COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 84, nº 1 (1992), p. 101-113
<http://www.numdam.org/item?id=CM_1992__84_1_101_0>

Compositio Mathematica **84**: 101–113, 1992.

Some decomposition lemmas of diffeomorphisms

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Received 9 April 1991; accepted 14 February 1992

Abstract. Let Ω be a volume element on an open manifold, M, which is the interior of a compact manifold $\bar{M}$. We will give conditions for a non-compact supported Ω-preserving diffeomorphism to decompose as a finite product of Ω-preserving diffeomorphisms with supports in locally finite families of disjoint cells.

A widely used and very powerful technique in the study of some subgroups of the group of diffeomorphisms of a differentiable manifold, Diff(M), is the decomposition of its elements as a finite product of diffeomorphisms with support in cells (See for example [1], [5], [7]).

It was in a paper of Palis and Smale [7] that first appeared one of those decompositions for the case of a compact manifold, M. They proved that for any finite open covering of M, any diffeomorphism of M close to the identity can be decomposed as a finite product of diffeomorphisms with supports in such open sets. They used this decomposition to prove the structural stability of some elements of Diff(M).

There is not such a general result for diffeomorphisms preserving a volume element. But, there is a decomposition in some cases. In particular, if M is a compact n-manifold and Ω is a volume element on M, Thurston in [9] stated that any element in the kernel of the flux homomorphism (ϕ_c: $\mathrm{Diff}_0^\Omega(M) \to H^{n-1}(M; \mathbb{R})/\Gamma$) decomposes as a finite product of diffeomorphisms preserving Ω and with supports in cells, and gave a sketch of a proof. This fact allowed him to transfer his result that ker ϕ_c is simple when M is a torus to a similar result for an arbitrary compact manifold M. (Banyaga [1] gave the same result for symplectic diffeomorphisms).

It is clear that when the manifold M is not compact and the diffeomorphism f has not compact support it is not possible to decompose f as a finite product of diffeomorphisms with supports in cells. In this case, the best we can expect is a decomposition as a finite product of diffeomorphisms with supports in disjoint unions of locally finite families of cells. Mascaró [4] proved that this is true for any volume preserving diffeomorphism of $M = \mathbb{R}^n$ for $n \geqslant 3$, and used it to give a lattice of the normal subgroups of $\mathrm{Diff}^\Omega(\mathbb{R}^n)$.

The authors were partially supported by the CICYT grant n. PS87-0115-C03-01.

This note is mainly devoted to give conditions for a non-compactly supported volume-preserving diffeomorphism, f, of a manifold M that is the interior of a compact one $\bar{M}$ to decompose as a finite product of volume preserving diffeomorphisms with supports in locally finite disjoint union of cells.

We thank the referee for his suggestions on clarifying some parts of this paper.

0. Uniform topologies

The topology on $\mathrm{Diff}^{\Omega}(M)$ that has proved adequate for our purpose is the C^{∞}-uniform topology. Thus, Section 0 is devoted to the definition and elementary properties of this topology.

DEFINITION 1. *Let M, N be differentiable manifolds and let d be a metric on N compatible with the topology of N.*

We define the C^0-uniform topology on $C^k(M,N)$, $k = 0, 1, \ldots$ as follows:

As a basis of neighbourhoods of $f \in C^k(M, N)$ we take the sets

$$U(f, \varepsilon, d) = \{h \in C^k(M, N)\colon d(h(x), f(x)) < \varepsilon, \text{ for any } x \in M\}.$$

On the other hand, there is a natural embedding

$$j^r\colon C^k(M, N) \to C^0(M, J^r(M, N))$$

where $J^r(M,N)$ represents the space of r-jets from M to N, given by $(j^r)(f)(m) = (j^r f)(m)$ the r-jet of f at m. If we consider the C^0-uniform topology on the space $C^0(M, J^r(M, N))$; then, the C^r-uniform topology on $C^k(M, N)$, for $0 < r \leqslant k$, is the topology induced by the above embedding (See [6] for more details).

We define the C^{∞}-uniform topology on $C^{\infty}(M,N)$ as the direct limit of the C^r-uniform topologies.

In this work we consider the C^{∞}-uniform topology on $\mathrm{Diff}^{\Omega}(M)$.

NOTE. (i) The C^r-uniform topology depends on the compatible metric d chosen on N.

(ii) It is clear that the C^{∞}-uniform topology is finer than the C^{∞}-compact open topology and coarser than the C^{∞}-Whitney topology.

(iii) With this topology, $\mathrm{Diff}^{\Omega}(M)$ is not a topological group but if $h \in \mathrm{Diff}^{\Omega}(M)$ is any fixed element, the right translation

$$r_h\colon \mathrm{Diff}^{\Omega}(M) \to \mathrm{Diff}^{\Omega}(M)$$

defined by $r_h(f) = f \circ h$ is continuous.

Since the path-component of the identity in $\mathrm{Diff}^{\Omega}(M)$ with respect to the C^{∞}-uniform topology is not a normal subgroup we will denote by $\mathrm{Diff}_1^{\Omega}(M)$ the normal subgroup of $\mathrm{Diff}^{\Omega}(M)$ generated by the path-component of the identity.

Let f be any element of $\mathrm{Diff}^{\Omega}(M)$, we will say that f is Ω-isotopic to the identity if there is a differentiable map $F: M \times [0, 1] \to M \times [0, 1]$ such that for any $t \in [0, 1]$ the map $F_t: M \equiv M \times \{t\} \to M \times \{t\} \equiv M$ preserves the volume element Ω and $F_1 = f$ and F_0 agrees with the identity.

We denote by $\mathrm{Diff}_0^{\Omega}(M)$ the normal subgroup of $\mathrm{Diff}^{\Omega}(M)$ of all elements Ω-isotopic to the identity. Clearly, we have $\mathrm{Diff}_1^{\Omega}(M) \subset \mathrm{Diff}_0^{\Omega}(M)$ and these two subgroups are different in general.

1. Decomposition of some diffeomorphisms of $X \times \mathbb{R}^+$

Let X be a closed manifold, $\mathbb{R}^+$ the interval $[0, \infty)$ and Ω a volume element on $X \times \mathbb{R}^+$. We denote by $\mathrm{Diff}^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ the group of all diffeomorphisms of $X \times \mathbb{R}^+$ that are the identity on a neighbourhood of $X \times \{0\}$ and preserve the volume element Ω.

Notice that any element of $\mathrm{Diff}^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ is Ω-isotopic to the identity.

We will prove here that if X has trivial first real homology group, then, any element of $\mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ can be decomposed as a finite product of diffeomorphisms with supports in disjoint unions of locally finite families of cells.

Let us consider on $X \times \mathbb{R}^+$ the metric compatible with the product topology given by

$$d((x, s), (x', s')) = \sup\{d'(x, x'), |s - s'|\}$$

where d' denotes a fixed metric on X.

THEOREM 1. *Let X be a closed connected $(n-1)$-manifold with trivial first real homology group. Then any element $f \in \mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ can be decomposed as $f = f_1 \circ \cdots \circ f_m$ with $f_i \in \mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ having support in a locally finite union of disjoint cells, for any $i = 1, \ldots, m$.*

Proof. Without loss of generality we can assume that f is an element of the path-component of the identity with respect to the C^{∞}-uniform topology.

We carry out the proof in three steps:

Step 1. Reduction to the case where f is close to the identity.

Let $\alpha: [0, 1] \to \mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ be a uniform path between f and the identity and let $U(id, \varepsilon, d)$ be any neighbourhood of the identity. Then, if we denote by $\alpha_t = \alpha(t)$, let δ be the Lebesgue number of the covering,

$\{\alpha^{-1}(U(\alpha_t, \varepsilon, d))\}$. We take a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that $|s_i - s_{i-1}| < \delta$ for any $i = 1, \ldots, m$. Thus, for any $i = 1, \ldots, m$ the elements α_{s_i}, $\alpha_{s_{i-1}}$ lie in some $U(\alpha_t, \varepsilon, d)$.

Therefore, we could write

$$f = \alpha_{s_m} = (\alpha_{s_m} \circ (\alpha_{s_{m-1}})^{-1}) \circ (\alpha_{s_{m-1}} \circ (\alpha_{s_{m-2}})^{-1}) \circ \cdots \circ (\alpha_{s_0} \circ (\alpha_{s_{-1}})^{-1})$$

where α_{s-1} is the identity and for any $i = 1, \ldots, m$, $(\alpha_{s_i} \circ (\alpha_{s_{i-1}})^{-1})$ is an element of $U(id, \varepsilon, d)$. Furthermore, we could define a path from the identity to $(\alpha_{s_i} \circ (\alpha_{s_{i-1}})^{-1})$ inside $U(id, \varepsilon, d)$ as follows:

$$\gamma: [0, 1] \to \mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$$

where $\gamma(t) = \alpha(ts_i + (1 - t)s_{i-1}) \circ (\alpha(s_{i-1}))^{-1}$.

Step 2. By Step 1 we could assume without loss of generality that there is a uniform path in $U(id, \varepsilon, d)$ from f to the identity.

We will prove that $f = f_1 \circ f_2$ with $f_i \in \mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ and isotopic to the identity by an Ω-isotopy with support in a locally finite union of disjoint compacts for $i = 1, 2$. We will also get f_1, f_2 near the identity.

Let $\alpha: [0, 1] \to U(id, \varepsilon, d)$ be the uniform path between f and the identity. It defines an Ω-isotopy, $F: (X \times \mathbb{R}^+) \times [0, 1] \to X \times \mathbb{R}^+$, by $F((x, s), t) = \alpha(t)(x, s)$.

Inductively, we construct a sequence of positive numbers $0 < \lambda_1 < \lambda_2 < \cdots$, such that:

(i) $F((X \times [0, \lambda_{2i+1}]) \times [0, 1]) \subset X \times [0, \lambda_{2i+2})$ for any $i = 0, 1, 2, \ldots$.
(ii) $F((X \times [\lambda_{2i+1}, \infty) \times [0, 1]) \subset X \times (\lambda_{2i}, \infty)$ for any $i = 1, 2, 3, \ldots$.

It is clear that we can take $\lambda_1 = 1$ and $\varepsilon < \lambda_{i+1} - \lambda_i < 2\varepsilon$.

Since $X \times \{\lambda_{2i+1}\}$ is compact and F is continuous there is a connected open neighbourhood, V_{2i+1}, of $X \times \{\lambda_{2i+1}\}$ such that $F(V_{2i+1} \times [0, 1]) \subset X \times (\lambda_{2i}, \lambda_{2i+2})$ for any $i = 0, 1, \ldots$. So, for any i, we can extend the above embeddings [3] to Ω-isotopies

$$F^i: (X \times (\lambda_{2i}, \lambda_{2i+2})) \times [0, 1] \to X \times (\lambda_{2i}, \lambda_{2i+2})$$

such that:

(i) F^i agrees with F on a neighbourhood of $X \times \{\lambda_{2i+1}\}$.
(ii) For any $t \in [0, 1]$, F^i_t agrees with the identity on a neighbourhood of $X \times \{\lambda_{2i}, \lambda_{2i+2}\}$.
(iii) For any $t \in [0, 1]$, $F^i_t \in U(id, \varepsilon, d)$.

So, they define an Ω-isotopy $H: (X \times \mathbb{R}^+) \times [0, 1] \to X \times \mathbb{R}^+$, with support in $\bigsqcup_i X \times (\lambda_{2i}, \lambda_{2i+2})$ and such that H_t is in $U(id, \varepsilon, d)$ for any $t \in [0, 1]$.

For any m we can consider the diffeomorphism $\varphi^m: X \times [0, \lambda_2] \to X \times [\lambda_{2m}, \lambda_{2m+2}]$ given by

$$\varphi^m(x, s) = \left(x, \lambda_{2m} + \frac{\lambda_{2m+2} - \lambda_{2m}}{\lambda_2} s\right).$$

Then we have

$$\begin{aligned} d(\varphi^m(x_1, s_1), \varphi^m(x_2, s_2)) &= \sup\left\{d'(x_1, x_2), \frac{\lambda_{2m+2} - \lambda_{2m}}{\lambda_2} |s_1 - s_2|\right\} \\ &\leqslant \sup\left\{d'(x_1, x_2), \frac{4\varepsilon}{\lambda_2} |s_1 - s_2|\right\} \\ &\leqslant \frac{4\varepsilon}{\lambda_2} d((x_1, s_1), (x_2, s_2)). \end{aligned}$$

If we define $f_1 = H_1$ we have $f_1 \in \mathrm{Diff}_1^\Omega(X \times \mathbb{R}^+, \operatorname{rel} X \times \{0\})$ and with $f_2 = f \circ f_1^{-1}$ we get the desired decomposition.

Step 3. Decomposition of the elements obtained in Step 2.

We could assume that f is an element of $\mathrm{Diff}_1^\Omega(X \times \mathbb{R}^+, \operatorname{rel} X \times \{0\})$, Ω-isotopic to the identity by an isotopy f_t near the identity and with support in $\bigsqcup_m X \times (\lambda_{m-1}, \lambda_m)$ with $\lambda_1 = 1$ and $\varepsilon < \lambda_{m+1} - \lambda_m < 2\varepsilon$.

To get the decomposition of f we will apply the Fragmenting Lemma of the Appendix to any restriction of f to $X \times [\lambda_{m-1}, \lambda_m]$. But being careful in choosing the triangulation at the beginning of the proof of Fragmenting Lemma we will obtain the same number of volume preserving diffeomorphisms on each compact $X \times [\lambda_{m-1}, \lambda_m]$. Therefore, they define volume preserving diffeomorphisms of $X \times \mathbb{R}^+$.

Furthermore, because we can assume $\lambda_1 = 1$ and $\varepsilon < |\lambda_m - \lambda_{m-1}| < 2\varepsilon$ we also get that any element in the decomposition is an element of $(\mathrm{Diff}_1^\Omega(X \times \mathbb{R}^+, \operatorname{rel} X \times \{0\})$. For these reasons, we construct the following triangulation.

Let us have the triangulation of $M_1 = X \times [0, \lambda_1]$,

$$T = \{\Delta_i^k : i \in I_k \text{ for } k = 0, 1, \ldots, n\}$$

and let $\{U_i^k, V_i^k : i \in I_k \text{ for } k = 0, 1, \ldots, n\}$ be the open coverings associated to the above triangulation T, defined in Appendix. Let

$$\varepsilon \leqslant \inf\{d(V_i^k, M_1 - U_i^k) : i \in I_k \text{ for } k = 0, 1, \ldots, n\} > 0.$$

(We could assume that f is an element of the neighbourhood of the identity given by ε^2).

On each compact, $M_m = X \times [\lambda_m, \lambda_{m+1}]$, we consider the triangulation given by:

$$\{\varphi^m(\Delta_i^k): i \in I_k \text{ for } k = 0, 1, \ldots, n\}$$

and the open coverings associated to the above triangulation

$$\{\varphi^m(U_i^k), \varphi^m(V_i^k): i \in I_k \text{ for } k = 0, 1, \ldots, n\}$$

We have for any $i \in I_k$, $k = 0, \ldots, n$ and $m \in N$, that

$$\varepsilon^2 < d(V_i^k, M_1 - U_i^k) \leqslant d(\varphi^m(V_i^k), M_m - \varphi^m(U_i^k)).$$

Therefore, if f is an element of $U(id, \varepsilon^2, d)$ we have $f(\varphi^m(V_i^k)) \subset \varphi^m(U_i^k)$ for any $i \in I_k$, $k = 0, \ldots, n$ and $m \in N$.

For any $m \in N$ we have the flux homomorphism

$$\phi_c: \mathrm{Diff}_{co}^{\Omega}(X \times (\lambda_{m-1}, \lambda_m)) \to H_c^{n-1}(X \times (\lambda_{m-1}, \lambda_m); \mathbb{R})$$

and, using Lefschetz duality theorem [8], we have

$$\begin{aligned} H_c^{n-1}(X \times (\lambda_{m-1}, \lambda_m); \mathbb{R}) &\cong H^{n-1}(X \times [\lambda_{m-1}, \lambda_m], X \times \{\lambda_{m-1}, \lambda_m\}; \mathbb{R}) \\ &\cong H_1(X \times [\lambda_{m-1}, \lambda_m]; \mathbb{R}) \end{aligned}$$

So, since $H_1(X; \mathbb{R}) = 0$, we have that each restriction of f to $X \times (\lambda_{m-1}, \lambda_m)$, f_m, is in the kernel of the flux homomorphism, and we can apply the Fragmenting Lemma of the Appendix to any compact $X \times [\lambda_m, \lambda_{m+1}]$ with the above triangulation and open coverings to get $f_m = f_m^0 \circ \cdots \circ f_m^n$ with $\mathrm{supp}(f_m^k) \subset \bigsqcup_i \varphi^m(U_i^k)$.

Since the number of diffeomorphisms that we get in the above decomposition depends only on the dimension of $X \times \mathbb{R}^+$, they define elements $f^k \in \mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ such that $f = f^0 \circ \cdots \circ f^n$, and each f^k has support in a locally finite family of disjoint cells.

Thus, we have proved that if X is a closed n-manifold with $n \geqslant 2$ and $H_1(X; \mathbb{R}) = 0$ then $\mathrm{Diff}_1^{\Omega}(X \times \mathbb{R}^+, \mathrm{rel}\, X \times \{0\})$ is generated by the elements with support in disjoint unions of locally finite families of cells.

2. Decomposition of some diffeomorphisms of M

Let M be the interior of a compact n-manifold $\bar{M}$ with non-empty boundary, $\partial \bar{M}$. Let $\partial_1 \bar{M}, \ldots, \partial_k \bar{M}$ be the connected components of $\partial \bar{M}$ and let Ω be any volume element on M.

First of all we recall the generalization of the flux homomorphism given by McDuff [5]. It is a homomorphism

$$\phi : \mathrm{Diff}_0^{\Omega}(M) \to H^{n-1}(M; \mathbb{R})$$

defined as follows: let $f \in \mathrm{Diff}_0^{\Omega}(M)$, then $\phi(f) = f^*(\omega) - \omega$ where ω is a $(n-1)$-form such that $d\omega = \Omega$. It is related to the flux by the following commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}_{co}^{\Omega}(M) & \xrightarrow{\phi_c} & H_c^{n-1}(M; \mathbb{R}) \cong H^{n-1}(\bar{M}, \partial\bar{M}; \mathbb{R}) \\ \big\downarrow & & \big\downarrow \alpha \\ \mathrm{Diff}_0^{\Omega}(M) & \xrightarrow{\phi} & H^{n-1}(M; \mathbb{R}) \cong H^{n-1}(\bar{M}; \mathbb{R}) \end{array} \qquad (*)$$

where α is induced from the inclusion.

The flux homomorphism plays an important role in the problem of extending a family of volume preserving embeddings $g_t : M_0 \to M$, with M_0 a compact submanifold of M, to a family of volume preserving diffeomorphisms which are the identity outside some neighbourhood of $\bigcup_{t \in I} g_t(M_0)$. This is not possible in general as can be seen in the following example.

On $M = S^{n-1} \times (-\infty, \infty)$ we consider the volume element $\Omega = \omega \wedge d\lambda$ where ω is a volume element on S^{n-1}. Let $f_t : M \to M$ be the translations $f_t(x, \lambda) = (x, \lambda + t)$. Clearly they preserve the volume element Ω.

Let $M_0 = S^{n-1} \times [1, 2]$, and we consider the embeddings $f_t \mid M_0 \to M$. There is no extension to a Ω-preserving isotopy such that it is the identity near $S^{n-1} \times \{0\} \cup S^{n-1} \times \{4\}$, because if there was such extension, $\bar{f}_t$, the Ω-volume of $S^{n-1} \times [0, 1]$ and $\bar{f}_t(S^{n-1} \times [0, 1]) = S^{n-1} \times [0, 1 + t]$ must be the same, and it is not true. That is due to the fact that there is some "mass' passing through $S^{n-1} \times \{0\}$.

The flux of f_t can be interpreted as a measure of this mass.

The obstruction to constructing a volume preserving isotopy $\bar{f}_t$ equal to f_t in a neighbourhood of M_0 is the element of $H^{n-1}(M, M_0; \mathbb{R})$ whose value on an n-chain c with boundary in M_0 is the integral

$$\int_c f_t'^* \Omega - \Omega,$$

where f_t' is a (possibly non volume-preserving) extension of $f_t | M_0$ ([2], [3]).

Thus if $f_t \in \ker \phi$ for any $t \in I$, we have

$$\int_c f_t'^* \Omega = \int_c f_t'^* \, \mathrm{d}\omega = \int_{\partial c} f_t'^* \omega = \int_{\partial c} f_t^* \omega = \int_{\partial c} \omega = \int_c \Omega.$$

and such obstruction does not exist.

Now, we prove the following decomposition

THEOREM 2. *Let M be an n-manifold as above with $n \geqslant 3$ and $H_1(\partial_i \bar{M}; \mathbb{R}) = 0$ for any $i = 1, \ldots, k$. Then, for any isotopy $f_t \in \mathrm{Diff}_1^\Omega(M) \cap \ker \phi$, f_1 can be written as a finite product of elements of* $\mathrm{Diff}_1^\Omega(M)$ *with support in locally finite families of cells.*

Proof. We take a collar of $\partial \bar{M}$ given by an embedding $\partial \bar{M} \times [0, 1] \to \bar{M}$ identifying $\partial \bar{M} \times \{1\}$ with $\partial \bar{M}$. Let us call $M_0 = \bar{M} - (\partial \bar{M} \times (0, 1])$.

We will divide the proof in three steps.

Step 1. Decomposition of f as a product $f = f_1 \circ f_2$, with $f_2 \in \mathrm{Diff}_{co}^\Omega(M)$ and $f_1 \in \mathrm{Diff}_1^\Omega(M, \mathrm{rel}\, M_0)$.

Since $f \in \mathrm{Diff}_1^\Omega(M)$ there is a uniform path $\{f_t\}$ between f and the identity such that each f_t preserves Ω. Let us consider the family of embeddings $f_t|: M_0 \to M$, since M_0 is compact, there is some $0 < \mu < 1$ such that the image of M_0 by the isotopy is included in

$$M' = M_0 \cup \bigsqcup_{i=1}^{k} \partial_i \bar{M} \times [0, \mu).$$

Since $f_t \in \ker \phi$, there is no obstruction to extend $f_t|: M_0 \to M'$ to a volume preserving isotopy that is the identity near $\partial_i \bar{M} \times \{\mu\}$ ([2], [3]). Then, there is a compactly supported Ω-isotopy $g_t: M \to M$ such that for any $t \in [0, 1]$, g_t agrees with f_t on M_0.

We define $f_2 = g_1$ and we have $f_2 \in \mathrm{Diff}_{co}^\Omega(M)$.

On the other hand, if we define $f_1 = f \circ f_2^{-1}$ we have that f_1 is the identity on M_0. Furthermore, the path $\sigma: [0, 1] \to \mathrm{Diff}^\Omega(M)$ given by $\sigma(t) = f_t \circ f_2^{-1}$ is a uniform path between f_1 and f_2^{-1}, and since f_2^{-1} is an element of $\mathrm{Diff}_{co}^\Omega(M) \subset \mathrm{Diff}_1^\Omega(M)$, we have $f_1 \in \mathrm{Diff}_1^\Omega(M)$.

Step 2. Decomposition of f_1 as a finite product of elements of $\mathrm{Diff}_1^\Omega(M)$ with support in disjoint unions of locally finite families of cells.

Since the support of f_1 is included in $\bigsqcup_i \partial_i \bar{M} \times [0, 1)$, the restrictions

$$f_1^i: \partial_i \bar{M} \times [0, 1) \to \partial_i \bar{M} \times [0, 1)$$

define elements $f_1^i \in \mathrm{Diff}_1^\Omega(\partial_i \bar{M} \times [0, 1), \mathrm{rel}\, \partial_i \bar{M} \times \{0\})$ for any $i = 1, \ldots, k$.

Therefore, since $H_1(\partial_i \bar{M}; \mathbb{R}) = 0$ for any $i = 1, \ldots, k$ we can apply Theorem 1 getting the desired decomposition of f_1.

Step 3. Decomposition of f_2 as a finite product of elements of $\mathrm{Diff}_1^\Omega(M)$ with support in locally finite families of cells.

If we can prove that $f_2 \in \ker \phi_c$ the result will follow from the Fragmenting Lemma of the Appendix. Therefore, since $f = f_1 \circ f_2$ and $f \in \ker \phi$ we have $\phi(f_1) + \phi(f_2) = 0$.

Since we have a decomposition of f_1 as finite product of diffeomorphisms with support in a locally finite family of disjoint cells, by [2] we know that f_1 is in the commutator subgroup of $\mathrm{Diff}^{\Omega}(\partial_i \bar{M} \times [0, 1), \mathrm{rel}\, \partial_i \bar{M} \times \{0\})$. Therefore, since the image of the flux homomorphism is a commutative subgroup, we have $\phi(f_1) = 0$.

Now, we consider the commutative diagram (*) and since

$$H_1(\partial \bar{M}; \mathbb{R}) = \prod H_1(\partial_i \bar{M}; \mathbb{R}) = 0.$$

the map α is a monomorphism. Then, $0 = \phi(f) = \phi(f_1) + \phi(f_2) = \phi(f_2) = \alpha \circ \phi_c(f_2)$ implies $f_2 \in \ker \phi_c$.

Appendix. The Fragmenting Lemma

Because we have not been able to find in the literature a proof of the Fragmenting Lemma stated in [9] we have written here a modification for the volume preserving case of the Banyaga's symplectic case [1].

We use the infinitesimal definition of the flux homomorphism, it is based on the Banyaga's definition for the symplectic case.

Let M be a connected smooth n-manifold, $n \geqslant 3$. And, let Ω be a volume element on M.

We denote by $\widetilde{\mathrm{Diff}}{}^{\Omega}_{co}(M)$ the universal covering of the group $\mathrm{Diff}^{\Omega}_{co}(M)$. The map

$$\tilde{\phi}_c \colon \widetilde{\mathrm{Diff}}{}^{\Omega}_{co}(M) \to H^{n-1}_c(M; \mathbb{R})$$

defined by $\tilde{\phi}_c(f, \{f_t\}) = \int_0^1 i(\dot{f}_t)\Omega\, dt$ is a group epimorphism.

If we denote by Γ the image by $\tilde{\phi}_c$ of the subgroup $\pi_1(\mathrm{Diff}^{\Omega}_{co}(M))$ of $\widetilde{\mathrm{Diff}}{}^{\Omega}_{co}(M)$, then, on the quotient, we get the epimorphism

$$\phi_c \colon \mathrm{Diff}^{\Omega}_{co}(M) \to H^{n-1}_c(M; \mathbb{R})/\Gamma.$$

The details of the proof of the above facts and some interesting properties of the flux homomorphism, as well as the equivalence between the infinitesimal and geometric definitions of flux, can be found in [2].

We will need the following result in the proof of the fragmenting lemma.

LEMMA. *Let* $\{f_t\}_{t \in I}$ *be an isotopy in* $\ker \phi_c$. *Then, for any* $t \in I$, *the* $(n-1)$*-form* $i(\dot{f}_t)\Omega$ *is exact.*

Proof. We consider the following commutative diagram

$$\begin{array}{ccc}
\pi_1(\mathrm{Diff}^{\Omega}_{co}(M)) & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{Diff}^{\Omega}_{co}}(M) & \xrightarrow{\tilde{\phi}_c} & H^{n-1}_c(M;\mathbb{R}) \\
p\downarrow & & \downarrow \\
\mathrm{Diff}^{\Omega}_{co}(M) & \xrightarrow{\phi_c} & H^{n-1}_c(M;\mathbb{R})/\Gamma
\end{array}$$

Since $p^{-1}(\ker\phi_c) = \pi_1(\mathrm{Diff}^{\Omega}_{co}(M))\cdot\ker\tilde{\phi}_c$, given the isotopy $\{f_t\}_{t\in I}$ in $\ker\phi_c$ there is a unique lifting $\{\tilde{f}_t\}_{t\in I}$, such that $\tilde{f}_0 = Id$, and $\tilde{\phi}_c(\tilde{f}_t)\in\Gamma$, for any $t\in I$. The map $t\mapsto\tilde{\phi}_c(\tilde{f}_t)$ is a path in Γ, which is a discrete subgroup (since Ω is a volume form); so the above map is constant.

Then $\tilde{\phi}_c(\tilde{f}_t) = \tilde{\phi}_c(\tilde{f}_0) = 0$ for any $t\in I$, and we get that, $\{\tilde{f}_t\}_{t\in I}\in\ker\tilde{\phi}_c$.

By definition of the universal covering, we can get, for any $t\in I$, an isotopy $\{F_{s,t}\}_{s\in I}$ in $\ker\tilde{\phi}_c$, representing $\tilde{f}_t$ (i.e. $\tilde{f}_t = (f_t, \{F_{s,t}\})$).

But, for any $t\in I$, the map given by:

$$(s, v)\mapsto F_{v(s-1)+1,((1-v)s+v)t}(s, v)\in I\times I$$

is an isotopy between the paths $s\mapsto F_{s,t}$ and $s\mapsto\tilde{f}_{st}$. So, we get $\tilde{\phi}_c(\tilde{f}_{st}) = \tilde{\phi}_c(F_{s,t}) = 0$.

Then, for any $t\in I$, the $(n-1)$-form

$$\tilde{\phi}_c(\tilde{f}_{st}) = \int_0^1 i(X_{st})\Omega\,\mathrm{d}s = \int_0^t i(\dot{f}_s)\Omega\,\mathrm{d}s$$

is exact, where X_{st} is the family of vector fields given by

$$X_{st}(m) = \frac{\mathrm{d}f_{st}}{\mathrm{d}s}(f_{st}^{-1}(m)),\quad \text{for any } m\in M.$$

So, there is a uniparametric family of $(n-2)$-forms $\{\alpha_t\}$ such that

$$\int_0^t i(\dot{f}_s)\Omega\,\mathrm{d}s = \mathrm{d}\alpha_t.$$

Finally, we get $i(\dot{f}_t)\Omega = \mathrm{d}\beta_t$, where $\beta_t = \dfrac{\partial\alpha_t}{\partial t}$.

Following Banyaga [1], now we introduce the concept of covering associated to a triangulation. Given a triangulation of M,

$$T = \{\Delta_i^k : i \in I_k \text{ for } k = 0, 1, \ldots, n\}$$

we will associate an open covering by cells $\mathscr{V} = \{V_i^k\}_{i \in I_k}$, for $k = 0, 1, \ldots, n$. We will construct it by recurrence on the skeleton.

Let V_i^0 be an open cell containing Δ_i^0 and such that $V_i^0 \cap V_j^0 = \emptyset$ if $i \neq j$. Now, let us suppose that we have construct the cells $\{V_i^l\}_{i \in I_l}$, for $l = 0, 1, \ldots, k-1$ which cover the $(k-1)$-skeleton and such that

$$\bar{\Delta}_i^k = \Delta_i^k - \bigcup_{j \leqslant k-1} V_i^j$$

is a retract of Δ_i^k. Let $\tilde{\Delta}_i^k$ be an expansion of $\bar{\Delta}_i^k$, then let V_i^k be a tubular neighbourhood of $\tilde{\Delta}_i^k$ such that $V_i^k \cap V_j^k = \emptyset$ if $i \neq j$.

FRAGMENTING LEMMA. *Let M be a closed, connected n-manifold, $n \geqslant 3$, and let Ω be a volume element on M. Let $\mathscr{W} = \{W_i\}$ be a finite covering of M by open cells. Then every isotopy $f_t \in \ker \phi_c$ can be written as a finite product of isotopies $f_t^i \in \ker \phi_c$ with support in the cells of $\mathscr{W}$.*

Proof. First, we consider a triangulation of M,

$$T_{\mathscr{W}} = \{\Delta_i^k : i \in I_k \text{ for } k = 0, 1, \ldots, n\}$$

such that the star of each simplex Δ_i^k is contained in some cell of $\mathscr{W}$.

We construct two open coverings associated to the above triangulation, $\mathscr{U} = \{U_i^k\}_{i \in I_k}$ and $\mathscr{V} = \{V_i^k\}_{i \in I_k}$, for $k = 0, 1, \ldots, n$, such that $\bar{V}_i^k \subset U_i^k$ and each open set U_i^k is contained in some cell of $\mathscr{W}$.

Each simplex Δ_i^k is covered by $\{V_r^j : \Delta_r^j \subset \Delta_i^k\}$. We denote $V^k = \bigcup_{i \in I_k} V_i^k$.

Let us assume that $f_t \in \ker \phi_c$ is in a small neighbourhood of the identity.

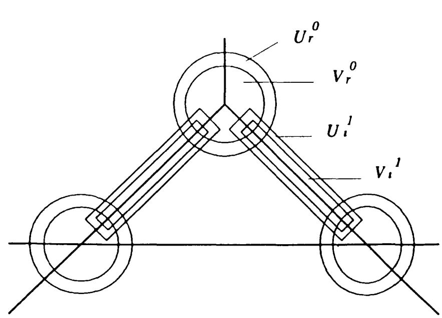

Fig. 1.

Inductively, we will construct diffeomorphisms $f_t^{(k)} \in \ker \phi_c$, for $1 \leqslant k \leqslant n$, such that

(1) $f_t^{(k)} = f_t^{(k-1)}$ near $M - U^k$,
(2) $f_t^{(k)} = f_t$ near $\bigcup_{j \leqslant k} V^j$.

Since $f_t^{(n)} = f_t$, if we define $f_t^{(-1)} = Id$ and $g_t^{(k)} = (f_t^{(k-1)})^{-1} \circ f_t^{(k)}$, for any $k = 1, \ldots, n$; then the result follows immediately from the fact that $f_t = g_t^{(1)} \circ \cdots \circ g_t^{(n)}$, and each diffeomorphism $g_t^{(k)}$ has support in U^k.

First of all, we construct $f_t^{(0)}$, for any $t \in I$.

Since $f_t \in \ker \phi_c$, by the previous Lemma, the $(n-1)$-form $i(\dot{f}_t)\Omega$ is exact for any $t \in I$, then let $\{\beta_t\}_{t \in I}$ be a uniparametric family of $(n-2)$-forms such that $i(\dot{f}_t)\Omega = \mathrm{d}\beta_t$, for any $t \in I$.

Let λ be a C^∞ real valued function with support in U^0, that is 1 on a neighbourhood of $V^{*0} = \bigcup_{t \in I} f_t(V^0)$. Let us consider the uniparametric family of $(n-2)$-forms $\{\lambda \cdot \beta_t\}_{t \in I}$, and let $\{\psi_t\}_{t \in I}$ be the isotopy we get by integrating the equation

$$i(\dot{\psi}_t)\Omega = d(\lambda \cdot \beta_t).$$

It is clear that $\psi_t \in \ker \phi_c$, it has support in U^0 and $\psi_t = f_t$ near V^0, for any $t \in I$. So, we define $f_t^{(0)} = \psi_t$, and it satisfies the desired conditions.

Let us suppose now that we have constructed the diffeomorphisms $f_t^{(j)}$, for $j < k$, satisfying all above conditions; since f_t is in a small neighbourhood of the identity, we can get each $f_t^{(j)}$ such small that if we define $V_i^{*j} = \bigcup_{t \in I} f_t(V_i^j)$ and $U_i^{*j} = M - \bigcup_{t \in I} f_t^{(j-1)}(M - U_i^j)$, then we have $V_i^{*j} \subset U_i^{*j}$ for $j \leqslant k$.

We construct $f_t^{(k)}$ in the following way.

Since $f_t^{(k-1)}$ and f_t are in $\ker \phi_c$, then there are two uniparametric families of $(n-2)$-forms such that:

$$i(\dot{f}_t^{(k-1)})\Omega = d(\beta_t^1) \quad \text{and} \quad i(\dot{f}_t)\Omega = d(\beta_t^2);$$

notice that we can choose β_t^1 and β_t^2 to coincide on $\bigcup_{j<k} f_t(V^j)$ for any $t \in I$, since we have that $f_t^{(k-1)} = f_t$ near $\bigcup_{j<k} V^j$.

Let (λ_1, λ_2) be a partition of unity subordinated to the open covering $\{(M - \bar{V}^{*k}), U^{*k}\}$. We consider the uniparametric family of $(n-2)$-forms $\{\lambda_1 \cdot \beta_t^1 + \lambda_2 \cdot \beta_t^2\}$, and integrating, as above, the equation

$$i(\dot{\psi}_t)\Omega = d(\lambda_1 \cdot \beta_t^1 + \lambda_2 \cdot \beta_t^2),$$

we get an isotopy $\{\psi_t\}$ in $\ker \phi_c$ equal to f_t on $\bigcup_{j \leqslant k} V^j$ and equal to $f_t^{(k-1)}$ on $M - U^k$. Then, we define $f_t^{(k)} = \psi_t$.

In the general case, if we suppose that the isotopy $\{f_t\}$ is not close to the identity, we can write $\{f_t\}$ as a finite product of small isotopies in $\ker \phi_c$ and the result follows.

Notice that the same proof works in the case that M is a compact manifold with boundary and $\{f_t\}$ is an isotopy of M that is the identity on a neighbourhood of ∂M.

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