effectively both on V and on C so that $\varphi \circ \sigma = \sigma \circ \varphi$ for every $\sigma \in G$. Let F be an arbitrary fibre of φ . Then by Corollary 4 the natural homomorphism $f^* \Omega_X^1 \otimes \mathcal{O}_F \rightarrow \Omega_F^1$ is surjective, and hence the restriction of f to F is unramified. This implies that if $\sigma \in G$ has a fixed point $v \in V$, then $\sigma(v') = v'$ for every point $v' \in \varphi^{-1}(\varphi(v))$. Finally from Remark to Proposition 4, we infer that the condition $R < 2g - 2$ holds. Thus we find that X is a variety of type Q_+ . Q.E.D.

REMARK. Let X be the same as in Definition of a variety of type Q_+ , and assume that the condition $R > 2g - 2$ holds in place of $R < 2g - 2$. Then we call X a *variety of type Q_-* . In Proof of Proposition 4, we can easily see that a variety X of type Q_- is a nonsingular variety with semiample tangent bundle such that $\kappa^{-1}(X) = 1$, where $\kappa^{-1}(X) = \kappa(\omega_X^*, X)$ is the *anti-Kodaira dimension* of X . Conversely, in the same manner as in Proof of Theorem II, we obtain the following:

THEOREM II'. *Let X be a nonsingular variety with semiample tangent bundle such that $\kappa^{-1}(X) = 1$. Then X is a variety of type Q_- .*

References

- [1] J. Bertin and P. Teller, Invariants symétriques et fibrés vectoriels sur les courbes. *Math. Ann.* 264 (1983) 423–436.
- [2] T. Fujita, Semipositive line bundles. *J. Fac. Sci. Univ. Tokyo* 30 (1983) 353–378.
- [3] D. Gieseker, p -ample bundles and their Chern classes. *Nagoya Math. J.* 43 (1971) 91–116.
- [4] R. Hartshorne, Algebraic Geometry. Graduate Texts in Math. 52, Springer, 1977.
- [5] S. Iitaka, On D-dimensions of algebraic varieties. *J. Math. Soc. Japan* 23 (1971) 356–373.
- [6] Y. Kawamata, Characterization of abelian varieties. *Compositio Math.* 43 (1981) 253–276.
- [7] K. Kodaira, On deformations of complex analytic structures I, II. *Ann. Math.* 67 (1958) 328–466.
- [8] K. Kodaira, On compact analytic surfaces II. *Ann. Math.* 77 (1963) 563–626.
- [9] I. Nakamura and K. Ueno, An addition formula for Kodaira dimensions of analytic fibre bundles whose fibres are Moisézon manifolds. *J. Math. Soc. Japan* 25 (1973) 363–371.
- [10] M. Reid, Bogomolov's theorem $c_1^2 \leq 4c_2$. *Intl. Symp. on Algebraic Geometry, Kyoto* (1977) 623–642.