COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 84, nº 1 (1992), p. 25-42
<http://www.numdam.org/item?id=CM_1992__84_1_25_0>

Compositio Mathematica **84**: 25–42, 1992.

On the structure of certain normal ideals

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Received 5 December 1990; accepted in revised form 4 June 1991

Introduction

Let I be a one-dimensional almost complete intersection prime ideal of height two in a regular local ring R. Mohan Kumar has shown that I is a set-theoretic complete intersection, actually that I^2 is already contained in a two-generated ideal, provided that I is (geometrically) linked to an ideal J with R/J a discrete valuation ring. (Recall that two prime ideals I and J of the same height are said to be *geometrically linked* if $I \neq J$ and $I \cap J$ is generated by a regular sequence.) On the other hand, as proven in [8, 1.9], the same conclusion on the set-theoretic generation of I holds under the assumption that I is *normal*, i.e., that all powers of I are integrally closed ideals.

Considering these results it seems natural to investigate the relation between the two assumptions of I being linked to a regular ideal versus I being normal. In this paper we will prove that the two above conditions are indeed equivalent, even without any restriction on the height of I (Corollary 1.7). Actually, as stated in our main result, we can even drop the assumption of I being a one-dimensional almost complete intersection (Theorem 1.6). To do so however, we have to replace "linkage" by "residual linkage", a notion that generalizes linkage to the case where the two "linked" ideals may not have the same height (Definition 1.4).

We are going to describe a technically simpler version of our main result. Let A be a Noetherian local ring with infinite residue class field, and write $\hat{A} = R/I$ where R is a regular local ring; then the *deviation* of A, $d(A)$, is the difference between the number of generators and the height of I, and A is said to be *strongly Cohen-Macaulay* if all the Koszul homology modules of a generating set for I are Cohen-Macaulay modules ([12], by [11, 1.6] this definition is independent of the particular presentation of $\hat{A}$, and by [2, p. 259] strong Cohen-Macaulayness is equivalent to Cohen-Macaulayness if $d(A) \leqslant 2$).

*The authors were partially supported by the NSF.

Now let A be as above an isolated strongly Cohen-Macaulay singularity with $d(A) = \dim A > 0$ (actually the singularity need not be isolated; it suffices to assume that A is reduced and that $d(A_{\mathfrak{p}}) < \dim A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ which are neither maximal nor minimal); then Theorem 1.6 asserts that the following are equivalent:

(a) I is normal.
(b) I^k is integrally closed for some $k > \dim A$.
(c) There exists an R-ideal J with R/J a discrete valuation ring, $I \not\subset J$, and the number of generators of $I \cap J$ being smallest possible, namely height $J = \dim R - 1$.

The equivalence of (a) and (b) is a curious fact showing that it suffices to consider one particular power of the defining ideal in order to check for normality. Part (c) can be paraphrased by saying that there exists a *geometric residual intersection* J of I with R/J a discrete valuation ring (Definition 1.4.b). If $d(A) = 1 = \dim A$, then the geometric residual intersection in (c) is simply a geometric link and we obtain the aforementioned Corollary 1.7. In general it is part (c) that yields structure theorems for certain normal ideals. For example, every one-dimensional self-radical almost complete intersection ideal I that is normal can be described explicitly as a Northcott-ideal (Corollary 1.7); this follows immediately once we have established that I is linked to a complete intersection. If in addition the characteristic of the residue class field is not two then I turns out to be a "symmetric" Northcott-ideal (Proposition 1.9). It is this symmetry that yields a simple proof of the aforementioned result from [8], and provides some hope for showing that one-dimensional normal self-radical almost complete intersection ideals are in fact set-theoretic complete intersections. Part (c) of the above Theorem 1.6 also determines the structure of the defining ideal in the next higher dimensional case, namely where $d(A) = \dim A = 2$; this will be the subject of a subsequent paper.

Our main result, Theorem 1.6, is stated in section 1. The proofs of the theorem and some generalizations thereof can be found in sections 2 through 4; they are largely based on the results in Proposition 2.2, Lemma 3.2, and Theorem 4.1.

1. The main result

In this section, we recall several definitions and state our main result (Theorem 1.6). By "ideal" we always mean a proper ideal. Let $(R, \mathbf{m})$ be a Noetherian local ring, I and R-ideal, and M a finitely generated R-module. Then $e(R)$ stands for the multiplicity of R, $v(M)$ denotes the minimal number of generators of M, $\lambda(M)$ its length, $S_k(M)$ stands for the kth symmetric power of M, height I is the height of I, grade I its grade, and $d(I) = v(I) -$ grade I is called the deviation of I. The ideal I is a complete intersection (almost complete intersection) if $d(I)=0$ ($d(I)\leqslant 1$ respectively), and it is said to be Cohen-Macaulay (Gorenstein, reduced, regular) if R/I has any of these properties. By $S(I)$ we denote the symmetric algebra of I, and by $R[It]$ (t a variable) its Rees algebra. Finally, $V(I)=\{P\in \operatorname{Spec}(R)\mid I\subset P\}$, if X is a finite set of variables over R then $R(X)=R[X]_{mR[X]}$, and for a matrix A with entries in R, $I_s(A)$ denotes the R-ideal generated by all the $s\times s$ minors of A.

DEFINITION 1.1 Let R be a Noetherian local ring, and let I be an R-ideal.

(a) The *integral closure* of I, written $\bar{I}$, is the set of all elements x in R satisfying a monic equation

$$x^n+\cdots+a_ix^i+\cdots+a_n=0$$

with $a_i\in I^i$ for $1\leqslant i\leqslant n$.

(b) I is *integrally closed* if $\bar{I}=I$.

(c) I is *normal* if $\overline{I^k}=I^k$ for all $k\geqslant 1$.

It is clear from the above definition that an ideal I in a normal domain R is normal if and only if its Rees algebra $R[It]$ is a normal domain. In detecting normality and other properties of the Rees algebra, the following two notions are often applied:

DEFINITION 1.2. Let R be a Noetherian local ring and let I be an R-ideal:

(a) ([12]) I is said to be *strongly Cohen-Macaulay* if all homology modules of the Koszul complex on some (and hence every) generating set of I are Cohen-Macaulay modules.

(b) ([1]) I is said to satisfy G_∞ if $v(I_{\mathfrak{p}})\leqslant \dim R_{\mathfrak{p}}$ for all $\mathfrak{p}\in V(I)$.

The connection to Rees algebra is provided by:

THEOREM 1.3. *Let R be a local Cohen-Macaulay ring, and let I be a strongly Cohen-Macaulay R-ideal with* grade $I>0$.

(a) ([7, 9.1]) *If I satisfies G_∞, then $S(I)\simeq R[It]$, and both algebras are Cohen-Macaulay.*

(b) ([7, 9.1] and [10, the proof of Proposition 2.1]) *If R is normal, I is reduced, and $v(I_{\mathfrak{p}})\leqslant \max\{\text{height } I, \dim R_{\mathfrak{p}}-1\}$ for all $\mathfrak{p}\in V(I)$, then $R[It]$ is normal.*

In the light of Theorem 1.3, to study normality of ideals, it seems reasonable to assume that the ideal is strongly Cohen-Macaulay and satisfies G_∞ but not the stronger numerical condition of 1.3.b. In fact, for our main result, Theorem 1.6, we will consider strongly Cohen-Macaulay ideals I which satisfy the condition of 1.3.b locally on the punctured spectrum and which are G_∞ on the nose locally at the maximal ideal. Before stating this result however, we need to recall the definition of residual intersection, a notion originally due to Artin and Nagata ([1]).

DEFINITION 1.4 ([14]). *Let R be* a Noetherian local ring, let I and J be R-ideals, and let $s \geqslant$ height I.

(a) J is called an *s-residual intersection* of I if there exists an R-ideal L contained in I such that $J = L\colon I$ and $v(L) \leqslant s \leqslant$ height J.
(b) J is called a *geometric s-residual intersection* of I if in addition height $(I + J) \geqslant s + 1$.

As already mentioned in the introduction, if I is an unmixed ideal and J is a prime ideal with height $I \leqslant$ height J, then J is a geometric s-residual intersection of I if $I \not\subset J$ and $v(I \cap J)$ is smallest possible as permitted by Krull's altitude formula, namely height J; in this case we may choose L to be $I \cap J$ and then $v(L) = s =$ height J. If R is a local Gorenstein ring and I is an unmixed R-ideal of height g, then (geometric) g-residual intersection simply corresponds to (geometric) linkage ([19]). One may take this as a definition of linkage; notice that L is a complete intersection in this case. There are numerous instances where residual intersections occur naturally ([1], [12], [16]), we just mention one more: Let R be a local Cohen-Macaulay ring and let I be a strongly Cohen-Macaulay R-ideal with grade $I > 0$ satisfying G_∞, then the extended Rees algebra $R[It, t^{-1}]$ is defined by a residual intersection of a hypersurface section on I ([12, 4.3]).

The next theorem contains several basic properties of residual intersections; once again, the conditions of Definition 1.2 play a crucial role.

THEOREM 1.5 ([14, 5.1]). *Let R be a local Gorenstein ring, let I be an R-ideal of height g that is strongly Cohen-Macaulay and satisfies G_∞, and let $J = L\colon I$ be an s-residual intersection of I. Then:*

(a) (cf. also [12, 3.1] and [9, 3.4]) *J is a Cohen-Macaulay ideal of height s,* depth $R/L = \dim R - s$, *and in case the residual intersection is geometric, $L = I \cap J$.*
(b) *$S_{s-g+1}(I/L)$ is the canonical module of R/J.*

We are now ready to state our main result.

THEOREM 1.6. *Let $(R, \mathbf{m})$ be a regular local ring with infinite residue class field, and let $I \neq \mathbf{m}$ be a reduced strongly Cohen-Macaulay R-ideal such that $v(I) = \dim R$ and $v(I_{\not p}) \leqslant \max\{\text{height}\, I, \dim R_{\not p} - 1\}$ for all $\not p \in V(I)\backslash\{\mathbf{m}\}$. Then the following are equivalent:*

(a) *I is a normal ideal.*
(b) *I^k is integrally closed for some $k > \dim R/I$.*
(c) *I has a residual intersection J with R/J a normal Gorenstein domain.*
(d) *I has a geometric residual intersection J with R/J a discrete valuation ring.*

The rest of the paper is mainly devoted to the proof of Theorem 1.6: In fact, that (b) implies (a) will follow from Theorem 2.13, that (c) implies (a) will be shown in Proposition 3.5, and that (a) implies (d) will follow from Corollary 4.11. However, as it turns out, all of the individual implications can be proven with weaker assumptions than Theorem 1.6, and they follow from other results that might be of independent interest. Before turning to the proofs, we give an immediate application of Theorem 1.6 to the case of almost complete intersections.

COROLLARY 1.7. *Let* $(R, \mathbf{m})$ *be a regular local ring with infinite residue class field, and let* I *be a reduced* R*-ideal with* $d(I) = 1 = \dim R/I$. *Then the following are equivalent*:

(a) I *is a normal ideal.*
(b) I^k *is integrally closed for some* $k > 1$.
(c) I *is geometrically linked to a regular ideal.*
(d) *There exists an* $n \times 1$ *matrix* $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ *with* $y_1, \ldots, y_n, y_{n+1}$ *forming a regular system of parameters and an* $n \times n$ *matrix* X *with entries in* R *and* $\det(X) \notin (y_1, \ldots, y_n)R$ *such that*

$$I = I_1(X \cdot Y) + I_n(X).$$

Proof. The ideal I is strongly Cohen-Macaulay since it is perfect and $d(I) \leqslant 1$ (e.g. [2, p. 259] or [12, 2.2]). Thus we may apply Theorem 1.6. Also, for the residual intersection J in Theorem 1.6.d, $\dim R/J = 1 = \dim R/I$, and hence J is simply a geometric link of I. Now Theorem 1.6 implies that (a), (b), and (c) are equivalent. On the other hand, the equivalence of (c) and (d) is well-known (e.g. [3, p. 193] or [13, 3.1]); we only indicate how to obtain the matrices Y and X in (d) assuming that (c) holds: Let $n = \text{height } I$, and let $a_1, \ldots, a_n$ be a regular sequence in I such that $J = (a_1, \ldots, a_n)$: I is a regular ideal. Now take $y_1, \ldots, y_n$ to be an arbitrary generating sequence of J and let X be any $n \times n$ matrix with entries in R giving an equation

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = X \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix};$$

then $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and X have all the properties asserted in (d). □

We now want to strengthen the statement of Corollary 1.7.d. To this end a general lemma is needed.

LEMMA 1.8. *Let R be a ring, and let $y_1, \ldots, y_n$ be elements in R.*

(a) *Let Z be an alternating $n \times n$ matrix with entries in the ideal $2(y_1, \ldots, y_n)R$; then $Z = U - U^*$ for some $n \times n$ matrix U whose rows are R-linear combinations of the Koszul relations among $y_1, \ldots, y_n$.*
(b) *Let X be an $n \times n$ matrix with entries in R such that X is symmetric modulo the ideal $2(y_1, \ldots, y_n)R$; then $X + U$ is symmetric for some $n \times n$ matrix U whose rows are R-linear combinations of the Koszul relations among $y_1, \ldots, y_n$.*

Proof. To prove the assertion of (b), simply apply part (a) to the matrix $Z = X^* - X$. In order to show (a), it suffices to consider the case where all entries of $Z = (z_{\nu\mu})$ are zero except for $z_{ij} = -z_{ji} = 2y_k$, with (i,j) a fixed pair, $i \neq j$, and k arbitrary. Let $\{e_1, \ldots, e_n\}$ be the standard basis of R^n. We take U to be $V_i + V_j + V_k$ where the ith row of V_i is $y_k e_j - y_j e_k$, the jth row of V_j is $y_i e_k - y_k e_i$, and the kth row of V_k is $y_i e_j - y_j e_i$, all other rows being zero (this also covers the cases $k = i$ and $k = j$). □

PROPOSITION 1.9. *If* char $R/\mathbf{m} \neq 2$ *then the matrix of Corollary* 1.7.*d can be chosen symmetric.*

Proof. We use the notation of Corollary 1.7.d. Since $R/2(y_1, \ldots, y_n)R$ is a discrete valuation ring, we may achieve that X is symmetric modulo the ideal $2(y_1, \ldots, y_n)R$. (This can be done by performing elementary row and column operations on X and Y that do not change any of the ideals $I_1(X \cdot Y)$, $I_n(X)$, $I_1(Y)$.) But then by Lemma 1.8.b, $X + U$ is symmetric for some $n \times n$ matrix U whose rows are relations among $y_1, \ldots, y_n$. Therefore $(X + U) \cdot Y = X \cdot Y$. Now it follows as in the proof of Corollary 1.7 that we may replace X by the symmetric matrix $X + U$. □

Corollary 1.7.d and Proposition 1.9 provide a strong structure theorem for reduced one-dimensional almost complete intersection R-ideals I having an integrally closed power $I^k \neq I$ (at least if $(R, \mathbf{m})$ is a regular local ring with $R/\mathbf{m}$ infinite and of characteristic different from two.) Especially the symmetry of the matrix X might be helpful in proving that such ideals are set-theoretic complete intersections. We illustrate this in the case where height $I = 2$, and thereby give a slightly different proof of [8, 1.9]: If height $I = 2$, then in the notation of Corollary 1.7.d and Proposition 1.9, $n = 2$ and

$$I = I_2\begin{pmatrix} \multicolumn{2}{c}{X} \\ \hline y_2 & -y_1 \end{pmatrix}$$

where X is symmetric. It follows from the linkage property of perfect height two ideals (e.g. [1, 3.2.b]), or by direct computation, that the product of the two ideals

$$I_2\left(\frac{X}{y_2 \quad -y_1}\right) \text{ and } I_2\left(X\,\middle|\,\begin{matrix} y_2 \\ -y_1\end{matrix}\right)$$

is contained in the ideal generated by the determinant of X and the determinant of

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{X} & \begin{matrix} y_2 \\ -y_1 \end{matrix} \\ \hline y_2 & -y_1 & 0 \end{array}\right].$$

However, by the symmetry of X,

$$I_2\left(X\,\middle|\,\begin{matrix} y_2 \\ -y_1\end{matrix}\right) = I_2\left(\frac{X}{y_2 \quad -y_1}\right) = I,$$

and therefore I^2 is contained in a complete intersection inside I. (The connection between the set-theoretic complete intersection property and the symmetry of certain matrices was first observed and systematically exploited by Ferrand, and Valla [20]; see also [5].)

We conclude this section with another consequence of Theorem 1.6.

COROLLARY 1.10. *Let R be a three-dimensional regular local ring, and let I be an R-ideal such that R/I is reduced and equidimensional and $e(R/I) \leqslant 5$. Then I is normal.*

Proof. We may assume that $v(I) >$ height $I = 2$. If $\mathbf{m}$ denotes the maximal ideal of R, we may further assume that $k = R/\mathbf{m}$ is infinite; then there exists an element $z \in \mathbf{m}\backslash\mathbf{m}^2$ such that z is regular on R/I and $e(R/(I, z)) = e(R/I)$. Let "$-$" denote reduction modulo zR. Now $\bar{R}$ is a two-dimensional regular local ring, $v(\bar{I}) > 2$, and $\lambda(\bar{R}/\bar{I}) \leqslant 5$. In this situation it follows from [8, Proof of 1.14] that $\bar{I}$ has a standard basis $\{f_1, f_2, f_3\}$ whose leading forms $\{L(f_1), L(f_2), L(f_3)\}$ in $gr_{\bar{\mathbf{m}}}\bar{R} = k[X, Y]$ are either $\{X^2, XY, Y^2\}$, or $\{X^2, XY, Y^3\}$, or $\{X^2, XY, Y^4\}$, or $\{X^3, XY, Y^3\}$, or $\{X^2 + \alpha XY + \beta Y^2, XY^2, Y^3\}$ where α and β are in k. For each of these five cases, one can easily write down the 2×3 matrix of relations among $L(f_1)$, $L(f_2)$, $L(f_3)$, and then lift those relations to obtain a 2×3 matrix of relations W among f_1, f_2, f_3. Furthermore, after elementary column operations, the entries of some column of W *generate* $\bar{\mathbf{m}}$. This implies that $\bar{I}$ and $\bar{\mathbf{m}}$ are linked. Since the property of being linked to a regular ideal is preserved under deformation, we conclude that I is also linked to a regular ideal. In addition $d(I) = 1 = \dim R/I$, and the normality of I follows from Theorem 1.6. □

Contrary to the proof of Corollary 1.10 however, it is not true that the property of being normal is preserved under deformation: For example let k be a field, $R = k[[X, Y, Z]]$, and $I = (X^2 - Y^2Z^3, Y^4 - XZ^2, Z^5 - XY^2)$; then Z is regular on R and on $R/I = k[[t^{16}, t^7, t^6]]$, and modulo ZR, the image of I in $k[[X, Y]]$ is a normal ideal ([23, p. 385]). However I is not normal ([21, p. 309]).

2. Ideals having an integrally closed power

The main result in this section is Theorem 2.13, which says that under suitable assumptions the integral closedness of one particular power of an ideal forces the ideal to be normal. In this and other results we will often replace the notion of integral closedness by the weaker concept of m-fullness, which is due to D. Rees ([22]). The latter notion has the advantage of being more amenable to computer algebra methods.

DEFINITION 2.1. Let $(R, \mathbf{m})$ be a Noetherian local ring, and let I be an R-ideal.

(a) If $R/\mathbf{m}$ is infinite, then I is called $\mathbf{m}$-*full* if there exists an element $y \in R$ with $\mathbf{m}I : y = I$.
(b) If $R/\mathbf{m}$ is finite, then I is called $\mathbf{m}$-*full* if there exists a Noetherian local ring $(R', \mathbf{m}')$ with R' faithfully flat over R, $\mathbf{m}' = \mathbf{m}R'$, and $R'/\mathbf{m}'$ infinite, such that IR' is $\mathbf{m}'$-full.

Every integrally closed ideal $I \neq \sqrt{0}$ is automatically $\mathbf{m}$-full ([6, 2.4]), and every $\mathbf{m}$-full ideal I has the *Rees property*, which means that $v(I) \geqslant v(J)$ for every R-ideal J with $I \subset J$ and $\lambda(J/I) < \infty$ ([6, 2.2], cf. also [22, Theorem 3]). It is only this property of $\mathbf{m}$-full or integrally closed ideals that we are going to use in our proofs.

Further recall that an ideal J in a Noetherian local ring $(R, \mathbf{m})$ is said to be a *reduction* of an R-ideal I, if $J \subset I$ and $I^n = JI^{n-1}$ for some $n \geqslant 1$. A reduction J is called a *minimal reduction* of I, if there is no reduction of I properly contained in J. Assuming grade $I > 0$, then J is a minimal reduction of I if and only if $J \subset I \subset \bar{J}$ and J is generated by analytically independent elements. Every R-ideal admits a minimal reduction provided that $R/\mathbf{m}$ is infinite ([18]).

The next proposition and its immediate consequences provide the technical background for the proof of Theorem 2.13, but they also have some other consequences, which might be of independent interest (Corollaries 2.6 through 2.11).

PROPOSITION 2.2. *Let* $(R, \mathbf{m})$ *be a Noetherian local ring, and let* I *be an* R-*ideal with* grade $I > 0$, *and let* J *be a minimal reduction of* I. *Write* $S = R[It]$. *Then one of the following two conditions hold:*

(a) $\mathbf{m}(J^k:\mathbf{m}) = \mathbf{m}J^k$ *for all* $k \geqslant 0$.

(b) *R is analytically irreducible, and there exists a prime ideal P* of *S containing* $\mathbf{m}$ *such that* S_P *is a discrete valuation ring and* $PS_P = \mathbf{m}S_P$.

Proof. Let $T = R[Jt]$, then $\mathbf{m}T$ is a prime ideal since J is generated by analytically independent elements. We first prove that if $T_{\mathbf{m}T}$ is not a discrete valuation ring then (a) holds.

So let $(\ )^{-1}$ denote inverse fractional ideals in the total ring of quotients. We assume that $T_{\mathbf{m}T}$ is not a discrete valuation ring. Then the maximal ideal $\mathbf{m}T_{\mathbf{m}T}$ is not invertible, and hence

$$\mathbf{m}T_{\mathbf{m}T}(\mathbf{m}T_{\mathbf{m}T})^{-1} = \mathbf{m}T_{\mathbf{m}T}. \tag{2.3}$$

On the other hand, $R[t]$ is contained in the total ring of quotients of T, and therefore

$$\mathbf{m}T(\mathbf{m}T)^{-1} \supset \mathbf{m}T(T:_{R[t]}\mathbf{m}T) \supset \mathbf{m}T. \tag{2.4}$$

Now (2.4), combined with (2.3) and the fact that $\mathbf{m}T$ is prime, implies

$$\mathbf{m}T(T:_{R[t]}\mathbf{m}T) = \mathbf{m}T,$$

or equivalently

$$\mathbf{m}(R[Jt]:_{R[t]}\mathbf{m}) = \mathbf{m}R[Jt]. \tag{2.5}$$

Reading the graded pieces of (2.5), we conclude that for all $k \geqslant 0$,

$$\mathbf{m}(J^k:_R\mathbf{m}) = \mathbf{m}J^k,$$

which is the condition in (a).

Now assuming that (a) does not hold, we have seen that $T_{\mathbf{m}T}$ is a discrete valuation ring. We wish to prove that (b) is satisfied. First, R is a subring of $T_{\mathbf{m}T}$, and hence a domain. All our assumptions are preserved as we pass from R to $\hat{R}$, and therefore also $\hat{R}$ is a domain.

Now S is a birational integral extension of T and $T_{\mathbf{m}T}$ is integrally closed. Therefore $S \subset T_{\mathbf{m}T}$. Write $P = S \cap \mathbf{m}T_{\mathbf{m}T}$. Then P is a prime ideal of S containing $\mathbf{m}$. Furthermore, the inclusions $T \subset S \subset T_{\mathbf{m}T}$ and $\mathbf{m}T \subset P \subset \mathbf{m}T_{\mathbf{m}T}$ imply that $S_P = T_{\mathbf{m}T}$ and $PS_P = \mathbf{m}T_{\mathbf{m}T}$. In particular, S_P is a discrete valuation ring with $PS_P = \mathbf{m}S_P$. □

COROLLARY 2.6. *Let* $(R, \mathbf{m})$ *be a Noetherian local ring, let I be an R-ideal with* grade $I > 0$, *and let J be a minimal reduction of I, and suppose that for some*

$k \geqslant 0$, depth $R/J^k = 0$ *and* J^k *is* **m**-*full* (*the latter assumption is satisfied if* J^k *is integrally closed*). *Write* $S = R[It]$. *Then* R *is analytically irreducible, and there exists a prime ideal* P *of* S *containing* $\mathbf{m}$ *such that* S_P *is a discrete valuation ring with* $PS_P = \mathbf{m}S_P$.

Proof. Since $(J^k{:}_R\mathbf{m})/J^k$ is a module of finite length and J^k is **m**-full it follows from [6, 2.2] (cf. also [22, Theorem 3]) that

$$v(J^k{:}_R\mathbf{m}) \leqslant v(J^k). \tag{2.7}$$

Now suppose that the assertion of the corollary is false, then by Proposition 2.2,

$$\mathbf{m}(J^k{:}_R\mathbf{m}) = \mathbf{m}J^k. \tag{2.8}$$

Combining (2.7) and (2.8), we conclude that $J^k{:}_R\mathbf{m} = J^k$, which is impossible since depth $R/J^k = 0$. □

If in Corollary 2.6, I happens to be generated by analytically independent elements, then $J = I$ and $\mathbf{m}S$ is prime. Hence we obtain the following result.

COROLLARY 2.9. *Let* $(R, \mathbf{m})$ *be a Noetherian local ring, let* I *be an* R-*ideal with* grade $I > 0$ *that is generated by analytically independent elements, and suppose that for some* $k \geqslant 0$, depth $R/I^k = 0$ *and* I^k *is* **m**-*full* (*the latter assumption is satisfied if* I^k *is integrally closed*). *Write* $S = R[It]$. *Then* R *is analytically irreducible, and* $S_{\mathbf{m}S}$ *is a discrete valuation ring.*

Before continuing, we record two immediate consequences of Corollary 2.9.

COROLLARY 2.10. *Let* $(R, \mathbf{m})$ *be a universally catenary local ring of dimension* d, *and let* I *be an* R-*ideal with* grade $I > 0$ *that is generated by less than* d *analytically independent elements. Then the following are equivalent*:

(a) depth $R/I^k > 0$.
(b) I^k *is* **m**-*full.*

Proof. If depth $R/I^k > 0$, then we may take any regular element on R/I^k to be the element y in Definition 2.1. Therefore I^k is **m**-full.

We only need to show that (b) implies (a). So assuming (b) suppose that depth $R/I^k = 0$. Now we may apply Corollary 2.9. In particular, R is a domain and therefore

$$\dim S_{\mathbf{m}S} = \dim S - \dim S/\mathbf{m}S = d + 1 - v(I) \geqslant 2.$$

On the other hand, again by Corollary 2.9, $\dim S_{\mathbf{m}S} = 1$, which yields a contradiction. □

The next result has been known if $k = 1$ ([6, 3.1]) or if k is sufficiently large ([17, Théorème 3]).

COROLLARY 2.11. *Let* $(R, \mathbf{m})$ *be a Noetherian local ring with* $\operatorname{depth} R > 0$, *and let* I *be an ideal generated by a system of parameters of* R. *Then the following are equivalent:*

(a) I *is normal.*
(b) I^k *is integrally closed for some* $k \geqslant 1$.
(c) R *is regular and* $\mathbf{m}/I$ *is cyclic.*

Proof. If I^k is integrally closed for some $k \geqslant 1$ then by Corollary 2.9, R is analytically irreducible and $S_{\mathbf{m}S}$ is a discrete valuation ring. Now our assertion follows from Goto's theorem ([6, 3.1]). □

Before proving the main result of this section, we need to recall the following result from [15]:

LEMMA 2.12 ([15, 2.7]). *Let* R *be a local Gorenstein ring, and let* I *be a perfect* R*-ideal that is strongly Cohen-Macaulay and satisfies* G_∞. *Then for every* $k > d(I)$, $\operatorname{depth} R/I^k = \dim R - v(I)$.

THEOREM 2.13. *Let* R *be a local Gorenstein ring, and let* $I \neq 0$ *be a perfect* R*-ideal that is strongly Cohen-Macaulay and* G_∞. *Then the following are equivalent:*

(a) I *is normal.*
(b) I^k *is integrally closed for some* $k > d(I)$.
(c) *For every* $\not p \in V(I)$ with $V(I_{\not p}) = \dim R_{\not p}$ *there exists* $k > d(I_{\not p})$ *such that* $I^k_{\not p}$ *is* $\not p R_{\not p}$*-full.*

Proof. First notice that grade $I > 0$ and write $S = R[It]$. Since I is strongly Cohen-Macaulay and G_∞, it follows from Theorem 1.3.a that the Rees algebra S and the symmetric algebra $S(I)$ are Cohen-Macaulay and that these algebras coincide. In particular, I can be generated by analytically independent elements.

It suffices to prove that (c) implies (a). Thus assuming (c) we will now show by induction on $\dim R$ that the ring S is integrally closed in $R[t]$ (for this proof we also allow I to be R).

So let y be an element of $R[t]$ that is integral over S, and let $C = S:_S y$ be the conductor of y in S. Suppose that $y \notin S$, then by the Cohen-Macaulayness of S, there exists a prime ideal P in S with $P \supset C$ and $\dim S_P \leqslant 1$. Write $\not p = P \cap R$. If $\not p \not\supset I$ then $y/1 \in R_{\not p}[t] = R_{\not p}[I_{\not p}t]$, whereas if $\mathbf{m} \neq \not p \supset I$, then $y/1 \in R_{\not p}[I_{\not p}t]$ by induction hypothesis. (Notice that the latter case cannot occur for $\dim R = 1$.) In either case, $C_P = S_P$, which is impossible. Therefore $\not p = \mathbf{m}$, and hence P contains the prime ideal $\mathbf{m}S$.

Furthermore, $\dim S = \dim R + 1$ because $S = R[It]$ with grade $I > 0$, and $\dim S/\mathbf{m}S = v(I)$ because $S \simeq S(I)$. Since S is also Cohen-Macaulay,

$$1 \geqslant \dim S_P \geqslant \dim S_{\mathbf{m}S} = \dim S - \dim S/\mathbf{m}S = \dim R + 1 - v(I) \geqslant 1,$$

where the latter inequality follows from the G_∞ assumption. In particular

$$\dim S_P = \dim S_{\mathbf{m}S} = \dim R + 1 - v(I) = 1.$$

From this we conclude that

$$P = \mathbf{m}S \tag{2.14}$$

and

$$v(I) = \dim R. \tag{2.15}$$

Now by (2.15) and Lemma 2.12, depth $R/I^k = 0$ for all $k > d(I)$, and hence by (2.15) and (c), depth $R/I^k = 0$ for some k where I^k is $\mathbf{m}$-full. Then Corollary 2.9 implies that R is a domain and that $S_{\mathbf{m}S}$ is a discrete valuation ring. Since furthermore $R[t]$ is contained in the quotient field of $S_{\mathbf{m}S}$, we then conclude that $y \in S_{\mathbf{m}S}$. Hence $y \in S_P$ by (2.14), which is impossible since $C_P \neq S_P$. □

Now it is clear that Theorem 2.13 implies the equivalence of parts (a) and (b) in Theorem 1.6, since there, $v(I) = \dim R$ and hence $d(I) = \dim R/I$.

We wish to point out that if under the assumptions of Theorem 2.13, I^k is integrally closed for some $k \geqslant 1$, then I^n is integrally closed for all n with $1 \leqslant n \leqslant k$. This is clear since $\mathrm{gr}_I(R)$ is Cohen-Macaulay ([7, 9.1]), hence there exists an element $x \in I \backslash I^2$ whose leading form is regular on $\mathrm{gr}_I(R)$. Now the inclusion $x^{k-n}\overline{I^n} \subset \overline{I^k} = I^k$ implies that $\overline{I^n} = I^n$.

On the other hand, the assumption $k > d(I)$ in Theorem 2.13 is sharp as can be seen from the next remark.

REMARK 2.16. Let $(R, \mathbf{m})$ be a regular local ring, and let $0 \neq I \subset \mathbf{m}^3$ be a reduced R-ideal such that I is strongly Cohen-Macaulay, $v(I) = \dim R$, and $v(I_{\not{p}}) \leqslant \max\{\text{height}\, I, \dim R_{\not{p}} - 1\}$ for every $\not{p} \in V(I)\backslash\{\mathbf{m}\}$. (For example, one may take I to be a sufficiently general specialization of a generic perfect height two ideal with at least four generators.) Then for all $k \geqslant 1$ and all $\not{p} \in V(I)\backslash\{\mathbf{m}\}$, $\operatorname{depth}(R/I^k)_{\not{p}} > 0$, and also for all $1 \leqslant k \leqslant d(I)$, depth $R/I^k > 0$ ([7, 5.1]). Hence for every $1 \leqslant k \leqslant d(I)$, the power I^k and the symbolic power $I^{(k)}$ coincide, and in particular, $I^k = \overline{I^k}$. On the other hand, I is not normal, since $v(I) = \dim R$ and $I \subset \mathbf{m}^3$ ([21, 1.1]).

3. Ideals admitting a residual intersection that is regular

In this section we will address the question of how certain properties of a residual intersection of an ideal I imply that I is normal (Proposition 3.5). The connection between a residual intersection and the Ress algebra of an ideal is provided by Lemma 3.2. To formulate it we need to recall a definition from [13].

DEFINITION 3.1. Let A and B be Noetherian local rings.

(a) B is a *deformation* of A if $B/(\mathbf{x}) \simeq A$ for some B-regular sequence $\mathbf{x} = x_1, \ldots, x_n$.
(b) B is *essentially a deformation* of A if there exist Noetherian local rings B_i, $1 \leqslant i \leqslant n$, such that $B_1 = A$, $B_n = B$, and for every $1 \leqslant i \leqslant n-1$ one of the following holds:

(i) $B_{i+1} \simeq (B_i)_P$ for some $P \in \operatorname{Spec}(B_i)$.
(ii) B_{i+1} is a deformation of B_i.
(iii) $B_{i+1}(Z) \simeq B_i(Y)$ for some finite sets of variables Z over B_{i+1}, Y over B_i.

Note that most ring theoretic properties are preserved or can only improve under essentially a deformation.

LEMMA 3.2. *Let R be a local Cohen-Macaulay ring, let I be an R-ideal with* $\operatorname{grade} I > 0$ *that is strongly Cohen-Macaulay and satisfies G_∞, and write $S = R[It]$. Further let $J = L:I$ be a residual intersection of I with I/L being cyclic. Then $S_{\mathbf{m}S}$ is essentially a deformation of R/J.*

Proof. Let $v(L) = n-1$ and write $L = (f_1, \ldots, f_{n-1})$. Then we may assume that $I = (f_1, \ldots, f_n)$, and hence $J = (f_1, \ldots, f_{n-1}):(f_n)$ with height $J \geqslant n-1$.

We will work with the presentation

$$\begin{aligned} S &= R[f_1 t, \ldots, f_n t] \\ &\simeq R[T_1, \ldots, T_n] \Big/ \left(\sum_{i=1}^{n} a_i T_i \,\middle|\, \sum_{i=1}^{n} a_i f_i = 0,\ a_i \in R \right). \end{aligned} \tag{3.3}$$

Now consider the sequence of elements in S, $\mathbf{x} = x_1, \ldots, x_{n-1} = f_1 t, \ldots, f_{n-1} t$. From (3.3) we see that

$$\begin{aligned} S/(\mathbf{x}) &\simeq R[T_n]/(a_n T_n \mid a_n f_n \in (f_1, \ldots, f_{n-1}),\ a_n \in R) \\ &= R[T]/JTR[T], \end{aligned} \tag{3.4}$$

where $T = T_n$. Since by the definition of residual intersection. $J \subset \mathbf{m}$, it follows that $S/(\mathbf{m}, \mathbf{x}) \simeq R/\mathbf{m}[T]$, which is a one-dimensional domain. Hence $P = (\mathbf{m}, \mathbf{x})$ is a prime ideal in S, and $\dim S_P = \dim S - \dim S/P = \dim R$.

Now $S_{\mathbf{m}S}$ is a localization of S_P, and to prove the assertion of the lemma it suffices to show that S_P is a deformation of $(R/J)(T)$.

To see this, notice that $\mathbf{x} \subset P$ and that by (3.4),

$$\begin{aligned} S_P/(\mathbf{x}) &\simeq (R[T]/JTR[T])_{(\mathbf{m})} \\ &= (R[T]/JR[T])_{(\mathbf{m})} \\ &= (R/J)(X). \end{aligned}$$

On the other hand, S_P is Cohen-Macaulay, and $\dim S_P = \dim R$, and

$$\dim S_P/(x_1, \ldots, x_{n-1}) = \dim(R/J)(X) = \dim R - \operatorname{ht} J \leqslant \dim R - (n-1).$$

Thus it follows that $\mathbf{x} = x_1, \ldots, x_{n-1}$ form an S_P-regular sequence. □

PROPOSITION 3.5. *Let* $(R, \mathbf{m})$ *be a regular local ring, and let* I *be a reduced strongly Cohen-Macaulay* R*-ideal such that* $v(I) \leqslant \dim R$ *and* $v(I_{\not p}) \leqslant \max\{\text{height}\, I, \dim R_{\not p} - 1\}$ *for all* $\not p \in V(I)\backslash\{\mathbf{m}\}$. *If there exists a residual intersection* J *of* I *with* R/J *a Gorenstein normal domain, then* I *is a normal ideal.*

Proof. First notice that $g = \text{grade}\, I > 0$. Since $S = R[It]$ is Cohen-Macaulay it suffices to prove that this ring satisfies Serre's condition (R_1). So let $P \in \text{Spec}(S)$ with $\dim S_P \leqslant 1$ and write $\not p = P \cap R$. If $\not p \notin V(I)$ or if $\not p$ is minimal in $V(I)$ then $I_{\not p} = R_{\not p}$ or $I_{\not p} = \not p R_{\not p}$, and S_P is regular. If $\not p$ is non-minimal in $V(I)\backslash\{\mathbf{m}\}$, then by our assumption,

$$\begin{aligned} \dim R_{\not p} > v(I)_{\not p} &\geqslant \dim S \otimes_R (R_{\not p}/\not p R_{\not p}) \geqslant \dim(S \otimes_R R_{\not p})/P \\ &\geqslant \dim R_{\not p} + 1 - 1 = \dim R_{\not p}. \end{aligned}$$

Therefore this case cannot occur. Finally, if $\not p = \mathbf{m}$, then $P \supset \mathbf{m}S$ and hence $P = \mathbf{m}S$. It remains to prove that $S_{\mathbf{m}S}$ is normal.

Now suppose that J is an s-residual intersection of I with $J = L{:}I$ where $v(L) \leqslant s$. Then by Theorem 1.5.b, the canonical module of R/J is given by the symmetric power $S_{s-g+1}(I/L)$. In particular,

$$1 = r(R/J) = \binom{s - g + v(I/L)}{s - g + 1},$$

and hence $v(I/L) = 1$. But then it follows from Lemma 3.2 that $S_{\mathbf{m}S}$ is essentially a deformation of R/J. Now the normality of R/J implies that $S_{\mathbf{m}S}$ has the same property. □

Proposition 3.5 shows that in Theorem 1.6, part (c) implies part (a).

REMARK 3.6. (a) Appealing to deformation arguments as in [14] one can replace the assumption of Lemma 3.2 that I/L be cyclic by the weaker hypothesis that J is an s-residual intersection with $s \geqslant v(I) - I$.

(b) Using induction on $\dim R/J$ one can prove Proposition 3.5 under the more general assumption that R is a local Gorenstein ring, I is strongly Cohen-Macaulay and satisfies G_∞, and $I_{\mathfrak{p}}$ is a normal ideal for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus V(J)$.

4. Residual intersections of normal ideals

In this section we will prove that under suitable assumptions every normal ideal admits a geometric residual intersection that is regular (Corollary 4.11). This will follow from our next more general result. Recall that $e(A)$ denotes the multiplicity of a Noetherian local ring A.

THEOREM 4.1. *Let R be a local Cohen-Macaulay ring with infinite residue class field, let I be an R-ideal with* $\operatorname{grade} I > 0$ *that is strongly Cohen-Macaulay and satisfies G_∞, and write $S = R[It]$. Then there exists a geometric residual intersection J of I with* $\operatorname{height} J = v(I) - 1$ *such that $e(R/J) = e(S_{\mathbf{m}S})$.*

Proof. Set $k = R/\mathbf{m}$, $\bar{S} = S \otimes_R k$, $n = v(I)$, and $d = \dim R + 1 - n$. Then $\bar{S} \simeq S(I \otimes_R k)$ is a polynomial ring over k in n variables and that $\dim S_{\mathbf{m}S} = \dim S - \dim \bar{S} = d$, whereas $\dim S_{\mathbf{m}S} \otimes_R R/I < d$. Now since k is infinite, there exist d elements $\mathbf{z} = z_1, \dots, z_d$ in $\mathbf{m}$ such that $z_1, \dots, z_d$ generate a minimal reduction of $\mathbf{m}S_{\mathbf{m}S}$ and $z_1, \dots, z_{d-1}$ generate an ideal primary to the maximal ideal of $S_{\mathbf{m}S} \otimes_R R/I$. In other words, $\mathbf{m}^r S_{\mathbf{m}S} = (\mathbf{z})\mathbf{m}^{r-1} S_{\mathbf{m}S}$ for some r, and in particular $e(S_{\mathbf{m}S}) = \lambda(S_{\mathbf{m}S}/(\mathbf{z})S_{\mathbf{m}S})$, where λ denotes length; furthermore $\mathbf{m}^t S_{\mathbf{m}S} \subset (I, z_1, \dots, z_{d-1})S_{\mathbf{m}S}$ for some t.

For $0 \leqslant i \leqslant r + 1$ we define graded $\bar{S}$-modules L_i by

$$L_i = (\mathbf{m}^i, \mathbf{z})S/(\mathbf{m}^{i+1}, \mathbf{z})S \text{ for } 0 \leqslant i \leqslant r - 1$$

$$L_r = \mathbf{m}^r S/(\mathbf{m}^{r+1}, \mathbf{z}\mathbf{m}^{r-1})S$$

$$L_{r+1} = (\mathbf{m}^t, I, z_1, \dots, z_{d-1})W/(\mathbf{m}^{t+1}, I, z_1, \dots, z_{d-1})S$$

and we set $e_i = v(L_i \otimes_{\bar{S}} \bar{S}_0)$. Notice that

$$e_r = e_{r+1} = 0 \tag{4.2}$$

whereas

$$\sum_{i=0}^{r-1} e_i = \lambda(S_{\mathbf{m}S}/(\mathbf{z})S_{\mathbf{m}S}) = e(S_{\mathbf{m}S}). \tag{4.3}$$

Now consider homogeneous presentations

$$\bar{S}^{m_i} \xrightarrow{\varphi_i} \bar{S}^{n_i} \to L_i \to 0$$

and the Fitting ideals $F_i = I_{n_i - e_i}(\varphi_i)$ generated by all $n_i - e_i$ size minors of φ_i. Then F_i are homogeneous $\bar{S}$-ideals with

$$V(F_i) = \{\bar{P} \in \operatorname{Spec}(\bar{S}) \mid v(L_i \otimes_{\bar{S}} \bar{S}) > e_i\}.$$

In particular, $O \notin \bigcup_{i=0}^{r+1} V(F_i)$, and thus $\bigcup_{i=0}^{r+1} V(F_i) \cap \operatorname{Proj}(\bar{S})$ is a closed subset properly contained in $\operatorname{Proj}(\bar{S})$. Since k is infinite, there exists a k-rational projective point not contained in $\bigcup_{i=0}^{r+1} V(F_i)$. Choosing a suitable basis $T_1, \ldots, T_n$ of $I \otimes_R k$, we may assume that this point is $(0:\cdots:0:1)$. In other words, $(T_1, \ldots, T_{n-1})\bar{S} \notin \bigcup_{i=0}^{r+1} V(F_i)$, where we view $\bar{S}$ as $S(I \otimes_R k) = k[T_1, \ldots, T_n]$. Thus for all $0 \leqslant i \leqslant r + 1$,

$$v(L_i \otimes_{\bar{S}} \bar{S}_{(T_1,\ldots,T_{n-1})\bar{S}}) \leqslant e_i. \tag{4.4}$$

Let $f_i t \in It$ be the preimage of T_i under the natural epimorphism

$$\pi \colon R[It] \to \bar{S} = k[T_1, \ldots, T_n],$$

and consider the embedding $\pi^*\colon \operatorname{Spec}(\bar{S}) \hookrightarrow \operatorname{Spec}(S)$. Then $\pi^*((T_1, \ldots, T_{n-1})\bar{S}) = P$ where $P = (\mathbf{m}, f_1 t, \ldots, f_{n-1} t)$. Now (4.4) implies that for all $0 \leqslant i \leqslant r + 1$,

$$v(L_i \otimes_S S_P) \leqslant e_i. \tag{4.5}$$

Further notice that $I = (f_1, \ldots, f_n)$ and define the R-ideal

$$J = (f_1, \ldots, f_{n-1}) : (f_n) = (f_1, \ldots, f_{n-1}) : I.$$

As in (3.4) one sees that $S_P/(f_1 t, \ldots, f_{n-1} t) \simeq (R/J)(T)$, and hence by (4.5), $v(L_i \otimes_S (R/J)(T)) \leqslant e_i$. Writing $A = R/J$, using the definition of L_i, faithfully flat descent, and (4.2), it follows that

$$v((\mathbf{m}^i, z)A/(\mathbf{m}^{i+1}, \mathbf{z})A) \leqslant e_i \quad \text{for} \quad 0 \leqslant i \leqslant r - 1, \tag{4.6}$$

$$v(\mathbf{m}^r A/(\mathbf{m}^{r+1}, \mathbf{z}\mathbf{m}^{r-1})A) \leqslant e_r = 0, \tag{4.7}$$

and

$$v((\mathbf{m}^t, I, z_1, \ldots, z_{d-1})A/(\mathbf{m}^{t+1}, I, z_1, \ldots, z_{d-1})A) \leqslant e_{r+1} = 0. \tag{4.8}$$

From (4.7) and (4.8) we see that

$$\mathbf{m}^r A = (z_1, \ldots, z_d)\mathbf{m}^{r-1} A \tag{4.9}$$

and

$$\mathbf{m}^t \subset (I, z_1, \ldots, z_{d-1})A. \tag{4.10}$$

Since $\mathbf{m}A$ is the maximal ideal of A, (4.9) and (4.10) imply that $\dim A \leqslant d$ and $\dim A/IA \leqslant d - 1$. Therefore height $J \geqslant \dim R - d = n - 1$, whereas height $(I + J) \geqslant n$. Thus J is a geometric $(n - 1)$-residual intersection of I.

Now by Theorem 1.5.a, height $J = n - 1$ which yields $\dim A = d$. Thus $\mathbf{z} = z_1, \ldots, z_d$ form a system of parameters of A, and now (4.9) implies that

$$e(A) = \lambda(A/(\mathbf{z})A) = \sum_{i=0}^{r-1} v((\mathbf{m}^i, \mathbf{z})A/(\mathbf{m}^{i+1}, \mathbf{z})A).$$

Combining this with (4.6) and (4.3) it follows that

$$e(A) \leqslant \sum_{i=0}^{r-1} e_i = e(S_{\mathbf{m}S}).$$

Thus we have constructed a geometric $(n - 1)$-residual intersection J of I with $e(R/J) \leqslant e(S_{\mathbf{m}S})$.

To show the equality of the multiplicities we may complete R to assume that all rings in question are excellent. By Lemma 3.2, $S_{\mathbf{m}S}$ is essentially a deformation of R/J, and hence by [13, 2.3], $e(S_{\mathbf{m}S}) \leqslant e(R/J)$. □

The next corollary shows that in Theorem 1.6, part (a) implies part (d).

COROLLARY 4.11. *Let R be a local Cohen-Macaulay ring with infinite residue class field, let I be an R-ideal with* $\dim R = v(I) > \operatorname{grade} I > 0$ *that is strongly Cohen-Macaulay and satisfies G_∞. If I is normal, then there exists a geometric residual intersection J of I with R/J being a discrete valuation ring.*

Proof. The assertion follows from Theorem 4.1 observing that $S_{\mathbf{m}S}$ is a discrete valuation ring and R/J is a one-dimensional Cohen-Macaulay ring (cf. Theorem 1.5.a). □

REMARK 4.12. Without assuming $v(I) = \dim R$ but with all the other assumptions of Corollary 4.11 in place one could still prove that there exists a geometric residual intersection J of I with height $J = v(I) - 1$ and R/J a normal domain (at least if R contains $\mathbb{Q}$; the latter assumption is needed since one has to use [4, 4.3]).

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