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On the location of poles of the triple L-functions

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Introduction

Let $K$ be a semi-simple abelian algebra of degree 3 over a global field $k$. In [22], I. I. Piatetski-Shapiro and S. Rallis constructed the triple L-functions for irreducible cuspidal automorphic representations of $GL_2(K \otimes A_k)$ by means of Rankin-type integrals following P. B. Garrett [3]. The purpose of this paper is to determine the location of the poles of these L-functions. To describe our main result, assume, for simplicity, $K = k \oplus k \oplus k$. Let $\alpha$ be the standard ideal norm: $A_k^* \to \mathbb{R}^*$. Given three irreducible cuspidal automorphic representations $\pi_1, \pi_2, \pi_3$ of $GL_2(A_k)$, let $\omega$ be the product of the central quasi-characters of these representations. Let $\sigma$ be the 8-dimensional representation of the L-group $GL_2(C)^3$ obtained by the tensor product of the standard representations of $GL_2(C)$. The triple L-function $L(s, \Pi, \sigma)$ is the L-function associated to $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and $\sigma$. This is defined by the Euler product:

$$L(s, \Pi, \sigma) = \prod_v L(s, \Pi_v, \sigma).$$

If $k_v$ is non-archimedean and $\Pi_v$ is of class 1, then

$$L(s, \Pi_v, \sigma) = \det(1 - A_1 \otimes A_2 \otimes A_3 \cdot q_v^{-s})^{-1},$$

where $q_v$ is the order of the residue field of $k_v$, and $A_i$ is the Langlands class of $\pi_{i,v}$ ($i = 1, 2, 3$). Then our main theorem in the case $K = k \oplus k \oplus k$ can be stated as follows.

THEOREM 2.7. Suppose that $K = k \oplus k \oplus k$, and $L(s, \Pi, \pi)$ has a pole somewhere. Then the following two assertions hold:

(a) Let $\Pi', \omega'$ be the objects obtained by twisting $\pi_1$ by $\chi^{s_0}$, $s_0 \in \mathbb{C}$. Then $\omega'^2 = 1$, $\omega' \neq 1$, and $L(s, \Pi', \sigma)$ has a simple pole at $s = 1$, for some $s_0 \in \mathbb{C}$.

(b) Assume that $\omega^2 = 1$, $\omega \neq 1$, and $L(s, \Pi, \sigma)$ has a pole at $s = 1$. Let $K$ be the

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quadratic extension of $k$ corresponding to $\omega$ by class field theory. Let $\theta$ be the generator of $\text{Gal}(K/k)$. Then there exist quasi-characters $\chi_1$, $\chi_2$, and $\chi_3$ of $A_K^x/K^x$ such that $\pi_1 = \pi(\chi_1)$, $\pi_2 = \pi(\chi_2)$, $\pi_3 = \pi(\chi_3)$, and $\chi_1\chi_2\chi_3 = 1$. Moreover, the triple $L$-function is equal to

$$\zeta_K(s)L_K(s, \chi_1^{-1}\chi_2^\theta)L_K(s, \chi_2^{-1}\chi_3^\theta)L_K(s, \chi_3^{-1}\chi_1^\theta).$$

Note that our results are consistent with “the Langlands philosophy”. Assume that for each $\pi_i$, there is a 2-dimensional complex representation $\rho_i$ of $\text{Gal}(k/k)$ such that $L(s, \pi_i) = L(s, \rho_i)$. Then our main theorem implies that, up to twist by $\omega^{s_0}$ for some $s_0 \in \mathbb{C}$, $L(s, \Pi, \sigma)$ has a pole if and only if $\rho_1 \otimes \rho_2 \otimes \rho_3$ has a trivial constituent.

A significant point of this result is its possible application to the construction of the lift $\text{GL}_2 \times \text{GL}_2 \to \text{GL}_4$ of automorphic representations by means of “the converse theorem”. The author hopes to treat this problem in the future.

Let us now describe the contents of this paper. Section 1 is devoted to the theory of Eisenstein series on symplectic group $\text{Sp}_n$. Assume, for simplicity, $k$ is a number field. Consider the representation space $I(\omega, s)$ of the representation $\text{Ind}_{P_n}^{\text{Sp}_n} \omega^s$ induced from a quasi-character $\omega$ of the parabolic subgroup

$$P_n = \left\{ \begin{pmatrix} A & * \\ 0_n & tA^{-1} \end{pmatrix} \in \text{Sp}_n \right\}$$

of $\text{Sp}_n$. Let $f^{(s)}$ be a meromorphic section of $I(\omega, s)$, which roughly means that $f^{(s)}$ belongs to $I(\omega, s)$ for each $s \in \mathbb{C}$ and is meromorphic in $s$. In order to make use of the Rankin-Selberg convolution, we require that the family $\{f^{(s)}\}$ has the following properties:

(i) $E(h; f^{(s)})$ has finite number of poles.
(ii) The family $\{f^{(s)}\}$ is stable under the intertwining operator $M_{w_0}$ with respect to the longest Weyl group element $w_0$.
(iii) The family $\{f^{(s)}\}$ is the restricted tensor product of families of meromorphic sections $\{f_v^{(s)}\}$ of induced representations $I(\omega_v, s)$ on $\text{Sp}_n(k_v)$.
(iv) The family $\{f_v^{(s)}\}$ contains all holomorphic sections.

Moreover, to get a good local functional equation, we need a normalization $M^{*}_{w_0}$ of the local intertwining operator such that

(v) $M^{*}_{w_0} \circ M^{*}_{w_0} = \text{const.}$
(vi) The family $\{f_v^{(s)}\}$ is stable under the normalized intertwining operator $M^{*}_{w_0}$. 
We shall construct this normalized intertwining operator, and the family \( \{ f^{(s)} \} \) in Section 1.2. A function \( f^{(s)} \) in this family is called a good section. Our normalized intertwining operator is different from Langlands’s normalization [16, Appendix 2]. In Section 1.3 we shall determine the location of the poles of the Eisenstein series \( E(h; f^{(s)}) \) associated to a good section \( f^{(s)} \). In Section 1.4 we calculate the residue of the Eisenstein series \( E(h; f^{(s)}) \) at \( s = \frac{n-1}{2} \).

Section 2 is devoted to the theory of the triple L-functions. We shall define the local L-factor and \( \varepsilon \)-factor, and give the functional equation for the triple L-functions. The location of the poles is then determined. The key lemma is that if \( \omega = 1 \), then \( L(s, \Pi, \sigma) \) does not have a pole at \( s = 1 \) (Proposition 2.5). The main theorem will be proved by showing that the base change of \( \Pi \) to \( GL_2(\mathbb{A}_k)^3 \) is not cuspidal.

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Notation

The \( n \times n \) zero and identity matrices are denoted by \( 0_n \) and \( \mathbf{1}_n \), respectively. If \( X \) is a matrix, \( \det X \) stands for its determinant. For a function \( f \) on a group \( G \) and \( x \in G \), we denote by \( \rho(x)f \) the right translation of \( f \) by \( x \), i.e., \( \rho(x)f(y) = f(yx) \). When \( G \) is locally compact, the Schwartz-Bruhat space of \( G \) is denoted by \( \mathcal{S}(G) \). If \( G \) is an algebraic group defined over a field \( k \), the group of \( k \)-valued points of \( G \) is denoted by \( G(k) \) or \( G \). If \( \pi \) is a representation of \( G \), its contragredient is denoted by \( \pi^\vee \). When \( k \) is a global field, the adele ring (resp. the idele group) of \( k \) is denoted by \( \mathbb{A}_k \) or \( \mathbb{A} \) (resp. \( \mathbb{A}_k^\times \) or \( \mathbb{A}_k^\times \)). We fix a non-trivial additive character \( \psi \) of \( A/k \) (resp. \( k \)), if \( k \) is a global field (resp. local field). The standard idele norm: \( A^\times \to \mathbb{R}_+^\times \) is denoted by \( \| \) or \( \alpha \). When \( k \) is a local field, the normalized absolute value: \( k^\times \to \mathbb{R}_+^\times \) is denoted by \( \| \) or \( \alpha \). When \( k \) is a global (resp. local) field, a quasi-character \( \chi \) of \( \mathbb{A}_k^\times \) (resp. \( k^\times \)) is called principal if \( \chi = \alpha^{s_0} \) for some \( s_0 \in \mathbb{C} \). When \( k \) is a global function field, the order of the coefficient field of \( k \) is denoted by \( q \). When \( k \) is a non-archimedean local field, \( \mathcal{O}, \mathfrak{m} \), and \( q \) are the maximal order of \( k \), a prime element of \( \mathcal{O} \), and the order of the residue field of \( k \), respectively. The multiplicative Haar measure \( d^\times x \) of \( k^\times \) is normalized so that \( \text{Vol}(\mathcal{O}^\times) = 1 \).
1. Analytic theory of Eisenstein series

1.1. Definitions

Let \( H_n \) be the symplectic group \( \text{Sp}_n \):

\[
H_n = \text{Sp}_n = \left\{ h \in \text{GL}_{2n} \mid h^t \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} h^t = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \right\}.
\]

We define parabolic subgroups \( P_n \) and \( B_n \) of \( H_n \) by

\[
P_n = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & t_A^{-1} \end{pmatrix} \in H_n \right\},
\]

\[
B_n = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & t_A^{-1} \end{pmatrix} \in P_n \mid A \text{ is upper triangular} \right\}.
\]

Let \( M_m \) (resp. \( T_n \)) be a Levi factor of \( P_n \) (resp. \( B_n \)) given by

\[
M_n = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & t_A^{-1} \end{pmatrix} \in \text{GL}_{2n} \right\},
\]

\[
T_n = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & t_A^{-1} \end{pmatrix} \mid A \text{ is diagonal} \right\}.
\]

We denote by \( U_n \) (resp. \( N_n \)) the unipotent radical of \( P_n \) (resp. \( B_n \)):

\[
U_n = \left\{ \begin{pmatrix} I_n & B \\ 0_n & 1_n \end{pmatrix} \mid B = tB \right\},
\]

\[
N_n = \left\{ \begin{pmatrix} A & * \\ 0_n & t_A^{-1} \end{pmatrix} \in H_n \mid A \text{ is unipotent upper triangular} \right\}.
\]

Let \( P_n^- \) and \( B_n^- \) be the opposite parabolic subgroups of \( P_n \) and \( B_n \), respectively. We denote by \( U_n^- \) (resp. \( N_n^- \)) the unipotent radical of \( P_n^- \) (resp. \( B_n^- \)).
Let $x_i (1 \leq i \leq n)$ be the character of $T_n$ given by

$$
\begin{pmatrix}
t_1 \\
\vdots \\
t_n \\
t_1^{-1} \\
\vdots \\
t_n^{-1}
\end{pmatrix}
\mapsto t_i.
$$

Let $\text{Norm}(T_n)$ be the normalizer of $T_n$ in $H_n$. We denote the Weyl group $\text{Norm}(T_n)/T_n$ by $W_{H_n}$. We shall often use the same symbol for an element of $\text{Norm}(T_n)$ and its image in $W_{H_n}$. Let $\Phi_{H_n}$ (resp. $\Phi_{M_n}$) be the set of roots of $H_n$ (resp. $M_n$) with respect to $T_n$. We denote by $N_a$ the unipotent group associated to a root $\alpha \in \Phi_{H_n}$. Each $N_a$ is isomorphic to $k$ in the natural way (by the coordinate). We denote by $w_\alpha$ the reflection determined by $\alpha$. Let $\alpha_i$ be the simple root:

$$
\alpha_i = x_i - x_{i+1}, \quad (1 \leq i \leq n-1)
$$

$$
\alpha_n = 2x_n.
$$

Let $\Omega_n$ be the complete set of representatives for $W_{H_n}/W_{M_n}$ obtained by choosing the unique element of minimal length in each coset. For each subset $I = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, we define an element $w_I$ of $W_{H_n}$ by

$$
x_1 \mapsto x_{j_1}, \ldots, x_{n-k} \mapsto x_{j_k},
$$

$$
x_{n-k+1} \mapsto -x_{i_k}, \ldots, x_n \mapsto -x_{i_1},
$$

where $J = \{j_1, j_2, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} - I$, $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_{n-k}$. The element $w_I$ belongs to $\Omega_n$ and each element of $\Omega_n$ is obtained in this way (cf. [20]). We also denote by $\Omega_n$ a set of representatives of $\Omega_n$ in $\text{Norm}(T_n)$. The length $l(w_I)$ of $w_I$ is given by

$$
l(w_I) = \#\{x \in \Phi_{H_n} | x > 0, w_I x < 0\}
= \sum_{r=1}^{k} (n + 1 - i_r).$$
Put

\[ w_0 = w_{\{1, 2, \ldots, n\}} = \begin{pmatrix} \mathbf{0}_n & -1 \\ -1 & 1 \\ \vdots & \vdots \\ 1 & \mathbf{0}_n \end{pmatrix} \]

This is the longest element in $\Omega_n$. For $w \in \text{Norm}(T_n)$ and a character $\chi$ of $T_n$, we put

\[ \chi^w(t) = \chi(w^{-1}tw). \]

Obviously $\chi^w$ depends only upon the class of $w$ in $W_{H_n}$, so we shall use the same notation $\chi^w$ for $w \in W_{H_n}$. We often regard a character of $T_n$ as a character of $B_n$ by the isomorphism $B_n/N_n \simeq T_n$.

1.2. Local theory

In this subsection, $k$ is a local field. We define the standard maximal compact subgroup $K_n$ of $H_n$ as follows.

When $k$ is non-archimedean, we put $K_n = H_n(\mathcal{O})$. When $k = \mathbb{R}$, we put

\[ K_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in H_n \mid A'B' = B'A, \quad A'A + B'B = \mathbf{1}_n \right\}. \]

When $k = \mathbb{C}$, we put

\[ K_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in H_n \mid A'B' = B'A, \quad \overline{A'A} + \overline{B'B} = \mathbf{1}_n \right\}. \]

When $k$ is non-archimedean, we put $R = \mathbb{C}[q^s, q^{-s}]$. When $k$ is archimedean, we let $R$ be the ring of entire functions on $\mathbb{C}$. Let $\omega$ be a quasi-character of $k^\times$ and let $s$ denote a complex number. Let $I(\omega, s) = \text{Ind}_{P_n}^{H_n}(\omega \omega^*)$ be the space of functions $f$ on $H_n$ which satisfy the following two conditions:

(i) $f$ is right $K_n$-finite.

(ii) For any $p = \begin{pmatrix} A & \ast \\ 0_n & \ast A^{-1} \end{pmatrix} \in P_n$,

\[ f(ph) = \omega(\det A) |\det A|^{s+(n+1)/2} f(h). \]
We say that a function $f^{(i)}(h)$ on $H_n \times \mathbb{C}$ is a holomorphic section of $I(\omega, s)$ if the following three conditions are satisfied:

1. For each $s \in \mathbb{C}$, $f^{(i)}(h)$ belongs to $I(\omega, s)$ as a function of $h \in H_n$.
2. For each $h \in H_n$, $f^{(i)}(h)$ belongs to $R$ as a function of $s \in \mathbb{C}$.
3. $f^{(i)}(h)$ is right $K_n$-finite.

We say that a meromorphic function $f^{(i)}(h)$ on $H_n \times \mathbb{C}$ is a meromorphic section of $I(\omega, s)$, if there is $a(s) \in R$ such that $a(s) \neq 0$, and $a(s)f^{(i)}(h)$ is a holomorphic section of $I(\omega, s)$. Note that a holomorphic section of $I(\omega, s)$ is determined by its restriction to $K_n \times \mathbb{C}$. We say that a holomorphic section $f^{(i)}(h)$ is a standard section if its restriction to $K_n \times \mathbb{C}$ does not depend on $s \in \mathbb{C}$. Obviously the space of holomorphic sections is generated by standard sections over $R$.

For a quasi-character $\chi$ of $T_n$, we define $\text{Ind}_{B_n}^{H_n}(\chi)$ to be the space of right $K_n$-finite functions $f(h)$ on $H_n$ such that

$$f(bh) = \chi(b)\delta_{B_n}^{1/2}(b)f(h),$$

where $\delta_{B_n}$ is the modulus quasi-character of $B_n$. Put

$$\chi_s(t) = \prod_{i=1}^{n} \omega(t_i)|t_i|^{s-(n+1)/2+i},$$

Then $I(\omega, s) \subseteq \text{Ind}_{B_n}^{H_n}(\chi_s)$. We define holomorphic sections, meromorphic sections, and standard sections of $\text{Ind}_{B_n}^{H_n}(\chi_s)$ similarly.

For $w \in \text{Norm}(T_n)$ and a quasi-character $\chi$ of $T_n$, we define the intertwining operator

$$M_w = M(w, \chi): \text{Ind}_{B_n}^{H_n}(\chi) \to \text{Ind}_{B_n}^{H_n}(\chi^w)$$

by

$$M_w f(h) = \int_{N_e w N_e w^{-1}} f(w^{-1}nh)dn.$$

Here the Haar measure $dn$ is determined as follows. For each $\alpha \in \Phi_{H_n}$, the Haar measure $dn_\alpha$ on $N_\alpha$ is given by the self dual measure on $k$ with respect to $\psi$ by the natural isomorphism $N_\alpha \simeq k$. Then the measure $dn$ is the product measure: $dn = \Pi dn_\alpha$. The integral is absolutely convergent if $\chi$ belongs to some open set and can be meromorphically continued to all $\chi$ (cf. [8], [25]).

If $l(w_1) + l(w_2) = l(w_1 w_2)$, then $M_{w_1} \circ M_{w_2} = M_{w_1 w_2}$. When $w = w_\alpha$ is a reflection with respect to a simple root $\alpha$, then $M(w, \chi)$ can be regarded as an intertwining
operator on $SL_2$ as follows: let $\tau: SL_2 \to H_n$ be a homomorphism corresponding to $\alpha$. We may assume $w = \tau\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$. Then for any $f \in \text{Ind}_{H_n}^{H_n}(\chi)$,

$$t^*_\alpha(M(w, \chi)f) = M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau^*\chi\right)(t^*_\alpha f),$$  \hspace{1cm} (1.2.1)

as a function on $SL_2$. Since $M(w, \chi)$ commutes with right translations (or actions of Hecke operators), it follows from (1.2.1) that the whole property of $M(w, \chi)$ is reduced to that of $M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau^*\chi\right)$. When $\omega$ is unramified, there exists a unique standard section $\phi_{\omega,s}$ of $I(\omega, s)$ such that $\phi_{\omega,s}|_{K_n} \equiv 1$. Similarly, there exists a unique standard section $\phi^{w}_{\omega,s}$ of $\text{Ind}_{H_n}^{H_n}(\chi_w)$ such that $\phi^{w}_{\omega,s}|_{K_n} \equiv 1$, for any $w \in \Omega_n$. Note that $\phi^{w}_{\omega,s} = \phi_{\omega,-1,-s}$.

Let us recall some known results concerning $SL_2 \simeq H_1$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$M_w = M(w, \omega) = M(w, \omega, s): I(\omega, s) \to I(\omega^{-1}, -s)$. Then:

1.2.2 $L(s, \omega)^{-1}M_w$ is holomorphic.

1.2.3 $M(w^{-1}, \omega^{-1}) \circ M(w, \omega) = \epsilon(s, \omega, \psi)^{-1}\epsilon(-s, \omega^{-1}, \psi)^{-1} \cdot \text{id}$.

1.2.4 If $\omega$ is unramified, and $\psi$ is of order 0,

$$M_w\phi_{\omega,s} = \frac{L(s, \omega)}{L(s+1, \omega)} \phi_{\omega^{-1},-s}.$$

1.2.5 If $k$ is non-archimedean and $\omega = 1$, the kernel and the image of $M(w, 1, 1)$: $I(1, 1) \to I(1, -1)$ are the Steinberg representation and the trivial representation, respectively.

1.2.6 If $k$ is non-archimedean and $\omega = 1$, the kernel and the image of $M(w, 1, -1)$: $I(1, -1) \to I(1, 1)$ are the trivial representation and the Steinberg representation, respectively.

1.2.7 If $\omega = 1$, then $\text{Res}_{s=0} M(w, 1, s)$ is a non-zero scalar multiplication.

If $w \in \Omega_n$, then the restriction of $M_w$ to $I(\omega, s) \subset \text{Ind}_{H_n}^{H_n}(\chi_s)$ is well defined (except for countably many values of $s$). If $f^{(\omega)}$ is a holomorphic section of $I(\omega, s)$, then $M_w f^{(\omega)}$ is a meromorphic section of $\text{Ind}_{H_n}^{H_n}(\chi_s)$. We denote this restriction by $M_w = M(w, \omega) = M_w(\omega, s)$, too. If $\omega$ is unramified, $w \in \text{Norm}(T_n) \cap K_n$, and $\psi$ is of order 0, then there exists a meromorphic function $c_w(s) = c_w(\omega, s)$ such that

$$M_w(\phi_{\omega,s}) = c_w(s)\phi^{w}_{\omega,s}.$$

$$c_w(s) = \prod_{\chi \in \Phi_{H_n} \atop w\chi < 0 \atop \alpha > 0} \frac{L(\langle \tilde{\alpha}, \chi_s \rangle)}{L(\langle \tilde{\alpha}, \chi_s \rangle + 1)}.$$
where $\langle , \rangle$ is a $W_\mu_n$-invariant inner product on $X^*(T_\mu) \otimes \mathbb{Z} \mathbb{C}$, and $\alpha = 2\bar{\alpha}/\langle \bar{\alpha}, \alpha \rangle$ is the coroot of $\alpha$.

In [20], the common denominator of $c_w(s)$ is calculated. Here we proceed in a slightly different way. Let $w = w_I$, $I = \{i_1, i_2, \ldots, i_k\}$. Put

$$N(w_I) = \{\alpha \in \Phi_{\mu_n} | \alpha > 0, w_I \alpha < 0\}$$

$$= \{2x_{n-m+1} | 1 \leq m \leq k\}$$

$$\cup \{x_m + x_{n-r+1} | 1 \leq r \leq k, i_r - r + 1 \leq m \leq n-r\}$$

We divide $N(w_I)$ into a disjoint union $\bigsqcup_{r=0}^{[n/2]} N_r(w_I)$:

$$N_r(w_I) = \begin{cases} \{2x_{n-m+1} | 1 \leq m \leq k\}, & \text{if } r = 0 \\ \emptyset, & \text{if } r > k \\ \{x_m + x_{n-r+1} | i_r - r + 1 \leq m \leq n-r\}, & \text{if } 1 \leq r \leq k, i_r \geq 2r \\ \{x_m + x_{n-r+1} | r \leq m \leq n-r\} \cup \{x_m + x_r | \mu_w(r) \leq m \leq n-r\}, & \text{if } 1 \leq r \leq k, i_r \leq 2r - 1. \end{cases}$$

Here

$$\mu_w(r) = \begin{cases} \min\{m | n-k+1 \leq m \leq n, i_r < i_{n-m+1}\}, & \text{if } 1 \leq r \leq n-k \\ r+1, & \text{if } n-k+1 \leq r \leq \left[\frac{n}{2}\right]. \end{cases}$$

Put

$$d^r(s) = \begin{cases} L\left(s + \frac{n+1}{2}, \omega\right), & \text{if } r = 0 \\ L(2s+n+1-2r, \omega^2), & \text{if } 1 \leq r \leq \left[\frac{n}{2}\right], \end{cases}$$

$$d^r_*(s) = \begin{cases} L\left(s + \frac{n+1}{2} - k, \omega\right), & \text{if } r = 0 \\ L(2s+n+1-2r, \omega^2), & \text{if } k < r \leq \left[\frac{n}{2}\right] \\ L(2s+i_r-2r+1, \omega^2), & \text{if } 1 \leq r \leq k, i_r \geq 2r \\ L(2s+n+r+\mu_w(r)-1, \omega^2), & \text{if } 1 \leq r \leq k, i_r \leq 2r - 1, \end{cases}$$

$$d(s) = \prod_{r=0}^{[n/2]} d^r(s), \quad a_w(s) = \prod_{r=0}^{[n/2]} a^r_w(s).$$
Then we have

\[ c_w(s) = \prod_{r=0}^{[n/2]} \frac{L(\langle \zeta, \chi_s \rangle)}{L(\langle \zeta, \chi_s \rangle + 1)} \]

Now, even when \( \omega \) is not unramified, we define \( c_w(s), d(s) \) etc. by formally substituting \( \omega \).

**DEFINITION.** The normalized intertwining operator

\[ M_{w_0}^{*} = M^{*}(w_0, \omega) = M^{*}(w_0, \omega; \psi) : I(\omega, s) \rightarrow I(\omega^{-1}, -s) \]

is given by

\[ M_{w_0}^{*} = \varepsilon' \left( s - \frac{n - 1}{2}, \omega, \psi \right) \cdot \prod_{r=1}^{[n/2]} \varepsilon(2s - n + 2r, \omega^2, \psi) \cdot M_{w_0}^{*}. \]

**LEMMA 1.1.**

\[ M^{*}(w_0^{-1}, \omega^{-1}; \psi) \circ M^{*}(w_0, \omega; \psi) = \omega(-1)^{n+1} \cdot \text{id}, \]

\[ M^{*}(w_0, \omega^{-1}; \overline{\psi}) \circ M^{*}(w_0, \omega; \psi) = \text{id}. \]

**Proof.** The second formula is just a reformulation of the first formula. We will prove the first formula. When \( n = 1 \), this is (1.2.3). Since

\[ \varepsilon(-s, \omega^{-1}, \psi) \varepsilon'(s+1, \omega, \psi) = \omega(-1), \]

the right-hand side of (1.2.3) is equal to

\[ \omega(-1) \frac{\varepsilon'(s+1, \omega, \psi)}{\varepsilon(s, \omega, \psi)} \cdot \text{id}. \]
For general $n$, take a minimal expression of $w_0$ in $W_H$ by simple reflections

$$w_0 = w_1 w_2 \cdots w_k.$$ 

By using (1.2.1) and (1.2.3) successively,

$$M_{w_0^{-1} \circ M_{w_0}} = M_{w_1^{-1} \circ \cdots \circ M_{w_2^{-1} \circ M_{w_1^{-1} \circ M_{w_2} \circ \cdots \circ M_{w_0}}}}$$

$$= \omega(-1)^n \prod_{\alpha \in \Phi_{w_1} \setminus \Phi_{w_0}} \epsilon'\left(\frac{\langle \alpha, c_s \rangle + 1, \psi}{\langle \alpha, c_s \rangle, \psi}\right) : \text{id}$$

$$= \omega(-1)^n \frac{\epsilon'(s + (n + 1)/2, \omega, \psi)}{\epsilon'(s - (n - 1)/2, \omega, \psi)}$$

$$\times \prod_{r=1}^{\lfloor n/2 \rfloor} \frac{\epsilon'(2s + n + 1 - 2r, \omega^2, \psi)}{\epsilon'(2s - n + 2r, \omega^2, \psi)} : \text{id}$$

$$= \omega(-1)^{n+1} \epsilon'\left(s - \frac{n - 1}{2}, \omega, \psi\right)^{-1} \epsilon'\left(-s - \frac{n - 1}{2}, \omega^{-1}, \psi\right)^{-1}$$

$$\times \prod_{r=1}^{\lfloor n/2 \rfloor} \epsilon'(2s - n + 2r, \omega^2, \psi)^{-1} \epsilon'(-2s - n + 2r, \omega^{-2}, \psi)^{-1} : \text{id}.$$ 

Hence the lemma.

DEFINITION. A meromorphic section $f^{(s)}(h)$ of $I(\omega, s)$ is a good section of $I(\phi_{\omega,s})$ if for any $w \in \Omega_n$,

$$[d(s)c_w(s)]^{-1} M_w f^{(s)}$$

is holomorphic.

In particular, if $\omega$ is unramified, $d(s)\phi_{\omega,s}$ is a good section of $I(\omega, s)$.

LEMMA 1.2. $f^{(s)}$ is a good section of $I(\omega, s)$ if and only if $M_{w_0} f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$.

Proof. It will suffice to prove that for each $w_j \in \Omega_n$, there exists an entire function $a(s)$ with no zeros such that

$$[d(\omega, s)c_{w_j}(\omega, s)]^{-1} M_{w_j} f^{(s)}(h)$$

$$= \epsilon(s)[d(\omega^{-1}, -s)c_{w_j}(\omega^{-1}, -s)]^{-1} M_{w_j} \circ M_{w_0} f^{(s)}(h). \quad (1.2.8)$$

We shall proceed by induction on $l(w_j)$. Obviously, (1.2.8) holds when $l(w_j) = 0$. 
Suppose $l(w_j) > 0$. There are two cases:

1. $j_{n-k} = n$.
2. $j_{n-k} = m < n$.

In case (1), put $I' = I \cup \{n\}$, $J' = J - \{n\}$. Then

$$l(w_I') = l(w_I) + 1, \quad l(w_J') = l(w_J) - 1,$$

$$w_I' = w_{an} \cdot w_I, \quad M_{w_I'} = M_{w_{an}} \circ M_{w_I},$$

$$w_J' = w_{an} \cdot w_J, \quad M_{w_J'} = M_{w_{an}} \circ M_{w_J},$$

$$c_{w_I'}(\omega^{-1}, -s) = c_{w_I}(\omega^{-1}, -s) \frac{L\left(-s + \frac{-n+1}{2} + k, \omega^{-1}\right)}{L\left(-s + \frac{-n+1}{2} + k + 1, \omega^{-1}\right)},$$

$$c_{w_J'}(\omega, s) = c_{w_J}(\omega, s) \frac{L\left(s + \frac{n+1}{2} - k, \omega\right)}{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)}.$$

On the other hand, by (1.2.1) and (1.2.3),

$$M_{w_{an}} \circ M_{w_I} = M_{w_{an}} \circ M_{w_{an}} \circ M_{w_I},$$

$$= C \cdot e'\left(s + \frac{n-1}{2} - k, \omega, \psi\right)^{-1} e'\left(-s - \frac{n-1}{2} + k, \omega^{-1}, \psi\right)^{-1} \cdot M_{w_I},$$

where $C$ is some non-zero constant. We have

$$[d(\omega, s)c_{w_I}(\omega, s)]^{-1} M_{w_I} f^{(s)}$$

$$= [d(\omega, s)c_{w_I}(\omega, s)]^{-1} \frac{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)}{L\left(s + \frac{n+1}{2} - k, \omega\right)}$$

$$\times C^{-1} \cdot e'\left(s + \frac{n-1}{2} - k, \omega, \psi\right) e'\left(-s - \frac{n-1}{2} + k, \omega^{-1}, \psi\right) \cdot M_{w_{an}} \circ M_{w_I} f^{(s)}.$$
By the induction assumption, this is equal to

$$
e_1(s) \frac{L\left(s + \frac{n+1}{2} - k, \omega\right)}{L\left(s + \frac{n+1}{2}, \omega\right)} \frac{L\left(1 - s - \frac{n-1}{2} + k, \omega^{-1}\right)}{L\left(s + \frac{n-1}{2} - k, \omega\right)}$$

$$\times \frac{L\left(s + \frac{n+1}{2} - k, \omega\right)}{L\left(-s - \frac{n-1}{2} + k, \omega^{-1}\right)}$$

$$\times [d(\omega^{-1}, -s)c_{w_j}(\omega^{-1}, -s)]^{-1}M_{w_{w_j}} \circ M_{w_j} \circ M_{w_0}^* f(s)$$

$$= e_1(s)[d(\omega^{-1}, -s)c_{w_j}(\omega^{-1}, -s)]^{-1}M_{w_j} \circ M_{w_0}^* f(s).$$

Here $e_1(s)$ is some entire function with no zeros.

In case (2), put $I' = I - \{m\} \cup \{m+1\}, J' = J - \{m+1\} \cup \{m\}$. Then

$$l(w_I) = l(w_I) + 1, \quad l(w_{I'}) = l(w_{I'}) - 1,$$

$$w_J = w_{z_m} \cdot w_J, \quad M_{w_{w_J}} = M_{w_{z_m}} \circ M_{w_J},$$

$$w_{I'} = w_{z_m} \cdot w_{I'}, \quad M_{w_{w_{I'}}} = M_{w_{z_m}} \circ M_{w_{I'}}.$$

By a calculation similar to case (1), (1.2.8) for $I$ is reduced to (1.2.8) for $I'$. Thus the lemma follows.

The following lemma is crucial for our theory.

**LEMMA 1.3.** Every holomorphic section of $I(\omega, s)$ is a good section.

**REMARK.** If $k \neq \mathbb{C}$, and $\omega$ is unramified, this lemma is nothing but [22, Theorem 4.2].

**Proof of Lemma 1.3.** Here we assume $k$ is non-archimedean. We may assume $\omega$ is ramified. If $\omega^2$ is ramified, then $d(s) = c_w(s) = 1$, for any $w \in \Omega_w$. Take a minimal expression of $w$ by simple reflections:

$$w = w_1 w_2 \cdots w_r, \quad M_w = M_{w_1} \circ M_{w_2} \circ \cdots \circ M_{w_r}.$$  

Each $M_{w_i}$ ($1 \leq i \leq r$) is holomorphic by (1.2.1) and (1.2.2). So the lemma is obvious in this case.
Now we assume $m$ is ramified and $W_2 = 1$. Let $w = w_I$, $I = \{i_1, i_2, \ldots, i_k\}$. Recall

$$a_w(s) = d(s)c_w(s) = \prod_{r=0}^{[n/2]} d_r^*(s).$$

It suffices to prove

$$\left[ \prod_{r=0}^{\min(k, [n/2])} d_r^*(s) \right]^{-1} M_w f(s)$$

is holomorphic. Put

$$A_w(s) = \prod_{r=0}^{\min(k, [n/2])} d_r^*(s).$$

We proceed by induction on $l(w)$. If $l(w) = 0$, (1.2.9) is obviously holomorphic.

(I) When $i_k = n$: put $I' = I - \{n\}$, $w' = w_{I'}$. Then

$$M_w = M_{w,n} \circ M_{w'}, \quad A_w(s) = A_w(s).$$

Since $M_{w,n}$ is entire, the holomorphy of (1.2.9) for $w$ is reduced to that for $w'$. By definition, we have

$$A_{w'}(s)A_w(s)^{-1} = \xi(2s + m - 2r + 2)\xi(2s + m - 2r + 1)^{-1},$$

$$M(w, \chi_s) = M(w_{s,m}, \chi_s^{w'}) \circ M(w', \chi_s).$$

Since $\xi(2s + m - 2r + 1)^{-1} M(w_{s,m}, \chi_s^{w'})$ is entire, it will suffice to prove that $2s \equiv -m + 2r - 2\left(\mod \frac{2\pi}{\log q} \mathbb{Z}\right)$ are not poles of (1.2.9). We now prove that the residue vanishes. By (1.2.7),

$$\xi(2s + m - 2r + 1)^{-1} M(w_{s,m}, \chi_s^{w'})$$

is holomorphic at these points. The residue is

$$\text{Res}_{2s \equiv -m + 2r - 2}(A_w(s)^{-1} M_w f^{(s)})$$

$$= c \cdot M(w_{s,m}, \chi_s^{w'}) \circ \text{Res}_{2s \equiv -m + 2r - 2}[\xi(2s + m - 2r + 2)A_w(s)^{-1} M_w f^{(s)}]$$

$$= c' \cdot M(w_{s,m}, \chi_s^{w'}) \circ [A_w(s)^{-1} M_w f^{(s)}]_{2s \equiv -m + 2r - 2}.$$
for some non-zero constants \(c, c'\). By (1.2.6), it is sufficient to prove that

\[
[A_w(s)^{-1}M_w f^{(s)}]_{2s = -m + 2r - 2}
\]

is left \(\iota_{sm}(\text{SL}_2)\)-invariant. We first observe

\[
A_w(s)^{-1}M_w f^{(s)} = \zeta(2s + m - 2r + 3)\zeta(2s + m - 2r + 2)^{-1}A_w(s)^{-1}M(w_{m+1}, \chi_s^\omega)M(w^\prime, \chi_s) f^{(s)}.
\]

Since \(\zeta(2s + m - 2r + 3)\) and \(\zeta(2s + m - 2r + 2)^{-1}M(w_{m+1}, \chi_s^\omega)\) is holomorphic at

\[
2s \equiv -m + 2r - 2 \left( \mod \frac{2\pi\sqrt{-1}}{\log q} \right),
\]

this is equal to

\[
c'' \cdot [\zeta(2s + m - 2r + 2)^{-1}M(w_{m+1}, \chi_s^\omega)]_{2s = -m + 2r - 2} \circ A_w(s)^{-1}M(w^\prime, \chi_s) f^{(s)},
\]

for some non-zero constant \(c''\). By the induction assumption,

\[
A_w(s)^{-1}M(w^\prime, \chi_s) f^{(s)}
\]

is holomorphic. Moreover this is left \(\iota_{sm}(\text{SL}_2)\)-invariant since

\[
w^\prime - 1 \iota_{sm}(\text{SL}_2)w^\prime \subseteq M_n.
\]

By (1.2.7),

\[
[\zeta(2s + m - 2r + 2)^{-1}M(w_{m+1}, \chi_s^\omega)]_{2s = -m + 2r - 2}
\]

is a scalar multiplication. Thus (1.2.10) is left \(\iota_{sm}(\text{SL}_2)\)-invariant.

(III) When \(i_k = n - 1, i_k - 1 = n - 2\): this case can be treated by the same technique as in the case (II) by putting

\[
I' = I - \{n - 1\} \cup \{n\}, \quad I'' = I - \{n - 2\} \cup \{n\}.
\]

(IV) When \(i_k < n - 1\). This case can be treated by a similar technique as in the case (II) by putting

\[
I' = I - \{i_k\} \cup \{i_k + 1\}, \quad I'' = I - \{i_k\} \cup \{i_k + 2\}.
\]

Now we may assume \(i_k = n - 1\), by (I) and (IV). Moreover, we may assume \(k \leq \left\lfloor \frac{n}{2} \right\rfloor\), since otherwise the assumption of (II) or (III) holds. To see this, assume
$k > \left[ \frac{9}{2} \right]$ and neither the assumption of (II) nor that of (III) holds. Then

$$i_k = n - 1, i_{k-1} \leq n - 3, \ldots, i_k \leq n - 2k + 2m - 1, \ldots, i_1 \leq n - 2k + 1 \leq 0.$$ 

This is a contradiction.

(V) When $k \leq \left[ \frac{9}{2} \right]$: put $I' = I \setminus \{ n - 1 \}$, $w' = w_{I'}$. Then

$$M_w = M(w_{a_n - 1}, \chi_{s}^{w_n}) \circ M(w_n, \chi_{s}^w) \circ M(w', \chi),$$

$$A_w(s) = A_w(s) \cdot \zeta(2s + n - 2k).$$

By the induction assumption, $A_w(s)^{-1} M_w f(s)$ is entire. Since both $M(w_n, \chi_n^{w_n})$ and $\zeta(2s + n - 2k)^{-1} M(w_{a_n - 1}, \chi_{s}^{w_n})$ are entire, $A_w(s)^{-1} M_w f(s)$ is entire. Thus the proof for non-archimedean local field is complete.

Appendix 1. Proof for Lemma 1.3 for archimedean case

In this appendix, we give a proof for Lemma 1.3 for an archimedean local field $k$. We may assume that $\omega$ is unitary.

Sublemma 1. If $w = w_0$, then (1.2.9) is holomorphic.

Proof. If $k = \mathbb{R}$, and $\omega = 1$, this is proved in [22 §4 Appendix 1]. Their proof is valid for $k = \mathbb{R}$, $\omega = \text{sgn}$. If $k = \mathbb{C}$, we have to show that the first part of [22 §4 Appendix 1, Theorem (p. 106)] holds for our situation, i.e., we have to show that

$$a_{w_0}(\omega, s)^{-1} \int_{\text{Sym}^n(\mathbb{C})} \phi(z) |\det z \zeta|^{s-(n+1)/2} \omega(\det z) \, dz$$

is entire for any $\phi \in \mathcal{S}'(\text{Sym}^n(\mathbb{C}))$. We may assume that $\omega(z) = z^k$ or $(\bar{z})^k$, $k \geq 0$. But the case $\omega(z) = (\bar{z})^k$ is reduced to the case $\omega(z) = z^k$ by taking complex conjugate. Put

$$\partial = \det
\begin{vmatrix}
\frac{\partial}{\partial z_{11}} & 1 & \cdots & 1 \\
\frac{1}{2} & \frac{\partial}{\partial z_{12}} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{\partial}{\partial z_{12}} & \cdots & \frac{\partial}{\partial z_{1n}} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{nn}}
\end{vmatrix}.$$
Then it is known that

\[ \partial((\det z\overline{z})^{k}\det z)^{k}) = \prod_{i=0}^{n-1} \left( s + k + \frac{i}{2} \right) \cdot (\det z\overline{z})^{\nu}(\det z)^{k-1}. \]

Repeating partial integration, we have

\[ \prod_{j=1}^{m} \prod_{i=0}^{n-1} \left( s + k + j + \frac{i-n-1}{2} \right) \int_{\text{Sym}^r(\mathbb{C})} \phi(z)(\det z\overline{z})^{-(n+1)/2} (\det z)^{k} \, dz \]

\[ = (-1)^m \int_{\text{Sym}^r(\mathbb{C})} \partial^m \phi(z)(\det z\overline{z})^{-(n+1)/2} (\det z)^{k+m} \, dz \]

for \( \text{Re}(s) > 0 \). Since the right-hand side is absolutely convergent for \( \text{Re}(s) > \frac{n-k-m-1}{2} \), we have

\[ \prod_{i=0}^{n-1} \Gamma \left( s + k - \frac{i}{2} \right) \int_{\text{Sym}^r(\mathbb{C})} \phi(z)(\det z\overline{z})^{-(n+1)/2} (\det z)^{k} \, dz \]

is entire. So (1.2.11) is entire.

Let \( Q \) (resp. \( Q' \)) be the maximal parabolic subgroup of \( \text{GL}_n \) given by

\[
Q = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \middle| a_1 \in \text{GL}_{n-1}, \ a_2 \in k^\times \right\} \\
\text{resp. } Q' = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \middle| a_1 \in k^\times, \ a_2 \in \text{GL}_{n-1} \right\}.
\]

Let \( I_Q(\omega, s) \) (resp. \( I_{Q'}(\omega, s) \)) be the representation of \( \text{GL}_n \) induced from the character of \( Q \) (resp. \( Q' \)) given by

\[
\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto \omega(\det a_1)|\det a_1|^{\nu/2} |a_2|^{-(n-1)/2} s
\]

\[
\text{resp. } \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto \omega^{-1}(\det a_2)|a_1|^{(n-1)/2} n|\det a_2|^{-s/2} \).
\]

We define standard sections, holomorphic sections, and meromorphic sections as usual. We define the intertwining operator \( M_w : I_Q(\omega, s) \mapsto I_{Q'}(\omega^{-1}, -s) \).
(resp. $M_w': I_Q(\omega, s) \rightarrow I_Q(\omega^{-1}, -s)$). Here

$$w = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}, \quad w' = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}. \]

**SUBLEMMA 2.** $L\left(s - \frac{n-2}{2}, \omega\right)^{-1} M(w, s)$ and $L\left(s - \frac{n-2}{2}, \omega\right)^{-1} M(\omega', s)$ are holomorphic.

**Proof.** This can be proved in the same way as [22, §4]. (See also [12 §5].)

**SUBLEMMA 3.**

$$M(w', \omega^{-1}) \circ M(w, \omega)$$

$$= \omega(-1)^n + 1 e' \left(s - \frac{n-2}{2}, \omega, \psi\right)^{-1} e' \left(-s - \frac{n-2}{2}, \omega^{-1}, \psi\right)^{-1} \cdot \text{id.}$$

**Proof.** This can be proved in the same way as the proof of Lemma 1.1.

We now return to the proof of Lemma 1.3. Let $w = w_I$ be an element of $\Omega_n$. We prove that

$$[d(\omega, s)c_w(\omega, s)]^{-1} M_w f(s)$$

is holomorphic. $M_w$ can be considered as an intertwining operator of $I\left(\omega, s + \frac{i_1 - 1}{2}\right)$ on $\text{Sp}_{n-i_1+1}$. We may assume $i_1 = 1$ by replacing $n$ by $n-i_1+1$ and $I$ by $\{i_r-i_1+1 | 1 \leq r \leq k\}$. We proceed by the induction on $\delta(w)=n-k$. When $n=k$, this is Sublemma 1. Assume $n-k \geq 1$. Put

$$m = \max\{r | i_r < n-k+r\},$$

$$I' = I \cup \{n-k+m\},$$

$$w' = w_{I'}.$$ Then $\#I' = k+1$, $l(w') = l(w) + k - m + 1$ and

$$w' = w_{\alpha_n}w_{\alpha_{n-1}} \cdots w_{\alpha_{n-k+m}} w.$$
Put
\[ w_{(0)} = w, \]
\[ w_{(r)} = w' = w_{a_{n-k+m+r-1}} \cdots w_{a_{n-k+m+1}} w_{a_{n-k+m}}, \quad 1 \leq r \leq k - m + 1. \]

Then
\[
M_{w_{(r)}} = M(w_{a_{n-k+m+r-1}}, \chi_s^{w_{(r-1)}}) \circ M_{w_{(r)}}, \quad 1 \leq r \leq k - m + 1
\]
\[ c_{w_{(r)}}(s) = c_{w_{(r-1)}}(s) \times \begin{cases} 
\frac{L(2s + n - k - m - r, \omega^2)}{L(2s + n - k - m - r + 1, \omega^2)}, & 1 \leq r \leq k - m \\
L\left(s + \frac{n - 1}{2} - k, \omega\right), & r = k - m + 1
\end{cases}
\]

We have
\[
c_w(s) = \frac{L(2s + n - 2k, \omega^2)}{L(2s + n - k - m, \omega^2)} \frac{L\left(s + \frac{n + 1}{2} - k, \omega\right)}{L\left(s + \frac{n + 1}{2} - k, \omega\right)} c_w(s).
\]

It is easy to see that
\[
M(w_{a_{n-1}}, \chi_s^{w_{(a-k-m-1)}}) \circ \cdots \circ M(w_{a_{n-k+m}}, \chi_s^w)
\]
is an intertwining operator on $GL_{k-m}$. By (1.2.3) and Sublemma 3,
\[
M(w_{a_{n-k+m}}, \chi_s^{w_{(1)}}) \circ \cdots \circ M(w_{a_{n-1}}, \chi_s^{w_{(a-k-m)}}) \circ M(w_{a_{n}} \chi_s^w) \circ M w' \\
= \omega(-1)^{e'} \left(s + \frac{n - 1}{2} - k, \omega, \psi\right)^{-1} e'\left(-s - \frac{n - 1}{2} + k, \omega^{-1}, \psi\right)^{-1}
\times e'(2s + n - 2k, \omega^2, \psi)^{-1} e'(-2s - n + k + m + 1, \omega^{-2}, \psi)^{-1} M_w
\]
By (1.2.2), Sublemma 2, and the induction assumption,
\[
L\left(-s + \frac{n - 1}{2} + k, \omega^{-1}\right)^{-1} M(w_{a_{n}}, \chi_s^w),
\]
\[
L(-2s - n + k + m + 1, \omega^{-2})^{-1} M(w_{a_{n-k+m}}, \chi_s^{w_{(1)}}) \circ \cdots \circ M(w_{a_{n-1}}, \chi_s^{w_{(a-k-m)}})
\]
and

\[ [d(\omega, s)c_w(\omega, s)]^{-1}M_w \]

are holomorphic. Thus we have

\[
L\left(-s - \frac{n-3}{2} + k, \omega^{-1}\right)^{-1}L(-2s - n + 2k + 1, \omega^{-2})^{-1}[d(\omega, s)c_w(\omega, s)]^{-1}M_w
\]

is holomorphic.

On the other hand, put

\[
w_k = \begin{pmatrix}
1_{n-k} & -1_k \\
1_k & 1_{n-k}
\end{pmatrix},
\]

\[
w = w'w_k.
\]

Then \( M_w = M_w \circ M_{w_k} \). Here, as in [22 §4], \( M_w \) is an intertwining operator on certain induced representation of \( GL_n \). As in [22 §4], we can prove

\[
\prod_{r=1}^{k} L(2s + i_r - 2r + 1, \omega^2)^{-1}M_w
\]

is holomorphic (cf. [22, Remark 4.1]). As for \( M_{w_k} \), by Sublemma 1,

\[
L\left(s + \frac{n+1}{2} - k, \omega\right)^{-1}\prod_{r=1}^{\lceil k/2 \rceil} L(2s + n - 2k + 2r, \omega^2)^{-1}M_{w_k}
\]

is holomorphic. Putting together, we can easily deduce

\[
\prod_{r=\lceil k/2 \rceil+1}^{k} L(2s + n - 2r, \omega^2)^{-1}[d(\omega, s)c_w(\omega, s)]^{-1}M_w
\]

is holomorphic. Since

\[
L\left(-s - \frac{n-3}{2} + k, \omega^{-1}\right)L(-2s - n + 2k + 1, \omega^{-2})
\]
has no poles in $\text{Re}(s) < -\frac{n}{2} + k + \frac{1}{2}$, and

$$\prod_{r=[k+1/2]}^{k} L(2s+n-2r, \omega^2)$$

has no poles in $\text{Re}(s) > -\frac{n}{2} + k$, it follows that

$$[d(\omega, s)c_\omega(\omega, s)]^{-1} M_\omega$$

is holomorphic. Thus Lemma 1.3 is proved.

REMARK. Our definition of good section is different from that of [22]. But we can prove that “germs” of good section of $I(\omega, s)$ at $s=s_0$ are generated by the following two families:

1) germs of holomorphic sections of $I(\omega, s)$ at $s=s_0$,
2) $\{ M_* \omega^i f^{(s)} | f^{(s)}$ is a germ of holomorphic section of $I(\omega^{-1}, -s)$ at $s=s_0 \}$.

In fact, we may assume $\omega$ is unitary and $\text{Re}(s_0) \geq 0$, by Lemma 1.2. Since $d(\omega, s)$ does not have zero at $s=s_0$, any good section of $I(\omega, s)$ is holomorphic at $s=s_0$. It is easy to see that when $k$ is non-archimedean, our definition agrees to that of [22] because there are essentially finite number of singularities.

Appendix 2. An interpretation of the normalizing factor

We give an interpretation of the normalizing factor $d(\omega, s)$ in terms of Arthur’s conjecture [1]. Let $G$ be a reductive group, $P$ be a maximal parabolic subgroup of $G$, $M$ be a Levi factor of $P$, $N$ be the unipotent radical of $P$, and $A$ be the maximal split torus of the center of $M$. Let $\pi_0$ be an irreducible discrete automorphic representation of $M$. Then, according to Arthur’s conjecture, $\pi$ is associated to a homomorphism

$$\varphi_\pi: \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{L}M.$$

Here $\mathcal{L}$ is the conjectual Langlands group. Let $\text{L}N$ be the Lie algebra of $\text{L}N$. Decompose $\text{L}N$ as in Shahidi [24].

$$\text{L}N = \prod_{i=1}^{r} \text{L}N_i.$$

Consider the induced representation $\text{Ind}_{\text{M}}^{G} \pi \varepsilon$. Here $\varepsilon$ is as in [24]. Let $\text{Ad}_{\text{L}N}$ be
the adjoint action of $L^\epsilon M$ on $L^\epsilon \mathcal{N}$. If $\pi$ is cuspidal and $\varphi_\pi$ is trivial on $SL_2(C)$, then the normalizing factor should be given by

$$\prod_{i=1}^{r} L(1 + is, \varphi_\pi \circ Ad_{\epsilon_i}).$$

(cf. Shahidi [24], Langlands [15].) Consider the general case where $\varphi_\pi \circ Ad_{\epsilon_i}$ is not trivial on $SL_2(C)$. In this case, decompose $\varphi_\pi \circ Ad_{\epsilon_i}$ into irreducible representation:

$$\varphi_\pi \circ Ad_{\epsilon_i} = \bigoplus_{j=1}^{m_i} \varphi_{ij} \otimes \text{sym}^{r_{ij}},$$

where $\varphi_{ij}$ is an irreducible representation of $\mathcal{L}$, and $\text{sym}^{r_{ij}}$ is the $r_{ij}$th symmetric power of the standard representation of $SL_2(C)$. Then we claim the normalizing factor should be

$$\prod_{i=1}^{r} \prod_{j=1}^{m_i} L(is + \frac{r_{ij}}{2} + 1, \varphi_{ij}).$$

In fact, the c-function $c_{w_0}(\pi, s)$ for the longest element $w_0$ of the Weyl group is given by

$$c_{w_0}(\pi, s) = \prod_{i=1}^{r} \frac{L(is, \varphi_\pi \circ Ad_{\epsilon_i})}{L(1 + is, \varphi_\pi \circ Ad_{\epsilon_i})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \frac{L(is, \varphi_{ij} \otimes \text{sym}^{r_{ij}})}{L(1 + is, \varphi_{ij} \otimes \text{sym}^{r_{ij}})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \prod_{a=0}^{r_{ij}} \frac{L(is - \frac{r_{ij}}{2} + a, \varphi_{ij})}{L(is - \frac{r_{ij}}{2} + a + 1, \varphi_{ij})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \frac{L(is - \frac{r_{ij}}{2}, \varphi_{ij})}{L(is + \frac{r_{ij}}{2} + 1, \varphi_{ij})},$$

at least up to bad primes. If $\pi$ is cuspidal, this is the only non-trivial c-function. This means at least when $\pi$ is cuspidal, our claim is justified, since the normalizing factor should be the least common denominator of the c-functions. One can expect that the least common denominator of the c-functions is equal to
the denominator of the c-function for the longest Weyl element even when \( n \) is not cuspidal.

Observe that in our case, \( G = \text{Sp}_n \), \( M = \text{GL}_n \), \( \pi = \omega \), \( \varphi_\pi = \omega \otimes \text{sym}^{n-1} \), \( \text{Ad}_{\lambda, 1} = \rho \), \( \text{Ad}_{\lambda, 2} = \Lambda^2 \rho \). Here \( \rho \) is the standard representation of \( \text{GL}_n \). Therefore,

\[
\varphi_\pi \circ \text{Ad}_{\lambda, 1} = \omega \otimes \text{sym}^{n-1}
\]

gives \( L \left( s + \frac{n+1}{2}, \omega \right) \), and

\[
\varphi_\pi \circ \text{Ad}_{\lambda, 2} = \bigotimes_{j=1}^{[n/2]} (\omega^2 \otimes \text{sym}^{2n-4j})
\]

gives \( \prod_{r=1}^{[n/2]} L(2s + n + 1 - 2r, \omega^2) \).

1.3. Eisenstein series

In this subsection, we assume \( k \) to be a global field. We will investigate the poles of Eisenstein series associated to good sections.

Let \( \omega \) be a quasi-character of \( \mathbb{A}^\times / k^\times \). Put \( K_n = \Pi_v K_{n,v} \). Let \( I(\omega, s) \) be the space of functions \( f(h) \) on \( H_n(A) \) which satisfy (1) and (2):

1. \( f \) is right \( K_n \)-finite.
2. For any \( p = \begin{pmatrix} A & * \\ 0_n & A^{-1} \end{pmatrix} \in P_n(A) \),

\[
f(ph) = \omega(\det A) |\det A|^{(s+n+1)/2} f(h).
\]

Clearly, \( I(\omega, s) = \bigotimes_v I(\omega_v, s) \). We also define holomorphic sections and meromorphic sections similarly. We say that a meromorphic section of \( I(\omega, s) \) is a good section if it is a finite sum of decomposable elements \( f^{(s)} = \Pi_v f_v^{(s)} \) satisfying following (i) and (ii).

(i) For almost all unramified \( v \), \( f_v^{(s)} = d(\omega_v, s) \phi_{\omega_v, s} \).
(ii) \( f_v^{(s)} \) is a good section of \( I(\omega_v, s) \) for all \( v \).

In other words, the space of global good sections is the restricted tensor product of the local good sections with respect to \( d(\omega_v, s) \phi_{\omega_v, s} \). Note that the product \( f^{(s)} = \Pi_v f_v^{(s)} \) is absolutely convergent for \( \text{Re}(s) > \frac{n+1}{2} \), and can be meromorphically continued to \( \mathbb{C} \).
We define the Eisenstein series $E(h; f^{(s)})$ associated to $f^{(s)}$ by

$$E(h; f^{(s)}) = \sum_{\gamma \in \mathcal{P}_\lambda \setminus \mathcal{H}_s} f^{(s)}(\gamma h).$$

This is absolutely convergent for $\text{Re}(s) \gg 0$, and can be meromorphically continued to $\mathbb{C}$. The functional equation of $E(h; f^{(s)})$ is given by

$$E(h; f^{(s)}) = E(h; M_{w_0} f^{(s)}).$$

Here $M_{w_0}$ is the global intertwining operator:

$$M_{w_0} = \bigotimes_v (M_{w_0})_v.$$

The global intertwining operator $M_{w_0}$ does not depend on the choice of representative of $w_0 \in W_{H_n}$ in $\text{Norm}(T_n)$.

**Lemma 1.4.** If $f^{(s)}$ is a good section of $I(\omega, s)$, then $M_{w_0} f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$.

**Proof.** Let $S$ be a finite set of places of $k$ such that if $v \notin S$, then $\omega_v$ is unramified, $\psi_v$ is of order 0, and $f^{(s)}_v = d(\omega_v, s) \phi_{\omega_v, v}$. Then

$$M_{w_0} f^{(s)} = \prod_{v \notin S} d(\omega_v, s) c_{w_0}(\omega_v, s) \phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0} f^{(s)}_v$$

$$= \prod_{v \notin S} a_{w_0}(\omega_v, s) \phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0} f^{(s)}_v$$

$$= \prod_{v \notin S} d(\omega_v^{-1}, -s) \phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0} f^{(s)}_v.$$

By Lemma 1.2, the lemma follows.

**Lemma 1.5.** Suppose that $n = 1$, and $\omega = 1$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the global intertwining operator $M_w: I(1, s) \rightarrow I(1, -s)$ is holomorphic at $s = 0$, and is equal to the scalar multiplication by $-1$ at $s = 0$.

**Proof.** Put $f^{(s)} = \prod_v \phi_{1, v}$, and $\xi(s) = |D|^{s/2} \zeta(s)$. Here $D$ is the discriminant of $k$ (resp. $D = q^{2g-2}$, $g$ is the genus of $k$) if $k$ is a number field (resp. if $k$ is a function field). Then

$$M_w f^{(s)} = \frac{\xi(s)}{\xi(s+1)} \prod_v \phi_{1, -s}.$$  \hfill (1.3.1)

Since $\xi(1-s) = \xi(s)$ and $\xi(s)$ has a simple pole at $s = 0, 1$, the right-hand side of
(1.3.1) is holomorphic at $s=0$, and

$$M_wf^{(0)} = -f^{(0)}.$$  

Since $I(1, s)$ is irreducible on some neighbourhood of $s=0$, the lemma follows.

**Proposition 1.6.** Suppose that $k$ is a number field. If $f^{(s)}$ is a good section of $I(\omega, s)$, then the pole of $E(h; f^{(s)})$ are at most simple. The set of possible poles is as follows.

1. When $\omega$ is principal: we may assume $\omega = 1$. Then the set of possible poles is:

$$\left\{ \frac{n+1}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n + 1, m \neq \frac{n+1}{2} \right\}$$

2. When $\omega$ is not principal, and $\omega^2$ is principal: we may assume $\omega^2 = 1$. Then the set of possible poles is:

$$\left\{ \frac{n-1}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n - 1, m \neq \frac{n-1}{2} \right\}$$

3. If $\omega^2$ is not principal, then $E(h; f^{(s)})$ is entire.

**Proof.** As in [22], the constant term $E^0(h; f^{(s)})$ of $E(h; f^{(s)})$ along $U_n(A)$ is given by

$$E^0(h; f^{(s)}) = \int_{U_n(k) \setminus U_n(A)} E(uh; f^{(s)}) du$$

$$= \sum_{w \in \Omega_n} M_wf^{(s)}.$$  

Let $S$ be as in the proof of Lemma 1.4. Then

$$M_wf^{(s)} = \prod_{v \notin S} d(\omega_v, s)c_w(\omega_v, s)^{\phi_w^v} \times \prod_{v \in S} M_wf^{(s)}$$

$$= d(\omega, s)c_w(\omega, s) \prod_{v \notin S} \phi_w^v$$

$$\times \prod_{v \in S} \left[ d(\omega_v, s)c_w(\omega_v, s) \right]^{-1} M_wf^{(s)}.$$  

Therefore the poles of $E(h; f^{(s)})$ comes from the poles of $d(\omega, s)c_w(\omega, s)$. In particular, if $\omega^2$ is not principal, $E(h; f^{(s)})$ is entire.

We may assume $\omega^2 = 1$, without loss of generality. When $\omega = 1$, (resp. $\omega^2 = 1$, ...
\( \omega \neq 1 \), the possible poles of \( d(\omega, s)c_w(\omega, s) \) are integral or half-integral points in

\[
\left[ -\frac{n+1}{2}, \frac{n+1}{2} \right] \text{ (resp. } \left[ -\frac{n-1}{2}, \frac{n-1}{2} \right]\).
\]

We first prove the proposition for the case \( n = 1 \) or \( n = 2 \). If \( n = 1 \), \( \omega \neq 1 \), then (2) is obvious since \( d(\omega, s)c_w(\omega, s) \) are entire. If \( n = 1 \), \( \omega = 1 \), then we have to show that \( s = 0 \) is not a pole of \( E^0(h; f^{(s)}) \). Note that \( f^{(s)} \) may have a simple pole at \( s = 0 \). Let \( w \) be as in Lemma 1.5. Then by Lemma 1.5,

\[
\lim_{s \to 0} \Re^0(h; f^{(s)}) = (1 + M_w) \left[ \lim_{s \to 0} sf^{(s)} \right]
\]

\( = 0. \)

Thus \( E^0(h; f^{(s)}) \) is holomorphic at \( s = 0 \).

If \( n = 2 \), the possible poles of \( d(\omega, s)c_w(\omega, s) \) are as follows:

<table>
<thead>
<tr>
<th>( I )</th>
<th>( l(w) )</th>
<th>( d(\omega, s)c_w(\omega, s) )</th>
<th>( \text{poles } (\omega = 1) )</th>
<th>( \text{poles } (\omega^2 = 1, \omega \neq 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( \emptyset )</td>
<td>( L(s + \frac{1}{2})\zeta(2s + 1) )</td>
<td>( -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0 )</td>
<td>( -\frac{1}{2}, 0 )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( {2} )</td>
<td>( L(s + \frac{1}{2})\zeta(2s + 1) )</td>
<td>( -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2} )</td>
<td>( -\frac{1}{2}, 0 )</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>( {1} )</td>
<td>( L(s + \frac{1}{2})\zeta(2s) )</td>
<td>( -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} )</td>
<td>( 0, \frac{1}{2} )</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>( {1, 2} )</td>
<td>( L(s - \frac{1}{2})\zeta(2s) )</td>
<td>( 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} )</td>
<td>( 0, \frac{1}{2} )</td>
</tr>
</tbody>
</table>

Here, \( L(s) = L(s, \omega) \). By functional equation, we may assume \( \Re(s) \geq 0 \), so what we have to prove are reduced to the following two statements.

(1.3.2) If \( \omega = 1 \),

\[
\lim_{s \to 1/2} (s - \frac{1}{2})^2(M_{w_3} + M_{w_4})f^{(s)} = 0.
\]

(1.3.3) If \( \omega^2 = 1 \),

\[
\lim_{s \to 0} s(1 + M_{w_2} + M_{w_3} + M_{w_4})f^{(s)} = 0.
\]

Proof of (1.3.2)

\[
\lim_{s \to 1/2} (s - \frac{1}{2})^2M_{w_4}f^{(s)} = \lim_{s \to 1/2} M(w_{27}, \zeta^w) \circ [(s - \frac{1}{2})^2M_{w_3}f^{(s)}].
\]

We know that \( (s - \frac{1}{2})^2M_{w_3}f^{(s)} \) is holomorphic at \( s = \frac{1}{2} \). Moreover, by (1.2.1) and
Lemma 1.5, $M(w_2, \chi_s^w)$ is holomorphic and is equal to the scalar multiplication by $-1$ at $s = \frac{1}{2}$. Hence (1.3.2).

Proof of (1.3.3). By the same way as above, we can prove

$$\lim_{s \to 0} s(M_{w_2} + M_{w_3})f^{(s)} = 0.$$ But the proof that

$$\lim_{s \to 0} s(1 + M_{w_4})f^{(s)} = 0$$

is more delicate. We have

$$M_{w_4}f^{(s)} = M(w_2, \chi_s^w) \circ M(w_2, \chi_s^w) \circ M(w_2, \chi_s^w) f^{(s)}.$$ By (1.2.1) and Lemma 1.5, $M(w_2, \chi_s^w)$ is holomorphic and is equal to the scalar multiplication by $-1$ at $s = 0$. Moreover, by (1.2.1), $M(w_2, \chi_0)$ (resp. $M(w_2, \chi_s^w)$) is essentially the intertwining operator

$$M \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s + \frac{1}{2} \right) : I(\omega, s + \frac{1}{2}) \to I(\omega, -s - \frac{1}{2})$$

(resp. $M \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -s - \frac{1}{2} \right) : I(\omega, -s - \frac{1}{2}) \to I(\omega, s + \frac{1}{2})$)

on $SL_2$. Moreover, these two are mutually the inverse of the other except for their singular points. Since the representations $I(\omega, s + \frac{1}{2})$ and $I(\omega, -s - \frac{1}{2})$ of $SL_2(A)$ are irreducible on some neighbourhood of $s = 0$, there is an integer $\alpha$ such that

$$s^{-\alpha}M \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s + \frac{1}{2} \right) \quad \text{and} \quad s^\alpha M \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -s - \frac{1}{2} \right)$$

are holomorphic, and are mutually the inverse of each other at $s = 0$. In fact, it is easy to see that $\alpha = \text{ord}_{s = 1/2} L(s, \omega)$. We have

$$\lim_{s \to 0} s M_{w_4}f^{(s)} = \lim_{s \to 0} \left[ s^\alpha M(w_2, \chi_s^w) \circ M(w_2, \chi_s^w) \circ [s^{-\alpha}M(w_2, \chi_0)](s f^{(s)}) \right].$$

Each term is holomorphic at $s = 0$, so the exchange of limit and the composition is possible. Hence (1.3.3).

Now we assume $n \geq 3$. By the functional equation, it is enough to investigate
the integral or half-integral points in \(0, \frac{n+1}{2}\). Note that \(f(s)\) is holomorphic on the right half plane \(\Re(s) > 0\) except for the case \(n\) is even and \(s=0\). In particular, if \(n\) is odd, \(s=0\) is not a pole of \(E(h; f^{(s)})\), by [16].

We recall the theory of degenerate Eisenstein series on \(GL_n\) (see [12, §5]). Let \(Q\) be the maximal parabolic subgroup of \(GL_n\) given by

\[
Q = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \middle| a_1 \in GL_{n-1}, a_2 \in k^\times \right\}.
\]

Let \(I_Q(s)\) be the representation of \(GL_n\) induced from the character of \(Q\) given by

\[
\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto |\det a_1|^{s/n}|a_2|^{-(n-1)s/n}.
\]

We define standard sections, holomorphic sections etc. as usual. For each prime \(v\) of \(k\), let \(F_{0,v}^{(s)}\) be the meromorphic section of \(I_{Q,v}(s)\) which takes value \(\zeta_v(s + \frac{n}{2})\) on the standard maximal compact subgroup of \(GL_{n,v}\).

Taking any finite set \(S\) of primes of \(k\), put

\[
F^{(s)} = \prod_{v \notin S} F_{0,v}^{(s)} \times \prod_{v \in S} F_v^{(s)}
\]

where \(F_v^{(s)}, v \in S\) are arbitrary holomorphic sections of \(I_{Q,v}(s)\). Define degenerate Eisenstein series on \(GL_n\) by

\[
E(g; F^{(s)}) = \sum_{\gamma \in Q \backslash GL_n} F^{(s)}(\gamma g).
\]

Then the possible poles of \(E(g; F^{(s)})\) are \(s = \pm \frac{n}{2}\). Moreover, each pole is at most simple and the residue is a constant function. The functional equation is given by

\[
E(g; F^{(s)}) = E(g; M_w F^{(s)}).
\]

Here

\[
w = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.
\]
$M_{w}F^{(s)}$ is a meromorphic section of the representation induced from the character

$$
\begin{pmatrix}
a_1 & * \\
0 & a_2
\end{pmatrix} \mapsto |a_1|^{-(n-1)s/n} |\det a_2|^{s/n}
$$

of the parabolic subgroup

$$Q' = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \middle| a_1 \in k^\times, a_2 \in \text{GL}_{n-1} \right\}.
$$

$M_{w}F^{(s)}$ has at most simple poles at $s = \frac{n}{2}, \frac{n}{2} - 1$.

We return to the proof of Proposition 1.6. Let

$$f^{(s)} = \prod_{v \in S} d(\omega_v, s) \phi_{\omega_v, s} \times \prod_{v \in S} f_v^{(s)}$$

be a good section. We may assume each $f_v^{(s)}$, $v \in S$ is a standard section, since $d(\omega_v, s)$ has no pole in $\text{Re}(s) \geq 0$.

Let $P_1^*$ be the parabolic subgroups of $H_n$ given by

$$P_1^* = \left\{ \begin{pmatrix} a & * & * & * \\
0 & A & * & * \\
0 & a^{-1} & 0 & * \\
0 & 0 & \tau A^{-1} & \end{pmatrix} \in H_n \middle| a \in k^\times, A \in \text{GL}_{n-1} \right\}.$$

Let $t = (t_1, t_2) \in \mathbb{C}^2$. Let $I_{P_1^*}(\omega_v, t)$, be the space of right $K_v$-finite function $f_{P_1^*}^{(t)}$ on $H_{n,v}$ such that

$$f_{P_1^*}^{(t)}(p_1, h) = \omega(a \det A)|a|^{t_1+n}|\det A|^{t_2+n/2} f_{P_1^*}^{(t)}(h),$$

where

$$p_1 = \begin{pmatrix} a & * & * & * \\
0 & A & * & * \\
0 & a^{-1} & 0 & * \\
0 & 0 & \tau A^{-1} & \end{pmatrix} \in P_1^*.$$

For each $v \in S$, let $f_v^{(t)}$ be a standard section (of two variables) of $I_{P_1^*}(\omega_v, t)$ defined by

$$f_v^{(t)}(p_1, k) = |a|^{n-1} \det A^{-1}|t_1-t_2|/n + 1/2 f_v^{(t)}(k),$$
where \( p_1 \) is as above, \( k \in K_v \), and
\[
s = \frac{t_1 + (n - 1)t_2}{n}.
\]

When \( v \not\in S \), let \( \phi_{P^*_v, \omega, \pi} \) be the standard section of \( I_{P^*_v}(\omega, t) \) which is identically 1 on \( K_v \). Put
\[
\tilde{f}^{(1)}(t) = \prod_{v \in S} L_v(t_1 + 1) \zeta_v \left( t_1 - t_2 + \frac{n}{2} \right) \zeta_v \left( t_1 + t_2 + \frac{n}{2} \right) L_v \left( t_2 + \frac{n}{2} \right) \prod_{r=1}^{(n-1)/2} \zeta_v(2t_2 + n - 2r)
\]
\[
\times \prod_{v \not\in S} \phi_{P^*_v, \omega, \pi} \times \prod_{v \in S} \tilde{f}^{(1)}(t).
\]

Here \( L_s(s) \) stands for \( L(\omega_v, s) \). Put
\[
E(h; \tilde{f}^{(1)}) = \sum_{\gamma \in P^*_v \backslash H_v} \tilde{f}^{(1)}(\gamma h)
\]
\[
= \sum_{\gamma \in P^*_v \backslash H_v} \sum_{\gamma \in P^*_v \backslash P_v} \tilde{f}^{(1)}(\gamma_1 \gamma h).
\]

The inner sum in the last expression is a degenerate Eisenstein series on \( GL_n \). In particular, the residue of this inner Eisenstein series along \( t_1 - t_2 = \frac{n}{2} \) is, up to non-zero constant, equal to
\[
L_s \left( s + \frac{n+1}{2} \right) \zeta(s+n-1)L_s \left( s + \frac{n-1}{2} \right) \prod_{r=1}^{(n-1)/2} \zeta(2s+n+1-2r)
\]
\[
\times \prod_{v \not\in S} \phi_{\omega_v, s} \times \prod_{v \in S} f_v(s)(\gamma h).
\]

Here \( s = t_2 + \frac{1}{2} \). So, the residue of \( E(h; \tilde{f}^{(1)}) \) along \( t_1 - t_2 = \frac{n}{2} \) is, up to non-zero constant, equal to
\[
\begin{cases}
L_s \left( s + \frac{n-1}{2} \right) \zeta(2s) E(h; f^{(s)}), & \text{if } n \text{ is even} \\
L_s \left( s + \frac{n-1}{2} \right) E(h; f^{(s)}), & \text{if } n \text{ is odd}.
\end{cases}
\]

Put
\[
D_1 = \left\{ (t_1, t_2) \mid \text{Re}(t_1) > \text{Re}(t_2) + \frac{n}{2}, \text{Re}(t_2) > \frac{n}{2} \right\}.
\]
Then $\tilde{f}^{(0)}$ is holomorphic on $D_1$, and the summation (1.3.4) is absolutely convergent on $D_1$, so $E(h; \tilde{f}^{(0)})$ is holomorphic on $D_1$. Put

$$P_2^* = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & A & * & B \\ 0 & 0 & a^{-1} & 0 \\ 0 & C & * & D \end{pmatrix} \in H_n \mid a \in k^\times, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H_{n-1} \right\}.$$

Then

$$E(h; \tilde{f}^{(0)}) = \sum_{\gamma \in P_1^* \setminus H_1} \sum_{\tilde{\gamma} \in P_1^* \setminus P_2^*} \tilde{f}^{(0)}(\gamma \gamma h).$$

The inner sum of (1.3.6) is

$$L_S(t_1 + 1) \zeta_S \left( t_1 - t_2 + \frac{n}{2} \right) \zeta_S \left( t_1 + t_2 + \frac{n}{2} \right)$$

times an Eisenstein series on $H_{n-1}$ associated to a good section of $I(\omega, t_2)$. By the induction assumption, the poles of this Eisenstein series is

$$\left\{ \begin{array}{c}
t_2 = \frac{n}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n, n \neq \frac{n}{2} \\
t_2 = \frac{n-2}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n - 2, n \neq \frac{n-2}{2}
\end{array} \right\} \text{ if } \omega = 1$$

and

$$\left\{ \begin{array}{c}
t_2 = \frac{n}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n - 2, n \neq \frac{n-2}{2} \end{array} \right\} \text{ if } \omega \neq 1$$

(1.3.7)

By the functional equation of the inner Eisenstein series, $E(h; \tilde{f}^{(0)})$ is holomorphic on the domain

$$D_2 = \left\{ (t_1, t_2) \mid \Re(t_1) > \Re(t_2) + \frac{n}{2}, \Re(t_1) > -\Re(t_2) + \frac{n}{2}, \Re(t_2) > \frac{n}{2} \right\}.$$

Therefore $E(h; \tilde{f}^{(0)})$ can be meromorphically continued to the convex closure of $D_1 \cup D_2$, and the singularities in this domain are given by (1.3.7).

Similarly, by the functional equation of degenerate Eisenstein series on $\text{GL}_n$, $E(h; \tilde{f}^{(0)})$ is holomorphic on the domain

$$D_3 = \left\{ (t_1, t_2) \mid \Re(t_1) > 1, \Re(t_2) > \Re(t_1) + \frac{n}{2} \right\}$$

and can be meromorphically continued to the convex closure of $D_1 \cup D_3$. The
singularities in this domain are given by
\[
\left\{ t_1 - t_2 = \pm \frac{n}{2} \right\}.
\] (1.3.8)

By the same reason, \(E(h; f^0)\) is holomorphic on
\[
D_4 = \left\{ (t_1, t_2) \mid \text{Re}(t_1) < -1, \text{Re}(t_2) > -\text{Re}(t_1) + \frac{n}{2} \right\}
\]
and can be meromorphically continued to the convex closure of \(D_2 \cup D_4\). The singularity in this domain is
\[
\left\{ t_1 + t_2 = \pm \frac{n}{2} \right\}.
\] (1.3.9)

Thus \(E(h; f^0)\) can be meromorphically continued to the convex closure of \(D_1 \cup D_2 \cup D_3 \cup D_4\) and the singularity in this domain is the union of (1.3.7), (1.3.8) and (1.3.9). Therefore (1.3.5) has at most simple poles at
\[
\left\{ s = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{n+1}{2} \right\}, \quad \text{if } n \text{ is even}
\]
\[
\left\{ s = \frac{1}{2}, 1, 2, \ldots, \frac{n+1}{2} \right\}, \quad \text{if } n \text{ is odd}
\]
for \(\text{Re}(s) \geq 0\). Here \(\frac{n+1}{2}\) is a pole only if \(\omega = 1\). If \(n\) is even, \(L_S \left( s + \frac{n-1}{2} \right) \) has neither poles nor zeros for \(\text{Re}(s) \geq 0\). If \(n\) is odd, \(L_S \left( s + \frac{n-1}{2} \right) \zeta_S(2s)\) has a simple pole at \(s = \frac{1}{2}\) and has no zero at positive integral or half-integral points. Note that we already know that \(s = 0\) is not a pole if \(n\) is odd. Thus we have proved Proposition 1.6.

**COROLLARY.** Let \(f^0\) be a global holomorphic section of \(I(\omega, s)\). Let \(S\) be a finite set of places of \(K\) such that \(f^0\) is invariant under \(K_v, v \notin S\). Then the set of poles of
\[
d_S(\omega, s)E(h; f^0)
\]
is given by Proposition 1.6.

This result is also proved in [14].

If \(K\) is a function field, we can prove the following proposition similarly.
PROPOSITION 1.7. Suppose \( k \) is a function field. If \( f^{(s)} \) is a good section of \( I(\omega, s) \), then the poles of \( E(h; f^{(s)}) \) are at most simple. The set of possible poles is as follows.

1) When \( \omega \) is principal: we may assume \( \omega = 1 \). The set of possible poles is:

\[
\left\{ \pm \frac{n+1}{2} + \frac{2\pi \sqrt{-1}}{\log q} \mathbb{Z} \right\} \\
\cup \left\{ \frac{n-1}{2} - m + \frac{\pi \sqrt{-1}}{\log q} \mathbb{Z} \mid m \in \mathbb{Z}, 0 \leq m \leq n-1, m \neq \frac{n-1}{2} \right\}
\]

2) When \( \omega \) is not principal, and \( \omega^2 \) is principal: we may assume \( \omega^2 = 1 \). Then the set of possible poles is:

\[
\left\{ \frac{n-1}{2} - m + \frac{\pi \sqrt{-1}}{\log q} \mathbb{Z} \mid m \in \mathbb{Z}, 0 \leq m \leq n-1, m \neq \frac{n-1}{2} \right\}
\]

3) If \( \omega^2 \) is not principal, then \( E(h; f^{(s)}) \) is entire.

REMARK. Proposition 1.6 or 1.7 implies that the possible poles of Langlands L-function of irreducible cuspidal automorphic representations of \( \text{Sp}_n \) attached to the standard representation of the L-group \( \text{Sp}_n \simeq SO(2n+1) \) are

\[
\{-n+1, -n+2, \ldots, n-1, n\}
\]

or

\[
\left\{ -n+1 + \frac{\pi \sqrt{-1}}{\log q} \mathbb{Z}, -n+2 + \frac{\pi \sqrt{-1}}{\log q} \mathbb{Z}, \ldots, n-1 + \frac{\pi \sqrt{-1}}{\log q} \mathbb{Z}, n + \frac{\pi \sqrt{-1}}{\log q} \mathbb{Z} \right\},
\]

and all of them are at most simple (cf. [14], [20], [21]).

1.4. Calculation of the residue at \( s = \frac{n-1}{2} \)

In this subsection, we assume \( \omega = 1 \). Then there exists a class 1 element of \( I(\omega, s) \). Take \( \phi_s \in I(\omega, s) \) such that \( \phi_s|_{\mathbb{C}^n} = 1 \). Put

\[
E(h, s) = E(h; \phi_s), \\
\tilde{E}(h, s) = \xi \left( s + \frac{n+1}{2} \right) \prod_{r=1}^{[n/2]} \xi(2s + n + 1 - 2r)E(h, s).
\]
\( \tilde{E}(h, s) \) satisfies the following functional equation:

\[
\tilde{E}(h, s) = \tilde{E}(h, -s).
\]

We will determine the residue of \( E(h; s) \) at \( s = \frac{n-1}{2} \). Let \( P_{n,r} \) be a parabolic subgroup of \( H_n \) given by

\[
P_{n,r} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & A & * & B \\ 0 & 0 & t^{a^{-1}} & 0 \\ 0 & C & * & D \end{pmatrix} \in H_n \Bigg| a \in \text{GL}_{n-r}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_r \right\}.
\]

Let \( s \in C \) and \( t = (t_1, t_2, \ldots, t_n) \in C^n = X^* (T_n) \otimes \mathbb{Z} C \). Let \( \phi(h; P_{n,r}; s), \phi(h; B_n; t) = \phi(h; B_n; t_1, t_2, \ldots, t_n) \) be the functions on \( H_n(A) \) given by

\[
\phi(pk; P_{n,r}; s) = |a|^{s+(n+r+1)/2}
\]

\[
\phi(bk; B_n; s) = \prod_{i=1}^{n} |b_i|^{t_i+n+1-i},
\]

where \( k \in K_n \).

\[
p = \begin{pmatrix} a & * & * & * \\ 0 & A & * & B \\ 0 & 0 & t^{a^{-1}} & 0 \\ 0 & C & * & D \end{pmatrix} \in P_{n,r}(A),
\]

\[
b = \begin{pmatrix} b_1 & * & & \\ b_2 & \ddots & & \\ & \ddots & \ddots & \\ 0 & b_n & \ddots & b_1^{-1} \\ \theta_n & & \ddots & b_2^{-1} \\ & & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & & \ddots & 0 \\ & & & & & \ddots & b_n^{-1} \end{pmatrix} \in B_n(A).
\]
Poles of the triple $L$-functions

For any $x \in \Phi_{x_0}^+$, let $l_x^+(t)$ and $\mathcal{F}_x^+$ be linear forms and hyperplanes of $\mathbb{C}^n$ given by

$$l_x^+(t) = \langle \tilde{x}, t \rangle - 1, \quad l_x^-(t) = \langle \tilde{x}, t \rangle + 1, \quad \mathcal{F}_x^+ = \{ t \in \mathbb{C}^n | l_x^+(t) = 0 \}, \quad \mathcal{F}_x^- = \{ t \in \mathbb{C}^n | l_x^-(t) = 0 \}.$$

It is easy to see that the residue along $\mathcal{F}_{x_1}^+, \ldots, \mathcal{F}_{x_{n-r+1}}^+, \mathcal{F}_{x_{n-r+1}}^-, \ldots, \mathcal{F}_{x_n}^+$ in the sense of [9, p. 195] is

$$R^{n-1} \prod_{i=2}^{n-r} \zeta(i)^{-1} \prod_{i=1}^{r} \zeta(2i)^{-1} E_{P_{x_0}} \left( h, t_{n-r} + \frac{n-r-1}{2} \right),$$

where $R = \text{Res}_{s=1} \zeta(s)$. Put

$$\tilde{E}_{B_n}(h, t) = \prod_{x \in \Phi_{x_0}^+} \zeta(\langle \tilde{x}, t \rangle + 1)^{E_{B_n}(h, t)}$$

$$= \prod_{1 \leq i < j \leq n} \zeta(t_i + t_j + 1) \zeta(t_i - t_j + 1) \prod_{i=1}^{n} \zeta(t_i + 1)^{E_{B_n}(h, t)}.$$

Then it is known that

$$\prod_{x \in \Phi_{x_0}^+} l_x^+(t) l_x^-(t)^{E_{B_n}(h, t)}$$

is entire and invariant under $t \rightarrow wtw^{-1}$ for any $w \in W_{H_n}$.

The value of (1.4.6) at $t = \left( s + \frac{n-1}{2}, s + \frac{n-3}{2}, \ldots, s + \frac{n-1}{2} \right)$ is

$$(2R)^{n-1} \prod_{i=2}^{n-1} \prod_{i=2}^{n} \{ (i-1)(i+1) \zeta(i) \}^{n-i} \times \prod_{i=1}^{n} \left( s + \frac{n+3}{2} - i \right) \left( s + \frac{n-1}{2} - i \right) \zeta \left( s + \frac{n+3}{2} - i \right) \times \prod_{1 \leq i < j \leq n} (2s+n+2-i-j)(2s+n-i-j) \zeta(2s+n+2-i-j) \times E_{P_{x_0}}(h, s).$$
So the value of (1.4.6) at \( t = (n - 1, n - 2, \ldots, 1, 0) \) is

\[
(2R)^{n-1} \prod_{i=2}^{n-1} (i(i+1)\zeta(i))^{n-i} \times (-R)n!(n-2)! \prod_{i=2}^{n} \zeta(i) \times 2\zeta(2) \prod_{i=2}^{n-1} \prod_{j=1}^{i} \zeta(i+j) \times 2 \text{ Res}_{s=(n-1)/2} E_{P_{n,n-1}}(h, s).
\]

On the other hand, the value of (1.4.6) at \( t = (s, n - 1, n - 2, \ldots, 1) \) is

\[
(2R)^{n-1} \prod_{i=2}^{n-1} (i(i+1)\zeta(i))^{n-i} \times \prod_{1 \leq i < j \leq n-1} (i+j+1)(i+j-1)\zeta(i+j) \times 2 \prod_{i=1}^{2n-1} (s-n+i+1)(s-n+i-1)\zeta(s-n+i+1) \times E_{P_{n,n-1}}(h, s).
\]

It follows that \( E_{P_{n,n-1}}(h, s) \) is holomorphic at \( s = 0 \), and the value of (1.4.6) at \( t = (0, n - 1, n - 2, \ldots, 1) \) is

\[
(2R)^{n-1} \prod_{i=2}^{n-1} (i(i+1)\zeta(i))^{n-i} \times \prod_{1 \leq i < j \leq n-1} (i+j+1)(i+j-1)\zeta(i+j) \times (-R^2)(n!)^2 ((n-2)!)^2 \prod_{i=2}^{n} \zeta(i) \prod_{i=2}^{n-1} \zeta(i) \times E_{P_{n,n-1}}(h, 0).
\]

Thus we get the following proposition.
PROPOSITION 1.8.

\[
\text{Res}_{s=(n-1)/2} E_{P_{n,0}}(h, s) = \frac{1}{2} R \prod_{i=1}^{[n/2]-1} \zeta(2i+1) \prod_{i=1}^{[n/2]} \zeta(2n-2i)^{-1} E_{P_{n-1}}(h, 0),
\]

or, equivalently

\[
\text{Res}_{s=(n-1)/2} \tilde{E}_{P_{n,0}}(h, s) = \frac{1}{2} R \zeta(n) \prod_{i=1}^{[n/2]-1} \zeta(2i+1) E_{P_{n-1}}(h, 0).
\]

LEMMA 1.9. \( I \left( 1, \frac{n-1}{2} \right) \) is generated by class 1 vectors.

Proof. Let \( \chi \) be a character of \( T_n \) given by

\[
\chi(t) = \prod_{i=1}^{n} |t_i|^{n-i}.
\]

Then \( I \left( 1, \frac{n-1}{2} \right) \) is a quotient of \( \text{Ind}_{H_n}^H \chi \). It is sufficient to prove that \( \text{Ind}_{H_n}^H \chi \) is generated by class 1 vectors. Let \( P \) be the standard parabolic subgroup of \( H_n \) corresponding to \( \alpha_n \). Then

\[
\text{Ind}_{H_n}^H \chi = \text{Ind}_P^H(\text{Ind}_{B_n}^P \chi).
\]

The restriction of \( \text{Ind}_{B_n}^P \chi \) to \( \iota_{2n}(SL_2) \) is an irreducible tempered representation. Let \( M \) be the standard Levi factor of \( P \) and \( w \) be the longest element of \( W_{M} \backslash W_{H_n} \), i.e.,

\[
w = \begin{pmatrix} \begin{array}{c} -1_{n-1} \\ 1 \\ 1_{n-1} \\ 1 \end{array} \end{pmatrix}.
\]

By the well-known theory of Langlands quotient, \( \text{Ind}_{H_n}^H(\text{Ind}_{B_n}^P \chi) \) is generated by any element \( f \) such that \( M_w f \neq 0 \). It is easy to check that a non-zero class 1 vector satisfies this condition.
Let $f^{(s)}$ be any good section of $I(1, s)$. Put

$$w = w_{[2, \ldots, n]}$$

$$= \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & \ddots \\
& \ddots & 1 \\
1 & & &
\end{pmatrix}. \quad (1.4.7)$$

It is easy to check that $M_w f^{(s)}$ has at most a simple pole at $s = \frac{n-1}{2}$ and

$$\text{Res}_{s = (n-1)/2} M_w f^{(s)}$$

is in $\text{Ind}^H_{H_{n,n-1}} 1$. An easy calculation shows

$$\text{Res}_{s = (n-1)/2} M_w \phi(h; P_{n,0}; s)$$

$$= R \prod_{i=1}^{[n/2]-1} \zeta(2i+1) \prod_{i=1}^{[n/2]} \zeta(2n-2i)^{-1} \phi(h; P_{n,n-1}; 0).$$

Thus by Proposition 1.8,

$$\text{Res}_{s = (n-1)/2} E_{P_{n,0}} (h, \phi(h; P_{n,0}; s))$$

$$= \frac{1}{2} E_{P_{n,n-1}} (h, \text{Res}_{s = (n-1)/2} M_w \phi(h; P_{n,0}; s)).$$

**PROPOSITION 1.10.**

$$\text{Res}_{s = (n-1)/2} E_{P_{n,0}} (h; f^{(s)}) = \frac{1}{2} E_{P_{n,n-1}} (h; \text{Res}_{s = (n-1)/2} M_w f^{(s)}).$$

**Proof.** By Proposition 1.8, this equation holds for a non-zero class 1 vector. Since both sides are $H_n$-equivariant, it holds for any $f^{(s)}$. 
2. Triple L-functions

Let $k$ be a global field. Let $K$ be a semi-simple abelian algebra of degree 3 over $k$. There are three cases:

Case (1) $K = k \oplus k \oplus k$.
Case (2) $K = k \oplus k'$, $k'$ is a quadratic extension of $k$.
Case (3) $K = k''$, $k''$ is a cubic extension of $k$.

Let $G$ be an algebraic group defined over $k$ given by

$$G = \{ g \in \text{GL}_2(K) \mid \text{det } g \in k^\times \}.$$ 

Thus $G$ is

Case (1) $\{(g^{(1)}, g^{(2)}, g^{(3)}) \in (\text{GL}_2)^3 \mid \text{det } g^{(1)} = \text{det } g^{(2)} = \text{det } g^{(3)}\},$
Case (2) $\{(g^{(1)}, g^{(2)}) \in \text{GL}_2 \times R_{k'/k} \text{GL}_2 \mid \text{det } g^{(1)} = \text{det } g^{(2)}\},$
Case (3) $\{g \in R_{k''/k} \text{GL}_2 \mid \text{det } g \in k^\times\}.$

As in [22, §0], we take an 8-dimensional representation $\sigma$ of the $L$-group of $\text{GL}_2(K)$. The $L$-group is the semi-direct product of $\text{GL}_2(C) \times \text{GL}_2(C) \times \text{GL}_2(C)$ and $W_k$. $W_k$ acts by permuting the three $\text{GL}_2(C)$ factors. The restriction of $\sigma$ to $\text{GL}_2(C) \times \text{GL}_2(C) \times \text{GL}_2(C)$ is $\sigma_2 \otimes \sigma_2 \otimes \sigma_2$, where $\sigma_2$ is the standard 2-dimensional representation of $\text{GL}_2(C)$. The restriction of $\sigma$ to $W_k$ is the permutation of the three factors.

We denote by $Z$ the connected component of the center of $G$. $Z$ is naturally isomorphic to $\text{GL}_1$. We embed $G$ into

$$\text{GSp}_3 = \left\{ h \in \text{GL}_6 \mid h \begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}, h = m(h) \begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}, m(h) \in k^\times \right\}$$

as in [22, §1]. We denote this embedding by $i$.

Let $\Pi$ be an irreducible cuspidal automorphic representation of $\text{GL}_2(A \otimes K)$, i.e.,

Case (1) $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, where $\pi_1$, $\pi_2$, and $\pi_3$ are irreducible cuspidal automorphic representation of $\text{GL}_2(A_k)$,
Case (2) $\Pi = \pi_1 \otimes \pi_2$, where $\pi_1$ (resp. $\pi_2$) is an irreducible cuspidal automorphic representation of $\text{GL}_2(A_k)$ (resp. $\text{GL}_2(A_k')$),
Case (3) $\Pi$ is an irreducible cuspidal automorphic representation of $\text{GL}_2(A_k')$.

Let $\Omega_{\Pi}$ be the central quasi-character of $\Pi$, and $\omega_{\Pi}$ be the restriction of $\Omega_{\Pi}$ to
Z(A). Put $\omega = \omega_{II}$. Let $\mathcal{W}(\Pi, \psi)$ be the Whittaker model of $\Pi$, i.e.,

Case (1) $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi)$,
Case (2) $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi \circ tr_{k/k})$,
Case (3) $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\Pi, \psi \circ tr_{k/k})$.

If $\varphi$ is a cusp form belonging to $\Pi$, then there exists $W \in \mathcal{W}(\Pi, \psi)$ such that

$$\varphi(g) = \sum_{x \in K^v} W \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g.$$

We assume that $W$ is decomposable: $W = \Pi_v W_v$. Here, $v$ runs over all places of $k$. Put

$$P = \left\{ \begin{pmatrix} m & * \\ 0 & 1 \end{pmatrix} \in GSp_3 \right\}.$$

By [22, §1], the double cosets $P \backslash GSp_3 / I(G)$ contains one open coset and the other cosets are all negligible in the terminology of [20]. We choose a representative $\eta_0$ of the open double coset and put

$$R_0 = \{ g \in G | \eta_0 I(g) \eta_0^{-1} \in P \}.$$

We can choose $\eta_0$ so that

$$R_0 = \left\{ \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \in GL_2(K) | a \in k^*, tr_{k/k} n = 0 \right\}.$$

Let $v$ be a place of $k$. Let $J(\omega_v, s)$ be the space of functions $f_v(h)$ on $GSp_3(k_v)$ which satisfy the following (i) and (ii):

(i) $f_v$ is right finite by the standard maximal compact subgroup of $GSp_3(k_v)$.

(ii) For $p = \begin{pmatrix} m & * \\ 0 & 1 \end{pmatrix} \in P(k_v)$,

$$f_v(ph) = \omega_v(m) |m|^{3s+(3/2)} \omega_v(det A) |det A|^{2s+1} f_v(h).$$

Observe that if $f_v \in J(\omega_v, s)$, then $f_v|_{Sp_3(k_v)} \in I(\omega_v, 2s-1)$. We define holomorphic sections and meromorphic sections of $J(\omega_v, s)$ in the same way as in Section 1. The intertwining operator $M_w$ can be defined similarly. We define a meromorphic section $f_v^{(s)}$ is good if

$$[d(\omega_v, 2s-1)c_w(\omega_v, 2s-1)]^{-1} M_w f_v^{(s)}$$
is holomorphic for all \( w \in \Omega_3 \). Obviously this condition is equivalent to say that 
\[ \rho(\phi)f_v^{(s)}|_{GSp_3(k_v)} \] is a good section of \( I(\omega_v, 2s - 1) \) for each Hecke operator \( \phi \) on 
\( GSp_3(k_v) \). By Lemma 1.2, \( f_v^{(s)}(h) \) is a good section of \( J(\omega_v, s) \) if and only if
\[ \omega_v(m(h))M^*_w f_v^{(s)}(h) \] is a good section of \( J(\omega_v^{-1}, 1 - s) \), where \( m(h) \) is the multiplier of \( h \), and by Lemma 1.3, any holomorphic section of \( J(\omega_v, s) \) is a good section.

For each meromorphic section \( f_v^{(s)} \in J(\omega_v, s) \), and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \), put

\[
\Psi_s(f_v^{(s)}, W_v) = \int_{R_{\omega_v} \backslash G_v} f_v^{(s)}(\eta_v)(g)W_v(g) \, dg.
\]

In [7], [22], it is proved that \( \Psi_s(f_v^{(s)}, W_v) \) is absolutely convergent for \( \text{Re}(s) > 0 \), and has meromorphic continuation to \( \mathbb{C} \), and if \( v \) is non-archimedean, 
\( \Psi_s(f_v^{(s)}, W_v) \) is a rational function of \( q_v^{-s} \). By [22, Proposition 3.3], for each \( s_0 \in \mathbb{C} \), there exists a holomorphic section \( f_v^{(s)} \) of \( J(\omega_v, s) \), and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \) such that

\[ \Psi_{s_0}(f_v^{(s_0)}, W_v) \neq 0. \]

Put \( \tilde{W}_v(g) = \Omega_v (\det g)^{-1} W_v(g) \), where \( \Omega_v \) is the central quasi-character of \( \Pi_v \).

Then \( \tilde{W}_v \in \mathcal{W}(\Pi_v, \psi_v) \). It is proved in [7], [22], that there exists a meromorphic function \( \varepsilon(s, \Pi_v, \sigma, \psi_v) \) such that

\[ \Psi_{1-s}(\omega_v(m(h))M^*_w f_v^{(s)}, \tilde{W}_v) = \varepsilon(s, \Pi_v, \sigma, \psi_v) \Psi_s(f_v^{(s)}, W_v). \]

For a non-archimedean place \( v \), we consider the fractional ideal \( I_v \) of 
\( R_v = \mathbb{C}[q_v^{-s}, q_v^s] \), generated by \( \Psi_s(f_v^{(s)}, W_v) \) attached to good sections \( f_v^{(s)} \) of 
\( J(\omega_v, s) \) and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \). Then by [22, Appendix 3 to §3], \( I_v \) admits a common denominator and \( 1 \in I_v \). Thus \( I_v \) has a generator of the form \( P(q_v^{-s})^{-1} \),

\[ P(X) \in \mathbb{C}[X], \quad P(0) = 1. \]

We let

\[ L(s, \Pi_v, \sigma) = P(q_v^{-s})^{-1}, \]

\[ \varepsilon(s, \Pi_v, \sigma, \psi_v) = \varepsilon(s, \Pi_v, \sigma, \psi_v)L(s, \Pi_v, \sigma)L(1-s, \Pi_v, \sigma)^{-1}, \]

then \( \varepsilon(s, \Pi_v, \sigma, \psi_v) \) is of the form \( aq_v^{bs} \), \( a \in \mathbb{C}, \ b \in \mathbb{Z} \), and

\[
\frac{\Psi_{1-s}(\omega_v(m(h))M^*_w f_v^{(s)}, \tilde{W}_v)}{L(1-s, \Pi_v, \sigma)} = \varepsilon(s, \Pi_v, \sigma, \psi_v) \frac{\Psi_s(f_v^{(s)}, W_v)}{L(s, \Pi_v, \sigma)}. \tag{2.1}
\]

When \( v \) is unramified, this definition agrees to usual definition 
\( \text{det}(1_g - \sigma(g_v, Fr)q_v^{-s})^{-1} \), where \( g_v \) is the Langlands class of \( \Pi_v \). For a holomorphic section \( f_v^{(s)} \) and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \), a careful calculation of denominator of
Ψ_s(f_v^{(s)}; W_v) shows that the denominator divides \( \det(18 - \sigma(g_v, Fr)q_v^{-s}) \) (cf. [22, Appendix 3 to §3]). It follows that \( L(s, \Pi_v, \sigma)^{-1} \) is a divisor of \( d(\omega_v, 2s - 1)^{-1} \det(18 - \sigma(g_v, Fr)q_v^{-s}) \). On the other hand, there are a good section \( f_v^{(s)} \) of \( J(\omega_v, s) \) and \( W_v \in \mathcal{V}(\Pi_v, \psi_v) \) such that \( \Psi_s(f_v^{(s)}; W_v) = \det(18 - \sigma(g_v, Fr)q_v^{-s})^{-1} \). This shows that \( L(s, \Pi_v, \sigma)^{-1} \) is a multiple of \( \det(18 - \sigma(g_v, Fr)q_v^{-s}) \). Moreover we know

\[
\varepsilon'(s, \Pi_v, \sigma, \psi_v) = \frac{\det(18 - \sigma(g_v, Fr)q_v^{-s})}{\det(18 - \sigma(g_v, Fr)q_v^{s-1})}
\]

Since \( d(\omega_v, 2s - 1)^{-1} \) and \( d(\omega_v^{-1}, 1 - 2s)^{-1} \) have no common divisor, we have

\[
L(s, \Pi_v, \sigma) = \det(18 - \sigma(g_v, Fr)q_v^{-s})^{-1}
\]

as we expected.

When \( k_v \) is archimedean, we define L-factor \( L(s, \Pi_v, \sigma) \) as follows. The proof of [7, Proposition 5.1] shows that there is a meromorphic function \( \alpha(s) \neq 0 \) such that

\[
\alpha(s)^{-1}\Psi_s(f_v^{(s)}; W_v)
\]

is holomorphic for any holomorphic section \( f_v^{(s)} \) and \( W_v \in \mathcal{V}(\Pi_v, \psi_v) \). Though [7] has dealt with only case (1), it is not difficult to generalize the result to the case \( k_v = \mathbb{R}, K_v = \mathbb{R} \oplus \mathbb{C} \). We have only to use the local functional equation of Asai-type L-functions instead of the results of [8]. By Weierstrass theorem, there is a meromorphic function \( \lambda(s) \) such that

\[
\lambda(s)^{-1}\Psi_s(f_v^{(s)}; W_v)
\]

is holomorphic for any good section \( f_v^{(s)} \) and \( W_v \in \mathcal{V}(\Pi_v, \psi_v) \) and if \( \lambda'(s) \) is another function with this property, then \( \lambda(s)\lambda'(s)^{-1} \) is holomorphic. Obviously, for each \( s_0 \in \mathbb{C} \), there exists a good section \( f_v^{(s)} \) and \( W_v \in \mathcal{V}(\Pi_v, \psi_v) \) such that (2.2) does not have a zero at \( s = s_0 \). By Lemma 1.3 and [22, Proposition 3.3], \( \lambda(s) \) has no zeros. We define \( L(s, \Pi_v, \sigma) = \lambda(s) \). Then (2.1) holds with some entire function \( \varepsilon(s, \Pi_v, \sigma, \psi_v) \) which have no zeros. Note that \( L(s, \Pi_v, \sigma) \) and \( \varepsilon(s, \Pi_v, \sigma, \psi_v) \) is determined only up to entire functions which have no zeros.

Let \( v \) be any place of \( k \). Assume \( \Pi_v \) is unitary. We define a non-negative real number \( \lambda(\Pi_v) \) as follows.

Case (1) \( \Pi_v = \pi_1 \otimes \pi_2 \otimes \pi_3 \): When \( \pi_i \) is tempered, put \( \lambda(\pi_i) = 0 \). When \( \pi_i \) is the complementary series \( \pi(\mu \alpha^s, \mu \alpha^{-s}) \), \( (\mu) \) is a unitary character of \( k_v \), put \( \lambda(\pi_i) = |\lambda| \).

Put \( \lambda(\Pi_v) = \lambda(\pi_1) + \lambda(\pi_2) + \lambda(\pi_3) \).

Case (2) \( \Pi_v = \pi_1 \otimes \pi_2 \): let \( \lambda(\pi_i) \) be as above, and put \( \lambda(\Pi_v) = \lambda(\pi_1) + 2\lambda(\pi_2) \).

Case (3) \( \Pi_v = \pi_1 \): let \( \lambda(\pi_1) \) be as above, and put \( \lambda(\Pi_v) = 3\lambda(\pi_1) \).
LEMMA 2.1. If $\Pi_v$ is unitary, then $L(s, \Pi_v, \sigma)$ has no poles on the domain $\Re(s) > \lambda(\Pi_v)$.

Proof. By an argument similar to [7, Theorem 1], [22, Proposition 3.2], we can show that if $f_v^{(s)}$ is a holomorphic section of $J(\omega_v, s)$ and $W_v \in \mathcal{W}(\Pi_v, \psi_v)$, then $\Psi_s(f_v^{(s)}; W_v)$ is absolutely convergent for $\Re(s) > \lambda(\Pi_v)$. Since $d(\omega_v, s)$ has no poles for $\Re(s) > 0$, a good section $f_v^{(s)}$ is holomorphic for $\Re(s) > 0$. This proves the lemma.

LEMMA 2.2. Assume $K$ is not a cubic extension of $k$. Assume $\Pi_v$ is unitary. Assume each component is a subquotient of a principal series, and $\lambda(\Pi_v) < 1/2$. Then $L(s, \Pi_v, \sigma)$ (resp. $\varepsilon(s, \Pi_v, \sigma, \psi_v)$) agrees to $L$-factor (resp. $\varepsilon$-factor) associated to the 8-dimensional representation of the Weil group $W_{k_v}$ determined by $\Pi_v$ and $\sigma$.

Proof. By [7, Proposition 5.1], $\varepsilon'(s, \Pi_v, \sigma, \psi_v)$ coincides $\varepsilon'$-factor determined by the Weil group. The proof of [7] Proposition 5.1 works for case (2). By the assumption, $L(s, \Pi_v, \sigma)$ has no poles on the domain $\Re(s) > \lambda(\Pi_v)$ and $L(1-s, \Pi_v, \sigma)$ has no poles on the domain $\Re(s) < 1-\lambda(\Pi_v)$. This proves the lemma.

REMARK. By Lemma 2.2, we can identify the archimedean $L$-factors and usual $\Gamma$-factors if $\Pi$ is generated by Hilbert modular forms over a totally real field.

COROLLARY. Assume $K$ is not a cubic extension of $k$. Assume $\Pi_v$ is unitary. Assume no component is extraordinary, and $\lambda(\Pi_v) < 1/2$. Then the conclusion of Lemma 2.2 holds.

Proof. For simplicity, we assume $K = k \oplus k \oplus k$, $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$, and all of $\pi_{1,v}, \pi_{2,v}$ and $\pi_{3,v}$ are supercuspidal. $\pi_{i,v} = \pi(\chi_{i,v})$ for some quasi-character $\chi_{i,v}$ of some quadratic extension $K_{i,v}$ of $k_v$. Choose global quadratic extension $K_i$ of $k$ such that $K_{i,v} = K_{i,v}$. It is easy to check that there exists global quasi-character $\chi_i$ of $\mathbf{A}_K^*$ such that $v$-part of $\chi_i$ is $\chi_{i,v}$ and $\pi(\chi_i)$ is principal series outside of $v$ and all archimedean place. Put $\Pi = \pi(\chi_1) \otimes \pi(\chi_2) \otimes \pi(\chi_3)$. Then $L(s, \Pi, \sigma)$ is $L$-function associated to 8-dimensional representation of global Weil group. The conclusion of Lemma 2.2 holds outside $v$, so does at $v$.

We now consider the global theory. We say that a meromorphic section of $J(\omega, s)$ is a good section if it is a finite sum of decomposable elements $f^{(s)} = \Pi_v f_v^{(s)}$, satisfying the following two conditions:

(i) For almost all unramified places $v$, $f_v^{(s)}|_{K_v} = d(\omega_v, 2s - 1)$.
(ii) $f_v^{(s)}$ is a good section of $J(\omega_v, s)$ for all $v$.

Note that the infinite product $\Pi_v f_v^{(s)}$ is absolutely convergent for $\Re(s) > 0$, and can be meromorphically continued to $\mathbb{C}$. 
For each good section \( f^{(s)} \) of \( J(\omega, s) \), put

\[
E(h; f^{(s)}) = \sum_{\gamma \in F^1(\overline{\mathbf{Sp}_3})} f^{(s)}(\gamma h).
\]

Then the restriction of \( E(h; f^{(s)}) \) to \( \mathbf{Sp}_3(\mathbb{A}) \) is an Eisenstein series on \( \mathbf{Sp}_3(\mathbb{A}) \) investigated in Section 1.3. In [7], [22], it is proved that if \( f^{(s)} = \Pi_v f^{(s)}_v \) is decomposable, then

\[
\int_{Z(\mathbb{A})\mathbf{G}(k) \cdot \mathbf{G}(\mathbb{A})} E(t(g); f^{(s)}) \varphi(g) \, dg = \prod_v \Psi_s(f^{(s)}_v, W_v),
\]

(2.3)

for \( \text{Re}(s) \gg 0 \). Set

\[
L(s, \Pi, \sigma) = \prod_v L(s, \Pi_v, \sigma)
\]

and

\[
\varepsilon(s, \Pi, \sigma) = \prod_v \varepsilon(s, \Pi_v, \sigma, \psi_v).
\]

Then by Proposition 1.6, (2.1), and (2.3), we have the following propositions.

PROPOSITION 2.3. \( L(s, \Pi, \sigma) \) can be meromorphically continued to \( \mathbb{C} \). It is entire if \( \omega^2 \) is not a principal quasi-character. If \( \omega^2 = 1 \), and \( k \) is a number field, then \( L(s, \Pi, \sigma) \) has possible poles at \( s = 0, 1 \). If \( \omega^2 = 1 \), and \( k \) is a function field with constant field \( \mathbb{F}_q \), then \( L(s, \Pi, \sigma) \) has possible poles at

\[
s \in \frac{\pi \sqrt{-1}}{2 \log q} \mathbb{Z}, 1 + \frac{\pi \sqrt{-1}}{2 \log q} \mathbb{Z}.
\]

All the possible poles are at most simple.

PROPOSITION 2.4. \( L(s, \Pi, \sigma) \) satisfies the following functional equation:

\[
L(s, \Pi, \sigma) = \varepsilon(s, \Pi, \sigma)L(1 - s, \overline{\Pi}, \sigma).
\]

Now we investigate the poles of \( L(s, \Pi, \sigma) \). By Proposition 2.3, we may assume \( \omega^2 = 1 \) and \( s = 0 \) or \( 1 \). By the functional equation, \( s = 0 \) is reduced to \( s = 1 \). If \( L(s, \Pi, \sigma) \) has a pole at \( s = 1 \), then there exists a good section \( f^{(s)} \) of \( J(\omega, s) \) and a cusp form \( \varphi \) belonging to \( \Pi \) such that

\[
\int_{Z(\mathbb{A})\mathbf{G}(k) \cdot \mathbf{G}(\mathbb{A})} [\text{Res}_{s=1} E(t(g); f^{(s)})] \varphi(g) \, dg \neq 0.
\]

(2.4)

PROPOSITION 2.5. If \( \omega = 1 \), then \( L(s, \Pi, \sigma) \) is holomorphic at \( s = 1 \). In
particular, if \( k \) is a number field, \( L(s, \Pi, \sigma) \) is entire (cf. [22, Theorem 5.1]).

Proof. By Proposition 1.10, the restriction of \( \text{Res}_{s=1} E(h; f^{(a)}) \) to \( \text{Sp}_3 \) is an Eisenstein series associated to a function in the representation induced from the trivial character of the maximal parabolic subgroup \( P_{3,2} \). It is easy to see that each coset in \((i(G) \cap \text{Sp}_3) \setminus \text{Sp}_3 / P_{3,2} \) is negligible. It follows that (2.4) is identically zero.

We now assume that \( \omega^2 = 1, \omega \neq 1 \) and \( L(s, \Pi, \sigma) \) has a pole at \( s = 1 \). Let \( K \) be the quadratic extension of \( k \) corresponding to \( \omega \) by class field theory, and \( \theta \) be the non-trivial element of \( \text{Gal}(K/k) \).

Suppose that \( k' = k'' \) is a cubic extension of \( k \). Let \( \Pi_K \) be the base change of \( \Pi \) to \( \text{GL}_2(A'_k K) \) (cf. [18]). Consider the triple \( L \)-function \( L(s, \Pi_K, \sigma_K) \) of \( \Pi_K \) over \( K \). Here, \( \sigma_K \) is the restriction of \( \sigma \) to the semi-direct product of \( \text{GL}_2(C) \times \text{GL}_2(C) \times \text{GL}_2(C) \) and \( W_K \). Then an easy calculation shows

\[
L(s, \Pi_K, \sigma_K) = L(s, \Pi \otimes \vec{\omega}, \sigma)L(s, \Pi, \sigma).
\]

Here, \( \vec{\omega} \) is any extension of \( \omega \) to \( A'_k \). Note that \( G \) is a Levi subgroup of the quasi-split simply connected group \( \text{Spin}(8) \) of either type \( ^3D_4 \) or \( ^6D_4 \) according as \( k''/k \) is cyclic or not (see Shahidi [23]). Then [23, Theorem 5.1] implies

\[
L(1 + 2s, \omega)L(1 + s, \Pi \otimes \vec{\omega}, \sigma) \neq 0
\]

for \( \text{Re}(s) = 0 \). Since \( \omega \) is a non-trivial unitary character of \( A'_k \), this implies the non-vanishing of \( L(s, \Pi, \sigma) \) at \( s = 1 \). So, \( L(s, \Pi_K, \sigma_K) \) has a pole at \( s = 1 \). But since \( \omega_{\Pi_K} = 1, \Pi_K \) cannot be cuspidal by Proposition 2.5. It follows that there is a quasi-character \( \chi \) of \( A'_k K \) such that \( \Pi = \pi(\chi) \). By a simple calculation, the triple \( L \)-function \( L(s, \pi(\chi), \sigma) \) is given by

\[
L(s, \pi(\chi), \sigma) = L_K(s, \pi|_{A'_k K})L_{k''/k}(s, (\chi \circ N_{k''/k})\chi^{-1} \chi^\theta).
\]

Here, \( \theta \) is regarded as an element of \( \text{Gal}(k''K/k'' \cap k''/k'' \cap k) \), by the natural isomorphism \( \text{Gal}(k''K/k'') \cong \text{Gal}(K/k) \). This equality holds up to bad prime factors. But in fact, (2.5) is an equality of global \( L \)-functions. To see this, observe that

\[
\prod_{\text{res} \Sigma} \epsilon(s, \Pi_v, \sigma, \psi_v)
\]

has no zero on \( \text{Re}(s) > 0 \), and has no poles on \( \text{Re}(s) < 1 \), by comparing the functional equation as a triple \( L \)-function and that as a \( L \)-function associated to 8-dimensional representation of the Weil group. By Lemma 2.1,

\[
\prod_{\text{res} \Sigma} L(s, \Pi_v, \sigma)
\]
coincides with the product of L-factors of the right-hand side, since $\lambda(\Pi_\pi)=0$ for $\Pi=\pi(\chi)$. It follows that (2.5) is an equality of global L-functions.

Let us prove $\chi|_{\Lambda^*_{k}}=1$. First observe that $\chi|_{\Lambda^*_{k}}=1$, since $\omega\pi(\omega)=\omega\cdot\chi|_{\Lambda^*_{k}}$. Suppose $\chi|_{\Lambda^*_{k}}\neq1$. Then $L_{\kappa,K}(s, (\chi\circ N_{\kappa,K/K})\chi^{-1}\chi)$ has a pole at $s=1$, therefore we have

$$\chi\circ N_{\kappa,K/K}=\chi(\chi)^{-1}.$$ 

Put $I=\text{Im}(N_{\kappa,K/K}:\Lambda_{\kappa,K} \to \Lambda_{\kappa,K})$. Then the index $[\Lambda_{\kappa,K}:\Lambda_{\kappa,K}]$ is 1 or 3, by the class fields theory. Let $y\in\Lambda_{\kappa,K}$, $x=N_{\kappa,K/K}(y)$. Then

$$\chi(x)=\chi(y)\chi(x^{-1})$$

$$=\chi(x^{-1}).$$

It follows that

$$\chi(x^3)=\chi(N_{\kappa,K/K}(x))$$

$$=\chi(x^3)\chi(x)^{-1}$$

$$=\chi(x^2).$$

So $\chi$ is trivial on $I\cdot K^\times$. It follows that $\chi|_{\Lambda^*_{k}}=1$, since $I\cdot K^\times\cdot\Lambda_{\kappa,K}^\times=\Lambda_{\kappa,K}^\times$. Thus we have proved the following theorem.

**THEOREM 2.6.** Suppose that $K=k''$, $k''$ is a cubic extension of $k$, and $L(s, \Pi, \sigma)$ has a pole somewhere. Then

(a) Let $\Pi', \omega'$ be the objects obtained by twisting $\pi_1$ by $\omega'$, $s_0\in\mathbb{C}$. Then $\omega'^2=1$, $\omega' \neq 1$, and $L(s, \Pi', \sigma)$ has a simple pole at $s=1$, for some $s_0\in\mathbb{C}$.

(b) Assume that $\omega^2=1$, $\omega \neq 1$, and $L(s, \Pi, \sigma)$ has a pole at $s=1$. Let $K$ be the quadratic extension of $k$ corresponding to $\omega$ by class field theory. Let $\theta$ be the non-trivial element of $\text{Gal}(k''/K'')$. Then there exists a quasi-character $\chi$ of $\Lambda_{k,K}^\times/k''K^\times$ such that $\Pi=\pi(\chi)$ and $\chi|_{\Lambda_{k,K}}=1$. Moreover the triple L-function is given by

$$L(s, \pi(\chi), \sigma)=\zeta_k(s)L_{k''}(s, \chi^{-1}\chi).$$

Next, suppose that $K=k\oplus k \oplus k$, $\Pi=\pi_1 \otimes \pi_2 \otimes \pi_3$. By the assumption, $\omega_1\omega_2\omega_3=\omega$. Let $\pi_{i,K}$ ($i=1, 2, 3$) be the base change of $\pi_i$ to $\text{GL}_2(\mathbb{A}_k)$. Put $\Pi_K=\pi_{1,K} \otimes \pi_{2,K} \otimes \pi_{3,K}$. Then,

$$L(s, \Pi_K, \sigma_K)=L(s, \Pi \otimes \omega, \sigma)L(s, \Pi, \sigma).$$
Here, $\Pi \otimes \omega$ means $(\pi_1 \otimes \omega) \otimes \pi_2 \otimes \pi_3$. As is case (3), the left-hand side has a pole at $s=1$, and $\omega_{\Pi_\chi} = 1$. This time, we can deduce that one of $\pi_{i,K}$ ($i = 1, 2, 3$), say $\pi_{1,K}$, is not cuspidal. So there is a quasi-character $\chi$ of $A^\times_K / K^\times$ such that $\pi_1 = \pi(\chi)$. Observe that $\chi|_{A^\times_\chi} = \omega_2^{-1} \omega_3^{-1}$, since the central quasi-character of $\pi(\chi)$ is $\omega \cdot \chi|_{A^\times_\chi}$. The triple L-function $L(s, \Pi, \sigma)$ is given by

$$L(s, \Pi, \sigma) = L_K(s, (\pi_{2,K} \otimes \chi) \times \pi_{3,K}).$$

Let us now prove that neither $\pi_{2,K}$ nor $\pi_{3,K}$ are cuspidal. Suppose that $\pi_{2,K}$ or $\pi_{3,K}$, say $\pi_{2,K}$, is cuspidal. Then

$$\pi_{2,K} \otimes \chi \simeq \tilde{\pi}_{3,K}.$$  \hspace{1cm} (2.6)

In particular, $\pi_{3,K}$ is cuspidal, too. Since $\pi_{2,K}$ and $\pi_{3,K}$ are $\theta$-invariant,

$$\pi_{2,K} \otimes \chi^\theta \simeq \tilde{\pi}_{3,K}.$$  \hspace{1cm} (2.7)

Put $\varepsilon = \chi(\chi^\theta)^{-1}$. Since $\pi(\chi)$ is cuspidal, $\varepsilon \neq 1$. By (2.6) and (2.7), we have $\pi_{2,K} \otimes \varepsilon \simeq \pi_{2,K}$. It follows that $\varepsilon^2 = 1$. Since $\varepsilon^\theta = \varepsilon^{-1} = \varepsilon$, there is a character $\varepsilon'$ of $A^\times_K / k^\times$ such that $\varepsilon = \varepsilon' \circ N_{K/k}$. Taking the central quasi-character of (2.6), we have

$$(\omega_2 \circ N_{K/k}) \chi_2^2 = (\omega_3 \circ N_{K/k})^{-1}.$$

Put $I = \text{Im}(N_{K/k} : A^\times_K \to A^\times_k)$. Let $y \in A^\times_K$, $x = N_{K/k}(y)$. Then

$$\omega_2(x) = \omega_3(x)^{-1} \chi(y)^{-2}$$
$$= \omega_3(x)^{-1} \chi(y)^{-1} \chi(y^\theta)^{-1} \varepsilon(y)$$
$$= \omega_3(x)^{-1} \chi(x)^{-1} \varepsilon'(x).$$

It follows that

$$\omega_1(x) \omega_2(x) \omega_3(x) = \chi(x) \omega(x) \omega_3(x)^{-1} \chi(x)^{-1} \varepsilon(x) \omega_3(x)$$
$$= \omega(x) \varepsilon'(x).$$

This contradicts to the assumption $\omega_1 \omega_2 \omega_3 = \omega$, since $\varepsilon'$ is not trivial on $I$.

We have proved that there are quasi-characters $\chi_i$ ($i = 1, 2, 3$) of $A^\times_K$ such that $\pi_i = \pi(\chi_i)$. The triple L-function is given by

$$L(s, \Pi, \sigma) = L_K(s, \chi_1 \chi_2 \chi_3) L_K(s, \chi_1 \chi_2 \chi_3) L_K(s, \chi_1 \chi_2 \chi_3) L_K(s, \chi_1 \chi_2 \chi_3).$$
In this case, this equality holds for every local $L$-factor, by Lemma 2.2. Replacing $\chi_i$ by $\chi_i^\theta$ if necessary, we have $\chi_1 \chi_2 \chi_3 = 1$. We have proved the following theorem.

**THEOREM 2.7.** Suppose that $K = k \oplus k \oplus k$, and $L(s, \Pi, \sigma)$ has a pole somewhere. Then the following two assertions hold:

(a) Let $\Pi'$, $\omega'$ be the objects obtained by twisting $\pi_1$ by $\pi_0$, $s_0 \in \mathbb{C}$. Then $\omega'^2 = 1$, $\omega' \neq 1$, and $L(s, \Pi', \sigma)$ has a simple pole at $s = 1$, for some $s_0 \in \mathbb{C}$.

(b) Assume that $\omega^2 = 1$, $\omega \neq 1$, and $L(s, \Pi, \sigma)$ has a pole at $s = 1$. Let $K$ be the quadratic extension of $k$ corresponding to $\omega$ by class field theory. Let $\theta$ be the generator of $\text{Gal}(K/k)$. Then there exist quasi-characters $\chi_1$, $\chi_2$, and $\chi_3$ of $\mathbb{A}_K^\times /K^\times$ such that $\pi_1 = \pi(\chi_1)$, $\pi_2 = \pi(\chi_2)$, $\pi_3 = \pi(\chi_3)$, and $\chi_1 \chi_2 \chi_3 = 1$. Moreover, the triple $L$-function is equal to

$$\zeta_K(s)L_K(s, \chi_1^{-1}\chi_1^\theta)L_K(s, \chi_2^{-1}\chi_2^\theta)L_K(s, \chi_3^{-1}\chi_3^\theta).$$

Now, suppose that $K = k \oplus k'$, $k'$ is a quadratic extension of $k$, $\Pi = \pi_1 \otimes \pi_2$. Let $\omega_1$ and $\omega_2$ be the central quasi-characters of $\pi_1$ and $\pi_2$, respectively. By the assumption, $\omega_1 \cdot (\omega_2|A_k) = \omega$.

We first prove $K \neq k'$. Assume that $K = k'$. In this case we have, as in case (3),

$$L(s, \Pi \otimes \omega, \sigma)L(s, \Pi, \sigma) = L_K(s, \pi_{1,K} \times \pi_2 \times \pi_2^\theta),$$

and this has a pole at $s = 1$. Here, $\Pi \otimes \omega$ means $(\pi_1 \otimes \omega) \otimes \pi_2$. As in case (3), we can prove that $\pi_{1,K}$ is not cuspidal. It follows that there is a quasi-character $\chi$ of $K$ such that $\pi_1 = \pi(\chi)$. Then

$$L(s, \Pi, \sigma) = L_K(s, (\pi_2 \otimes \chi) \times \pi_2^\theta).$$

Therefore we have $\pi_2 \otimes \chi \simeq \pi_2^\theta$. Then $\pi_2 \otimes \varepsilon \simeq \pi_2$, where $\varepsilon = \chi(\chi^\theta)^{-1}$. As in case (1), we can prove that $\varepsilon^2 = 1$, $\varepsilon \neq 1$, $\varepsilon^\theta = \varepsilon$ and that there is a character $\varepsilon'$ of $\mathbb{A}_K^\times /k^\times$ such that $\varepsilon = \varepsilon' \circ N_{K/k}$. Taking the central character of $\pi_2 \otimes \chi \simeq \pi_2^\theta$, we have

$$\omega_2^2 \chi^2 = (\omega_2^\theta)^{-1}.$$

Let $I$, $x$ and $y$ be as in the case (1). Then

$$\omega_2(y) = \omega_2(y^\theta)^{-1}\chi(y)^{-2}$$

$$= \omega_2(y^\theta)^{-1}\chi(y)^{-1}\chi(y^\theta)^{-1}\vartheta(y)$$

$$= \omega_2(y^\theta)^{-1}\chi(x)^{-1}\varepsilon'(x).$$
It follows that
\[
\omega_1(x)\omega_2(x) = \chi(x)\omega(x)\omega_2(y y'^{\theta}) = \chi(x)\omega(x)\chi(x)^{-1}e'(x) = \omega(x)e'(x).
\]

This contradicts to the assumption $\omega_1 \cdot \omega_2|_{\Lambda_1^*} = \omega$, since $e'$ is non-trivial on $I$.

Thus we have proved $K \neq k'$.

Suppose $K \neq k'$. Let $\pi_{1,K}$ (resp. $\pi_{2,K}$) be the base change of $\pi_1$ (resp. $\pi_2$) to $GL_2(\mathbb{A}_K)$ (resp. $GL_2(\mathbb{A}_{k'K})$). In this case we can prove that at least one of $\pi_{1,K}$ and $\pi_{2,K}$ is not cuspidal as in case (1). We first prove that actually $\pi_{2,K}$ is not cuspidal. Suppose that $\pi_{2,K}$ is cuspidal. Then $\pi_{1,K}$ is not cuspidal, so there is a quasi-character $\chi$ of $\mathbb{A}_K^*$ such that $\pi_1 = \pi(\chi)$. Then the triple $L$-function is given by the Asai-$L$-function of $\pi_{2,K}$ twisted by $\chi$:

\[
L(s, \Pi, \sigma) = L_K(s, \pi_{2,K}, \chi)_{\text{Asai}}.
\]

Let $\eta$ be the character of $\mathbb{A}_{k'}^*/K^*$ corresponding to $k'K/K$ by class field theory. Then

\[
L_K(s, (\pi_{2,K} \boxtimes \chi) \times \pi_{2,K}^{\theta}) = L_K(s, \pi_{2,K}, \chi)_{\text{Asai}}L_K(s, \pi_{2,K}, \chi\eta)_{\text{Asai}}.
\]

Since $L_K(s, \pi_{2,K}, \chi\eta)_{\text{Asai}}$ is the triple $L$-function for $\pi(\chi\eta) \times \pi_2$, it does not have a zero at $s = 1$, so $L_K(s, (\pi_{2,K} \boxtimes \chi) \times \pi_{2,K}^{\theta})$ has a pole at $s = 1$. As in the case $K = k'$, this is impossible.

Thus $\pi_{2,K}$ is not cuspidal, so $\pi_2 = \pi(\chi)$ for some quasi-character $\chi$ of $\mathbb{A}_{k'}^*$. The triple $L$-function is given by

\[
L(s, \Pi, \sigma) = L(s, \pi_1 \times \pi(\chi|_{\mathbb{A}_{k'}^*}))L(s, \pi_1 \times \pi(\chi|_{\mathbb{A}_k^*}))
\]

up to finite number of Euler factors. Here, $K'$ is the quadratic extension of $k$, contained in $k'K$ different from $K$ and $k'$.

It follows that $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbb{A}_k^*})$ or $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbb{A}_{k'}^*})$, but the latter is impossible for the following reason. First we observe the central quasi-character of $\pi(\chi)$, $\pi(\chi^{-1}|_{\mathbb{A}_k^*})$, and $\pi(\chi^{-1}|_{\mathbb{A}_{k'}^*})$ are $\chi|_{\mathbb{A}_k^*} \cdot \omega_{k'/k}$, $\chi^{-1}|_{\mathbb{A}_k^*} \cdot \omega$, and $\chi^{-1}|_{\mathbb{A}_{k'}^*} \cdot \omega_{k'/k}$, respectively. Here, $\omega_{k'/k}$ (resp. $\omega_{k'/k}$) is the character of $\mathbb{A}_{k'/k}^*$ (resp. $\mathbb{A}_k^*/k^*$) of order 2 corresponding to $k'K/k'$ (resp. $K'/k$) by class field theory. If $\pi_1 \simeq \pi(\chi^{-1}|_{\mathbb{A}_k^*})$, we have

\[
\omega_1(x)\omega_2(x) = \chi^{-1}(x)\omega_{k'/k}(x)\chi(x)\omega_{k'/k}(x) = \omega_{k'/k}(x).
\]
This contradicts to the assumption $\omega_1 \cdot (\omega_2|_{A_k}) = \omega$, so one cannot have $\pi_1 \simeq \pi(\chi^{-1}|_{A_k})$.
Suppose $\pi_1 \simeq \pi(\chi^{-1}|_{A_k})$, and $\pi_2 \simeq \pi(\chi)$. Then an easy calculation shows that the triple $L$-function is equal to

$$\zeta_K(s) L_K(s, (\chi^{-1}\theta)|_{A_k}) L_K(s, \chi^{-1}\theta).$$

Here, $\theta$ is regarded as an element of $\text{Gal}(k'K/k')$, by the natural isomorphism $\text{Gal}(k'K/k') \simeq \text{Gal}(K/k)$. As in case (1), this equation holds for all place $v$.

Thus we have proved the following theorem.

**THEOREM 2.8.** Suppose that $K = k \oplus k'$, $k'$ is a quadratic extension of $k$, and $L(s, \Pi, \sigma)$ has a pole somewhere. Then the following two assertions hold:

(a) Let $\Pi'$, $\omega'$ be the objects obtained by twisting $\Pi$ by $z_0$, $s_0 \in \mathbb{C}$. Then $\omega'^2 = 1$, $\omega' \not\equiv 1$, $\omega'$ does not correspond to $k'/k$ by class field theory, and $L(s, \Pi', \sigma)$ has a simple pole at $s = 1$, for some $s_0 \in \mathbb{C}$.

(b) Assume that $\omega^2 = 1$, $\omega \not\equiv 1$, $\omega$ does not correspond to $k'/k$ by class field theory, and $L(s, \Pi, \sigma)$ has a simple pole at $s = 1$. Let $K$ be the quadratic extension of $k$ corresponding to $\omega$ by class field theory. Let $\theta$ be the generator of $\text{Gal}(k'K/k')$. Then there exists a quasi-character $\chi$ of $A_{k/K}^\times$ such that $\pi_1 \simeq \pi(\chi^{-1}|_{A_k})$, and $\pi_2 = \pi(\chi)$. Moreover, the triple $L$-function is equal to

$$\zeta_K(s) L_K(s, (\chi^{-1}\theta)|_{A_k}) L_K(s, \chi^{-1}\theta).$$

**References**

15. R. P. Langlands: *Euler products*, Yale University, New Haven.