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On the location of poles of the triple L-functions

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Introduction

Let K be a semi-simple abelian algebra of degree 3 over a global field k. In [22], I.I. Piatetski-Shapiro and S. Rallis constructed the triple L-functions for irreducible cuspidal automorphic representations of GL₂(K ⊗ A_k) by means of Rankin-type integrals following P. B. Garrett [3]. The purpose of this paper is to determine the location of the poles of these L-functions. To describe our main result, assume, for simplicity, K = k ⊕ k ⊕ k. Let α be the standard idele norm: A_k → R⁺. Given three irreducible cuspidal automorphic representations π_1, π_2, and π_3 of GL₂(A_k), let ω be the product of the central quasi-characters of these representations. Let σ be the 8-dimensional representation of the L-group GL₂(C)^3 obtained by the tensor product of the standard representations of GL₂(C). The triple L-function L(s, π, σ) is the L-function associated to \Pi = π_1 ⊗ π_2 ⊗ π_3 and σ. This is defined by the Euler product:

$$L(s, \Pi, \sigma) = \prod_v L(s, \Pi_v, \sigma).$$

If k_v is non-archimedean and \Pi_v is of class 1, then

$$L(s, \Pi_v, \sigma) = \det(1 - A_1 \otimes A_2 \otimes A_3 \cdot q_v^{-s})^{-1},$$

where q_v is the order of the residue field of k_v, and A_i is the Langlands class of \pi_{i,v} (i = 1, 2, 3). Then our main theorem in the case K = k ⊕ k ⊕ k can be stated as follows.

**THEOREM 2.7.** Suppose that K = k ⊕ k ⊕ k, and L(s, \Pi, \pi) has a pole somewhere. Then the following two assertions hold:

(a) Let \Pi', \omega' be the objects obtained by twisting \pi_1 by \omega^{s_0}, s_0 ∈ C. Then \omega'^2 = 1, \omega' ≠ 1, and L(s, \Pi', \sigma) has a simple pole at s = 1, for some s_0 ∈ C.

(b) Assume that \omega^2 = 1, \omega ≠ 1, and L(s, \Pi, \sigma) has a pole at s = 1. Let \mathcal{K} be the

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quadratic extension of $k$ corresponding to $\omega$ by class field theory. Let $\theta$ be the generator of $\text{Gal}(K/k)$. Then there exist quasi-characters $\chi_1$, $\chi_2$, and $\chi_3$ of $\mathbb{A}_k^\times/K^\times$ such that $\pi_1 = \pi(\chi_1)$, $\pi_2 = \pi(\chi_2)$, $\pi_3 = \pi(\chi_3)$, and $\chi_1\chi_2\chi_3 = 1$. Moreover, the triple $L$-function is equal to

$$\zeta_K(s)\zeta_k(s, \chi_1^{-1}\chi_1^0)\zeta_k(s, \chi_2^{-1}\chi_2^0)\zeta_k(s, \chi_3^{-1}\chi_3^0).$$

Note that our results are consistent with "the Langlands philosophy". Assume that for each $\pi_i$, there is a 2-dimensional complex representation $\rho_i$ of $\text{Gal}(k/k)$ such that $L(s, \pi_i) = L(s, \rho_i)$. Then our main theorem implies that, up to twist by $\chi^s$ for some $s_0 \in \mathbb{C}$, $L(s, \Pi, \sigma)$ has a pole if and only if $\rho_1 \otimes \rho_2 \otimes \rho_3$ has a trivial constituent.

A significant point of this result is its possible application to the construction of the lift $\text{GL}_2 \times \text{GL}_2 \rightarrow \text{GL}_4$ of automorphic representations by means of "the converse theorem". The author hopes to treat this problem in the future.

Let us now describe the contents of this paper. Section 1 is devoted to the theory of Eisenstein series on symplectic group $\text{Sp}_n$. Assume, for simplicity, $k$ is a number field. Consider the representation space $I(\omega, s)$ of the representation $\text{Ind}_{P_n}^{\text{Sp}_n} \omega \otimes^\sigma$ induced from a quasi-character $\omega$ of the parabolic subgroup

$$P_n = \left\{ \begin{pmatrix} A & * \\ 0_n & A^{-1} \end{pmatrix} \in \text{Sp}_n \right\}$$

of $\text{Sp}_n$. Let $f^{(s)}$ be a meromorphic section of $I(\omega, s)$, which roughly means that $f^{(s)}$ belongs to $I(\omega, s)$ for each $s \in \mathbb{C}$ and is meromorphic in $s$. In order to make use of the Rankin-Selberg convolution, we require that the family $\{f^{(s)}\}$ has the following properties:

(i) $E(h; f^{(s)})$ has finite number of poles.
(ii) The family $\{f^{(s)}\}$ is stable under the intertwining operator $M_{w_0}$ with respect to the longest Weyl group element $w_0$.
(iii) The family $\{f^{(s)}\}$ is the restricted tensor product of families of meromorphic sections $\{f_v^{(s)}\}$ of induced representations $I(\omega_v, s)$ on $\text{Sp}_n(k_v)$.
(iv) The family $\{f_v^{(s)}\}$ contains all holomorphic sections.

Moreover, to get a good local functional equation, we need a normalization $M_{w_0}^*$ of the local intertwining operator such that

(v) $M_{w_0}^* \circ M_{w_0}^* = \text{const}$.
(vi) The family $\{f_v^{(s)}\}$ is stable under the normalized intertwining operator $M_{w_0}^*$. 

We shall construct this normalized intertwining operator, and the family \( \{f^{(s)}(t)\} \) in Section 1.2. A function \( f^{(s)} \) in this family is called a good section. Our normalized intertwining operator is different from Langlands's normalization [16, Appendix 2]. In Section 1.3 we shall determine the location of the poles of the Eisenstein series \( E(h; f^{(s)}) \) associated to a good section \( f^{(s)} \). In Section 1.4 we calculate the residue of the Eisenstein series \( E(h; f^{(s)}) \) at \( s = \frac{n-1}{2} \).

Section 2 is devoted to the theory of the triple L-functions. We shall define the local \( L \)-factor and \( \varepsilon \)-factor, and give the functional equation for the triple \( L \)-functions. The location of the poles is then determined. The key lemma is that if \( \omega = 1 \), then \( L(s, \Pi, \sigma) \) does not have a pole at \( s = 1 \) (Proposition 2.5). The main theorem will be proved by showing that the base change of \( \Pi \) to \( GL_2(A_K)^3 \) is not cuspidal.

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Notation

The \( n \times n \) zero and identity matrices are denoted by \( 0_n \) and \( 1_n \), respectively. If \( X \) is a matrix, \( \det X \) stands for its determinant. For a function \( f \) on a group \( G \) and \( x \in G \), we denote by \( \rho(x)f \) the right translation of \( f \) by \( x \), i.e., \( \rho(x)f(y) = f(yx) \). When \( G \) is locally compact, the Schwartz-Bruhat space of \( G \) is denoted by \( \mathcal{S}(G) \). If \( G \) is an algebraic group defined over a field \( k \), the group of \( k \)-valued points of \( G \) is denoted by \( G(k) \) or \( G \). If \( n \) is a representation of \( G \), its contragredient is denoted by \( \overline{n} \). When \( k \) is a global field, the adele ring (resp. the idele group) of \( k \) is denoted by \( A_k \) or \( A \) (resp. \( \mathcal{A}_k \) or \( \mathcal{A} \)). We fix a non-trivial additive character \( \psi \) of \( A/k \) (resp. \( k \)), if \( k \) is a global field (resp. local field). The standard idele norm: \( A_k^\times \rightarrow \mathbb{R}_+^\times \) is denoted by \( \| \) or \( \alpha \). When \( k \) is a local field, the normalized absolute value: \( k^\times \rightarrow \mathbb{R}_+^\times \) is denoted by \( \| \) or \( \alpha \). When \( k \) is a global (resp. local) field, a quasi-character \( \chi \) of \( A^\times \) (resp. \( k^\times \)) is called principal if \( \chi = \alpha^{s_0} \) for some \( s_0 \in \mathbb{C} \). When \( k \) is a global function field, the order of the coefficient field of \( k \) is denoted by \( q \). When \( k \) is a non-archimedean local field, \( \mathcal{O} \), \( \mathfrak{m} \), and \( q \) are the maximal order of \( k \), a prime element of \( \mathcal{O} \), and the order of the residue field of \( k \), respectively. The multiplicative Haar measure \( d^\times x \) of \( k^\times \) is normalized so that \( \text{Vol}(\mathcal{O}^\times) = 1 \).
1. Analytic theory of Eisenstein series

1.1. Definitions

Let $H_n$ be the symplectic group $\text{Sp}_n$:

$$H_n = \text{Sp}_n$$

$$= \left\{ h \in \text{GL}_{2n} \left| h \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}^t h = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \right. \right\}.$$ 

We define parabolic subgroups $P_n$ and $B_n$ of $H_n$ by

$$P_n = \left\{ \begin{pmatrix} A & * \\ 0_n & tA^{-1} \end{pmatrix} \in H_n \right\},$$

$$B_n = \left\{ \begin{pmatrix} A & * \\ 0_n & tA^{-1} \end{pmatrix} \in P_n \left| A \right. \text{ is upper triangular} \right\}.$$ 

Let $M_m$ (resp. $T_n$) be a Levi factor of $P_n$ (resp. $B_n$) given by

$$M_n = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & tA^{-1} \end{pmatrix} \left| A \in \text{GL}_n \right. \right\},$$

$$T_n = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & tA^{-1} \end{pmatrix} \left| A \text{ is diagonal} \right. \right\}.$$ 

We denote by $U_n$ (resp. $N_n$) the unipotent radical of $P_n$ (resp. $B_n$):

$$U_n = \left\{ \begin{pmatrix} 1_n & B \\ 0_n & 1_n \end{pmatrix} \left| B = tB \right. \right\},$$

$$N_n = \left\{ \begin{pmatrix} A & * \\ 0_n & tA^{-1} \end{pmatrix} \in H_n \left| A \text{ is unipotent upper triangular} \right. \right\}.$$ 

Let $P_n^-$ and $B_n^-$ be the opposite parabolic subgroups of $P_n$ and $B_n$, respectively. We denote by $U_n^-$ (resp. $N_n^-$) the unipotent radical of $P_n^-$ (resp. $B_n^-$).
Let $x_i$ ($1 \leq i \leq n$) be the character of $T_n$ given by

$$
\begin{pmatrix}
    t_1 \\
    \vdots \\
    t_n \\
    t_1^{-1} \\
    \vdots \\
    t_n^{-1}
\end{pmatrix} \mapsto t_i.
$$

Let $\text{Norm}(T_n)$ be the normalizer of $T_n$ in $H_n$. We denote the Weyl group $\text{Norm}(T_n)/T_n$ by $W_{H_n}$. We shall often use the same symbol for an element of $\text{Norm}(T_n)$ and its image in $W_{H_n}$. Let $\Phi_{H_n}$ (resp. $\Phi_{M_n}$) be the set of roots of $H_n$ (resp. $M_n$) with respect to $T_n$. We denote by $N_a$ the unipotent group associated to a root $\alpha \in \Phi_{H_n}$. Each $N_a$ is isomorphic to $k$ in the natural way (by the coordinate). We denote by $w_a$ the reflection determined by $\alpha$. Let $\alpha_i$ be the simple root:

$$
\alpha_i = x_i - x_{i+1}, \quad (1 \leq i \leq n-1)
$$

$$
\alpha_n = 2x_n.
$$

Let $\Omega_n$ be the complete set of representatives for $W_{H_n}/W_{M_n}$ obtained by choosing the unique element of minimal length in each coset. For each subset $I = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, we define an element $w_I$ of $W_{H_n}$ by

$$
x_1 \mapsto x_{j_1}, \ldots, x_{n-k} \mapsto x_{j_k},
$$

$$
x_{n-k+1} \mapsto -x_{j_k}, \ldots, x_n \mapsto -x_{i_1},
$$

where $J = \{j_1, j_2, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} - I$, $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_{n-k}$. The element $w_I$ belongs to $\Omega_n$ and each element of $\Omega_n$ is obtained in this way (cf. [20]). We also denote by $\Omega_n$ a set of representatives of $\Omega_n$ in $\text{Norm}(T_n)$. The length $l(w_I)$ of $w_I$ is given by

$$
l(w_I) = \# \{ \alpha \in \Phi_{H_n} | \alpha > 0, w_I \alpha < 0 \}
$$

$$
= \sum_{r=1}^{k} (n + 1 - i_r).$$
Put

\[
  w_0 = w_{\{1,2,...,n\}} = \begin{pmatrix}
    0_n & -1 \\
    -1 & \ddots \\
    \ddots & 1 \\
    1 & 0_n
  \end{pmatrix}
\]

This is the longest element in $\Omega_n$. For $w \in \text{Norm}(T_n)$ and a character $\chi$ of $T_n$, we put

\[
  \chi^w(t) = \chi(w^{-1}tw).
\]

Obviously $\chi^w$ depends only upon the class of $w$ in $W_{H_n}$, so we shall use the same notation $\chi^w$ for $w \in W_{H_n}$. We often regard a character of $T_n$ as a character of $B_n$ by the isomorphism $B_n/N_n \simeq T_n$.

1.2. Local theory

In this subsection, $k$ is a local field. We define the standard maximal compact subgroup $K_n$ of $H_n$ as follows.

When $k$ is non-archimedean, we put $K_n = H_n(\mathcal{O})$. When $k = \mathbb{R}$, we put

\[
  K_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in H_n \mid A'B = B'A, A'^tA + B'B = 1_n \right\}.
\]

When $k = \mathbb{C}$, we put

\[
  K_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in H_n \mid A'B = B'A, A'^tA + B'B = 1_n \right\}.
\]

When $k$ is non-archimedean, we put $R = \mathbb{C}[q^s, q^{-s}]$. When $k$ is archimedean, we let $R$ be the ring of entire functions on $\mathbb{C}$. Let $\omega$ be a quasi-character of $k^*$ and let $s$ denote a complex number. Let $I(\omega, s) = \text{Ind}_{k_n}^{H_n}(\omega \chi^s)$ be the space of functions $f$ on $H_n$ which satisfy the following two conditions:

(i) $f$ is right $K_n$-finite.

(ii) For any $p = \begin{pmatrix} A & * \\ 0_n & A^{-1} \end{pmatrix} \in P_n$,

\[
  f(ph) = \omega(\det A)|\det A|^{s+(n+1)/2} f(h).
\]
We say that a function \( f^{(0)}(h) \) on \( H_n \times C \) is a holomorphic section of \( I(\omega, s) \) if the following three conditions are satisfied:

1. For each \( s \in C \), \( f^{(0)}(h) \) belongs to \( I(\omega, s) \) as a function of \( h \in H_n \).
2. For each \( h \in H_n \), \( f^{(0)}(h) \) belongs to \( R \) as a function of \( s \in C \).
3. \( f^{(0)}(h) \) is right \( K_n \)-finite.

We say that a meromorphic function \( f^{(0)}(h) \) on \( H_n \times C \) is a meromorphic section of \( I(\omega, s) \), if there is \( \alpha(s) \in R \) such that \( \alpha(s) \neq 0 \), and \( \alpha(s)f^{(0)}(h) \) is a holomorphic section of \( I(\omega, s) \). Note that a holomorphic section of \( I(\omega, s) \) is determined by its restriction to \( K_n \times C \). We say that a holomorphic section \( f^{(0)}(h) \) is a standard section if its restriction to \( K_n \times C \) does not depend on \( s \in C \). Obviously the space of holomorphic sections is generated by standard sections over \( R \).

For a quasi-character \( \chi \) of \( T_n \), we define \( \text{Ind}_{B_n}^{H_n}(\chi) \) to be the space of right \( K_n \)-finite functions \( f(h) \) on \( H_n \) such that

\[
f(bh) = \chi(b)\delta_{B_n}^{1/2}(b)f(h),
\]

where \( \delta_{B_n} \) is the modulus quasi-character of \( B_n \). Put

\[
\chi_s(t) = \prod_{i=1}^{n} \omega(t_i)|t_i|^{s - (n + 1)/2 + i},
\]

Then \( I(\omega, s) \subset \text{Ind}_{B_n}^{H_n}(\chi_s) \). We define holomorphic sections, meromorphic sections, and standard sections of \( \text{Ind}_{B_n}^{H_n}(\chi_s) \) similarly.

For \( w \in \text{Norm}(T_n) \) and a quasi-character \( \chi \) of \( T_n \), we define the intertwining operator

\[
M_w = M(w, \chi) : \text{Ind}_{B_n}^{H_n}(\chi) \to \text{Ind}_{B_n}^{H_n}(\chi^w)
\]

by

\[
M_w f(h) = \int_{N_e \cap wN_{w^{-1}}} f(w^{-1}nh)dn.
\]

Here the Haar measure \( dn \) is determined as follows. For each \( \alpha \in \Phi_{H_n} \), the Haar measure \( dn_{a} \) on \( N_{a} \) is given by the self dual measure on \( k \) with respect to \( \psi \) by the natural isomorphism \( N_{a} \cong k \). Then the measure \( dn \) is the product measure: \( dn = \Pi dn_{a} \). The integral is absolutely convergent if \( \chi \) belongs to some open set and can be meromorphically continued to all \( \chi \) (cf. [8], [25]).

If \( l(w_1) + l(w_2) = l(w_1w_2) \), then \( M_{w_1} \circ M_{w_2} = M_{w_1w_2} \). When \( w = w_{a} \) is a reflection with respect to a simple root \( \alpha \), then \( M(w, \chi) \) can be regarded as an intertwining
operator on $\text{SL}_2$ as follows: let $\iota_z : \text{SL}_2 \to H_n$ be a homomorphism corresponding to $\alpha$. We may assume $w = \iota_z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then for any $f \in \text{Ind}_{\text{B}_n}^{\text{H}_n}(\chi)$,

$$t_z^*(\iota_z^*(w, \chi)f) = \iota_z^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.2.1)$$

as a function on $\text{SL}_2$. Since $M(w, \chi)$ commutes with right translations (or actions of Hecke operators), it follows from (1.2.1) that the whole property of $M(w, \chi)$ is reduced to that of $\iota_z^* \chi$. When $\omega$ is unramified, there exists a unique standard section $\phi_{\omega,s}$ of $I(\omega, s)$ such that $\phi_{\omega,s}|_{K_n} \equiv 1$. Similarly, there exists a unique standard section $\phi_{\omega,s}^w$ of $\text{Ind}_{\text{B}_n}^{\text{H}_n}(\chi_s^w)$ such that $\phi_{\omega,s}^w|_{K_n} \equiv 1$, for any $w \in \Omega_n$. Note that $\phi_{\omega,s}^w = \phi_{\omega^{-1}, -s}$.

Let us recall some known results concerning $\text{SL}_2 \simeq H_1$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$M_w = M(w, \omega) = M(w, \omega, s) : I(\omega, s) \to I(\omega^{-1}, -s)$. Then:

(1.2.2) $L(s, \omega)^{-1} M_w$ is holomorphic.

(1.2.3) $M(w^{-1}, \omega^{-1}) M(w, \omega) = \epsilon'(s, \omega, \psi)^{-1} \epsilon(-s, \omega^{-1}, \psi)^{-1} \cdot \text{id}$.

(1.2.4) If $\omega$ is unramified, and $\psi$ is of order 0,

$$M_w \phi_{\omega,s} = \frac{L(s, \omega)}{L(s+1, \omega)} \phi_{\omega^{-1}, -s}.$$ 

(1.2.5) If $k$ is non-archimedean and $\omega = 1$, the kernel and the image of $M(w, 1, 1)$: $I(1, 1) \to I(1, -1)$ are the Steinberg representation and the trivial representation, respectively.

(1.2.6) If $k$ is non-archimedean and $\omega = 1$, the kernel and the image of $M(w, 1, -1)$: $I(1, -1) \to I(1, 1)$ are the trivial representation and the Steinberg representation, respectively.

(1.2.7) If $\omega = 1$, then $\text{Res}_{s=0} M(w, 1, s)$ is a non-zero scalar multiplication.

If $w \in \Omega_n$, then the restriction of $M_w$ to $I(\omega, s) \subset \text{Ind}_{\text{B}_n}^{\text{H}_n}(\chi_s)$ is well defined (except for countably many values of $s$). If $f^{(s)}$ is a holomorphic section of $I(\omega, s)$, then $M_w f^{(s)}$ is a meromorphic section of $\text{Ind}_{\text{B}_n}^{\text{H}_n}(\chi_s^w)$. We denote this restriction by $M_w = M(w, \omega) = M(w, \omega, s)$. If $\omega$ is unramified, $w \in \text{Norm}(T_n) \cap K_n$, and $\psi$ is of order 0, then there exists a meromorphic function $c_w(s) = c_w(\omega, s)$ such that

$$M_w(\phi_{\omega,s}) = c_w(s) \phi_{\omega,s}^w.$$

$$c_w(s) = \prod_{\alpha \in \Phi_{\text{H}_n}} \frac{L(\bar{\alpha}, \chi_s)}{L(\bar{\alpha}, \chi_s + 1)}.$$
were \( (\cdot, \cdot) \) is a \( W_{\mu_n} \)-invariant inner product on \( X^*(T_n) \otimes \mathbb{Z} \mathbb{C} \), and \( \tilde{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle \) is the coroot of \( \alpha \).

In [20], the common denominator of \( c_w(s) \) is calculated. Here we proceed in a slightly different way. Let \( w = w_I, I = \{i_1, i_2, \ldots, i_k\} \). Put

\[
N(w_I) = \{\alpha \in \Phi_{\mu_n} | \alpha > 0, w_I \alpha < 0\}
= \{2x_{n-m+1} | 1 \leq m \leq k\}
\cup \{x_m + x_{n-r+1} | 1 \leq r \leq k, i_r - r + 1 \leq m \leq n - r\}
\]

We divide \( N(w_I) \) into a disjoint union \( \bigcup_{r=0}^{[n/2]} N_r(w_I) \):

\[
N_r(w_I) = \begin{cases} 
\{2x_{n-m+1} | 1 \leq m \leq k\}, & \text{if } r = 0 \\
\emptyset, & \text{if } r > k \\
\{x_m + x_{n-r+1} | i_r - r + 1 \leq m \leq n - r\}, & \text{if } 1 \leq r \leq k, i_r \geq 2r \\
\{x_m + x_{n-r+1} | r \leq m \leq n - r\} \cup \{x_m + x_r | \mu_w(r) \leq m \leq n - r\}, & \text{if } 1 \leq r \leq k, i_r \leq 2r - 1.
\end{cases}
\]

Here

\[
\mu_w(r) = \begin{cases} 
\min\{m | n-k+1 \leq m \leq n, j_r < i_{n-m+1}\}, & \text{if } 1 \leq r \leq n-k \\
r + 1, & \text{if } n-k+1 \leq r \leq \left[\frac{n}{2}\right].
\end{cases}
\]

Put

\[
d^r(s) = \begin{cases} 
L\left(s + \frac{n+1}{2}, \omega\right), & \text{if } r = 0 \\
L(2s+n+1-2r, \omega^2), & \text{if } 1 \leq r \leq \left[\frac{n}{2}\right].
\end{cases}
\]

\[
a^r_w(s) = \begin{cases} 
L\left(s + \frac{n+1}{2} - k, \omega\right), & \text{if } r = 0 \\
L(2s+n+1-2r, \omega^2), & \text{if } k < r \leq \left[\frac{n}{2}\right] \\
L(2s+i_r-2r+1, \omega^2), & \text{if } 1 \leq r \leq k, i_r \geq 2r \\
L(2s-n+r+\mu_w(r)-1, \omega^2), & \text{if } 1 \leq r \leq k, i_r \leq 2r - 1,
\end{cases}
\]

\[
d(s) = \prod_{r=0}^{[n/2]} d^r(s), \quad a_w(s) = \prod_{r=0}^{[n/2]} a^r_w(s).
\]
Then we have
\[ c_w(s) = \prod_{r=0}^{[n/2]} \frac{L(\langle \vec{\alpha}, \chi_s \rangle)}{L(\langle \vec{\alpha}, \chi_s \rangle + 1)} \]
\[ = \prod_{r=0}^{[n/2]} \frac{d_w(s)}{d'(s)} \]
\[ = \frac{a_w(s)}{d(s)}. \]

Thus \( d(s) \) is the smallest common denominator of \( c_w(s), \omega \in \Omega_n \). Note that

\[ c_w(s) = \prod_{r=0}^{\min(k, [n/2])} \frac{a_w(s)}{d'(s)}. \]

Now, even when \( \omega \) is not unramified, we define \( c_w(s), d(s) \) etc. by formally substituting \( \omega \).

**DEFINITION.** The normalized intertwining operator

\[ M_{w_0}^* = M^*(w_0, \omega) = M^*(w_0, \omega; \psi): I(\omega, s) \rightarrow I(\omega^{-1}, -s) \]

is given by

\[ M_{w_0}^* = \varepsilon'(s - \frac{n-1}{2}, \omega, \psi) \cdot \prod_{r=1}^{[n/2]} \varepsilon(2s - n + 2r, \omega^2, \psi) \cdot M_{w_0}. \]

**LEMMA 1.1.**

\[ M^*(w_0^{-1}, \omega^{-1}; \psi) \circ M^*(w_0, \omega; \psi) = \omega(-1)^{n+1} \cdot \text{id}, \]
\[ M^*(w_0, \omega^{-1}; \psi) \circ M^*(w_0, \omega; \psi) = \text{id}. \]

**Proof.** The second formula is just a reformulation of the first formula. We will prove the first formula. When \( n = 1 \), this is (1.2.3). Since

\[ \varepsilon'(-s, \omega^{-1}, \psi) \varepsilon'(s + 1, \omega, \psi) = \omega(-1), \]

the right-hand side of (1.2.3) is equal to

\[ \omega(-1) \frac{\varepsilon'(s + 1, \omega, \psi)}{\varepsilon'(s, \omega, \psi)} \cdot \text{id}. \]
For general $n$, take a minimal expression of $w_0$ in $W_{n}$ by simple reflections

$$w_0 = w_1w_2 \cdots w_k.$$  

By using (1.2.1) and (1.2.3) successively,

$$M^{-1}_{w_0} \circ M_{w_0} = M^{-1}_{w_1} \circ \cdots \circ M^{-1}_{w_k} \circ M_{w_1} \circ M_{w_2} \circ \cdots \circ M_{w_k}$$

$$= \omega(-1)^n \prod_{\alpha \in \Phi_{k}, \alpha \neq w_n} \frac{\varepsilon'\left(\langle \bar{z}, \chi_s \rangle + 1, \psi \right)}{\varepsilon'\left(\langle \bar{z}, \chi_s \rangle, \psi \right)} \cdot \text{id}$$

$$= \omega(-1)^n \frac{\varepsilon'(s+(n+1)/2, \omega, \psi)}{\varepsilon'(s-(n-1)/2, \omega, \psi)}$$

$$\times \prod_{r=1}^{[n/2]} \frac{\varepsilon'(2s+n+1-2r, \omega^2, \psi)}{\varepsilon'(2s-n+2r, \omega^2, \psi)} \cdot \text{id}$$

$$= \omega(-1)^{n+1} \varepsilon'\left(s - \frac{n-1}{2}, \omega, \psi\right)^{-1} \varepsilon'\left(-s - \frac{n-1}{2}, \omega^{-1}, \psi^{-1}\right)^{-1}$$

$$\times \prod_{r=1}^{[n/2]} \varepsilon'(2s-n+2r, \omega^2, \psi)^{-1} \varepsilon'(-2s-n+2r, \omega^{-2}, \psi)^{-1} \cdot \text{id}.$$  

Hence the lemma.

**DEFINITION.** A meromorphic section $f^{(s)}(h)$ of $I(\omega, s)$ is a good section of $I(\omega, s)$ if for any $w \in \Omega_n$,

$$[d(s)c_w(s)]^{-1} M_w f^{(s)}$$

is holomorphic.

In particular, if $\omega$ is unramified, $d(s)\phi_{\omega,s}$ is a good section of $I(\omega, s)$.

**LEMMA 1.2.** $f^{(s)}$ is a good section of $I(\omega, s)$ if and only if $M_{w_0}^{*} f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$.

**Proof:** It will suffice to prove that for each $w_j \in \Omega_n$, there exists an entire function $a(s)$ with no zeros such that

$$[d(\omega, s)c_{w_j}(\omega, s)]^{-1} M_{w_j} f^{(s)}(h)$$

$$= \varepsilon(s)[d(\omega^{-1}, -s)c_{w_j}(\omega^{-1}, -s)]^{-1} M_{w_j}^{*} M_{w_0}^{*} f^{(s)}(h). \quad (1.2.8)$$

We shall proceed by induction on $l(w_j)$. Obviously, (1.2.8) holds when $l(w_j) = 0$.  


Suppose \( l(w_j) > 0 \). There are two cases:

1. \( j_{n-k} = n \).
2. \( j_{n-k} = m < n \).

In case (1), put \( I' = I \cup \{n\} \), \( J' = J - \{n\} \). Then

\[
l(w_{I'}) = l(w_j) + 1, \quad l(w_{J'}) = l(w_j) - 1,
\]

\[
w_{I'} = \mathcal{w}_{n} \cdot w_{J'}, \quad M_{w_{I'}} = M_{\mathcal{w}_{n}} \circ M_{w_{J'}},
\]

\[
w_{J'} = \mathcal{w}_{n} \cdot w_{I}, \quad M_{w_{J'}} = M_{\mathcal{w}_{n}} \circ M_{w_{I}},
\]

\[
c_{w_{I'}}(\omega^{-1}, -s) = c_{w_{I}}(\omega^{-1}, -s) \frac{L\left(-s + \frac{-n+1}{2} + k, \omega^{-1}\right)}{L\left(-s + \frac{-n+1}{2} + k + 1, \omega^{-1}\right)},
\]

\[
c_{w_{J'}}(\omega, s) = c_{w_{J}}(\omega, s) \frac{L\left(s + \frac{n+1}{2} - k, \omega\right)}{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)},
\]

On the other hand, by (1.2.1) and (1.2.3),

\[
M_{\mathcal{w}_{n}} \circ M_{w_{I'}} = M_{\mathcal{w}_{n}} \circ M_{\mathcal{w}_{n}} \circ M_{w_{I}}
\]

\[
= C \cdot \epsilon'\left(s + \frac{n-1}{2} - k, \omega, \psi\right)^{-1} \epsilon'\left(-s - \frac{n-1}{2} + k, \omega^{-1}, \psi\right)^{-1} \cdot M_{w_{I}},
\]

where \( C \) is some non-zero constant. We have

\[
[d(\omega, s)c_{w_{I'}}(\omega, s)]^{-1} M_{w_{I'}} f^{(s)}
\]

\[
= [d(\omega, s)c_{w_{I}}(\omega, s)]^{-1} \frac{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)}{L\left(s + \frac{n+1}{2} - k, \omega\right)}
\]

\[
\times C^{-1} \cdot \epsilon'\left(s + \frac{n-1}{2} - k, \omega, \psi\right) \epsilon'\left(-s - \frac{n-1}{2} + k, \omega^{-1}, \psi\right) \cdot M_{\mathcal{w}_{n}} \circ M_{w_{I}} f^{(s)}.
\]
By the induction assumption, this is equal to

\[
\varepsilon_1(s) \frac{L\left(s + \frac{n+1}{2} - k - 1, \omega\right)}{L\left(s + \frac{n+1}{2} - k, \omega\right)} \frac{L\left(1 - s - \frac{n-1}{2} + k, \omega^{-1}\right)}{L\left(s + \frac{n-1}{2} - k, \omega\right)} \\
\times \frac{L\left(s + \frac{n+1}{2} - k, \omega\right)}{L\left(-s - \frac{n-1}{2} + k, \omega^{-1}\right)} \\
\times [d(\omega^{-1}, -s)c_{w_j}(\omega^{-1}, -s)]^{-1} M_{w_{w_j} \circ w_j} \circ M_{w_0}^{f(s)} \\
= \varepsilon_1(s)[d(\omega^{-1}, -s)c_{w_j}(\omega^{-1}, -s)]^{-1} M_{w_j} \circ M_{w_0}^{f(s)}.
\]

Here \(\varepsilon_1(s)\) is some entire function with no zeros.

In case (2), put \(I' = I - \{m\} \cup \{m + 1\}, J' = J - \{m + 1\} \cup \{m\}\). Then

\[
l(w_{I'}) = l(w_{I}) + 1, \quad l(w_{J'}) = l(w_{J}) - 1,
\]

\[
w_{J'} = w_{w_{s_{m} \cdot w_{I}}}, \quad M_{w_{J'}} = M_{w_{s_{m} \cdot w_{I}}} \circ M_{w_{J}}.
\]

By a calculation similar to case (1), (1.2.8) for \(I\) is reduced to (1.2.8) for \(I'\). Thus the lemma follows.

The following lemma is crucial for our theory.

**LEMMA 1.3.** Every holomorphic section of \(I(\omega, s)\) is a good section.

**REMARK.** If \(k \neq \mathbb{C}\), and \(\omega\) is unramified, this lemma is nothing but [22, Theorem 4.2].

**Proof of Lemma 1.3.** Here we assume \(k\) is non-archimedean. We may assume \(\omega\) is ramified. If \(\omega^2\) is ramified, then \(d(s) = c_w(s) = 1\), for any \(w \in \Omega_w\). Take a minimal expression of \(w\) by simple reflections:

\[
w = w_1 w_2 \cdots w_r, \quad M_w = M_{w_1} \circ M_{w_2} \circ \cdots \circ M_{w_r}.
\]

Each \(M_{w_i} (1 \leq i \leq r)\) is holomorphic by (1.2.1) and (1.2.2). So the lemma is obvious in this case.
Now we assume \( \omega \) is ramified and \( \omega^2 = 1 \). Let \( w = w_I, I = \{i_1, i_2, \ldots, i_k\} \). Recall

\[
a_w(s) = d(s)c_w(s) = \prod_{r=0}^{\min(k, \lfloor n/2 \rfloor)} a^*_w(s).
\]

It suffices to prove

\[
\left[ \prod_{r=0}^{\min(k, \lfloor n/2 \rfloor)} a^*_w(s) \right]^{-1} M_w f^{(s)}
\]

is holomorphic. Put

\[
A_w(s) = \prod_{r=0}^{\min(k, \lfloor n/2 \rfloor)} a^*_w(s).
\]

We proceed by induction on \( l(w) \). If \( l(w) = 0 \), (1.2.9) is obviously holomorphic.

(I) When \( i_k = n \): put \( I' = I - \{n\}, w' = w_{I'} \). Then

\[
M_w = M_{w_{I_n}} \circ M_{w'}, \quad A_w(s) = A_w(s).
\]

Since \( M_{w_{I_n}} \) is entire, the holomorphy of (1.2.9) for \( w \) is reduced to that for \( w' \).

(II) When \( i_r + 2 = i_{r+1} + 1 < i_{r+2} \), for some \( 1 \leq r \leq k - 2 \): put \( i_r = m, I' = I - \{m + 1\} \cup \{m + 2\}, I'' = I - \{m\} \cup \{m + 2\}, w' = w_I, w'' = w_{I''}. \) We reduce the holomorphy of (1.2.9) for \( w \) to that for \( w' \). By definition, we have

\[
A_w(s) A_w(s)^{-1} = \zeta(2s + m - 2r + 1) \zeta(2s + m - 2r + 1)^{-1},
\]

\[
M(w, \chi_s) = M(w_{I_n}, \chi_s^{w}) \circ M(w', \chi_s).
\]

Since \( \zeta(2s + m - 2r + 1)^{-1} M(w_{I_n}, \chi_s^{w}) \) is entire, it will suffice to prove that \( 2s \equiv -m + 2r \mod \frac{2\pi \sqrt{-1}}{\log q} \) are not poles of (1.2.9). We now prove that the residue vanishes. By (1.2.7),

\[
\zeta(2s + m - 2r + 1)^{-1} M(w_{I_n}, \chi_s^{w})
\]

is holomorphic at these points. The residue is

\[
\text{Res}_{2s \equiv -m + 2r - 2} (A_w(s)^{-1} M_w f^{(s)})
\]

\[
= c \cdot M(w_{I_n}, \chi_s^{w}) \circ \text{Res}_{2s \equiv -m + 2r - 2} \left[ \zeta(2s + m - 2r + 1) A_w(s)^{-1} M_w f^{(s)} \right]
\]

\[
= c' \cdot M(w_{I_n}, \chi_s^{w}) \circ \left[ A_w(s)^{-1} M_w f^{(s)} \right]_{2s \equiv -m + 2r - 2},
\]
for some non-zero constants $c, c'$. By (1.2.6), it is sufficient to prove that

$$[A_{\omega}(s)^{-1}M_{\omega,f}(s)]_{2s = -m + 2r - 2}$$

is left $i_{\tau_{m}}(SL_{2})$-invariant. We first observe

$$A_{\omega}(s)^{-1}M_{\omega,f}(s)$$

$$= \zeta(2s + m - 2r + 3)\zeta(2s + m - 2r + 2)^{-1}A_{\omega}(s)^{-1}M(w_{2m+1}^{w}, \chi_{s}^{w})M(w, \chi_{s}^{w})f(s).$$

Since $\zeta(2s + m - 2r + 3)$ and $\zeta(2s + m - 2r + 2)^{-1}M(w_{2m+1}^{w}, \chi_{s}^{w})$ is holomorphic at $2s \equiv -m + 2r - 2 \left( \mod \frac{2\pi}{\log q} \mathbb{Z} \right)$, this is equal to

$$c''[\zeta(2s + m - 2r + 2)^{-1}M(w_{2m+1}^{w}, \chi_{s}^{w})]_{2s = -m + 2r - 2}A_{\omega}(s)^{-1}M(w, \chi_{s}^{w})f(s),$$

for some non-zero constant $c''$. By the induction assumption,

$$A_{\omega}(s)^{-1}M(w, \chi_{s}^{w})f(s)$$

is holomorphic. Moreover this is left $i_{\tau_{m}}(SL_{2})$-invariant since

$$w^{w_{2m+1}}i_{\tau_{m}}(SL_{2})w^{w} \subset M_n.$$

By (1.2.7),

$$[\zeta(2s + m - 2r + 2)^{-1}M(w_{2m+1}^{w}, \chi_{s}^{w})]_{2s = -m + 2r - 2}$$

is a scalar multiplication. Thus (1.2.10) is left $i_{\tau_{m}}(SL_{2})$-invariant.

(III) When $i_k = n - 1, i_{k-1} = n - 2$: this case can be treated by the same technique as in the case (II) by putting

$$I' = I - \{n-1\} \cup \{n\}, \quad I'' = I - \{n-2\} \cup \{n\}.$$ 

(IV) When $i_k < n - 1$. This case can be treated by a similar technique as in the case (II) by putting

$$I' = I - \{i_k\} \cup \{i_k+1\}, \quad I'' = I - \{i_k\} \cup \{i_k+2\}.$$ 

Now we may assume $i_k = n - 1$, by (I) and (IV). Moreover, we may assume $k \leq \left\lfloor \frac{n}{3} \right\rfloor$, since otherwise the assumption of (II) or (III) holds. To see this, assume
$k > \left[ \frac{n}{2} \right]$ and neither the assumption of (II) nor that of (III) holds. Then

$$i_k = n - 1, \ i_{k - 1} \leq n - 3, \ldots, i_k \leq n - 2k + 2m - 1, \ldots, i_1 \leq n - 2k + 1 \leq 0.$$  

This is a contradiction.

(V) When $k \leq \left[ \frac{n}{2} \right]$: put $I' = I - \{ n - 1 \}, \ w' = w_{I'}$. Then

$$M_w = M(w_{a_0 - 1}, x_{a_0}^{-w}) \cdot M(w_{a_0}, x_{a_0}^{-w}) \cdot M(w', x_d),$$

$$A_w(s) = A_w(s) \cdot \zeta(2s + n - 2k).$$

By the induction assumption, $A_w(s)^{-1} M_w f^{(\omega)}$ is entire. Since both $M(w_{a_0}, x_{a_0}^{-w})$ and $\zeta(2s + n - 2k)^{-1} \cdot M(w_{a_0 - 1}, x_{a_0}^{-w})$ are entire, $A_w(s)^{-1} M_w f^{(\omega)}$ is entire. Thus the proof for non-archimedean local field is complete.

**Appendix 1. Proof for Lemma 1.3 for archimedean case**

In this appendix, we give a proof for Lemma 1.3 for an archimedean local field $k$.

We may assume that $\omega$ is unitary.

SUBLEMMA 1. If $w = w_0$, then (1.2.9) is holomorphic.

**Proof.** If $k = \mathbb{R}$, and $\omega = 1$, this is proved in [22 §4 Appendix 1]. Their proof is valid for $k = \mathbb{R}, \ \omega = \text{sgn}$. If $k = \mathbb{C}$, we have to show that the first part of [22 §4 Appendix 1, Theorem (p. 106)] holds for our situation, i.e., we have to show that

$$a_{w_0}(\omega, s)^{-1} \int_{\text{Sym}(\mathbb{C})} \varphi(z) |\det z|^s (n + 1)^{1/2} \omega(\det z) \, dz \quad (1.2.11)$$

is entire for any $\varphi \in \mathcal{S}(\text{Sym}^n(\mathbb{C}))$. We may assume that $\omega(z) = z^k$ or $(\bar{z})^k, \ k \geq 0$. But the case $\omega(z) = (\bar{z})^k$ is reduced to the case $\omega(z) = z^k$ by taking complex conjugate. Put

$$\vartheta = \det \begin{vmatrix} \frac{\partial}{\partial z_{11}} & 1 & \frac{\partial}{\partial z_{12}} & \cdots & 1 & \frac{\partial}{\partial z_{1n}} \\ \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{22}} & \ddots & \vdots \\ \vdots & \vdots & \ddots \\ 1 & \frac{\partial}{\partial z_{2n}} & \cdots & \frac{\partial}{\partial z_{nn}} \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{nn}} & \end{vmatrix}$$
Then it is known that

\[ \partial(|\det z\bar{z}|^n |\det z|^k) = \prod_{i=0}^{n-1} \left( s + k + i - \frac{1}{2} \right) \cdot (|\det z\bar{z}|^n |\det z|^k)^{-1}. \]

Repeating partial integration, we have

\[
\prod_{j=1}^{m} \prod_{i=0}^{n-1} \left( s + k + j + \frac{i - n - 1}{2} \right) \int_{\text{Sym}^2(\mathbb{C})} \varphi(z) |\det z\bar{z}|^{-(n+1)/2} |\det z|^k \, dz
\]

\[
= (-1)^m \int_{\text{Sym}^2(\mathbb{C})} \varphi(z) |\det z\bar{z}|^{-(n+1)/2} |\det z|^k + m \, dz
\]

for \( \text{Re}(s) \gg 0 \). Since the right-hand side is absolutely convergent for \( \text{Re}(s) > \frac{n-k-m-1}{2} \), we have

\[
\prod_{i=0}^{n-1} \Gamma \left( s + k - i + \frac{1}{2} \right) \int_{\text{Sym}^2(\mathbb{C})} \varphi(z) |\det z\bar{z}|^{-(n+1)/2} |\det z|^k \, dz
\]

is entire. So (1.2.11) is entire.

Let \( Q \) (resp. \( Q' \)) be the maximal parabolic subgroup of \( \text{GL}_n \) given by

\[
Q = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \left| a_1 \in \text{GL}_{n-1}, a_2 \in k^\times \right. \right\}
\]

(resp. \( Q' = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \left| a_1 \in k^\times, a_2 \in \text{GL}_{n-1} \right. \right\} \)).

Let \( I_Q(\omega, s) \) (resp. \( I_{Q'}(\omega, s) \)) be the representation of \( \text{GL}_n \) induced from the character of \( Q \) (resp. \( Q' \)) given by

\[
\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto \omega(\det a_1)|\det a_1|^{s/n}|a_2|^{-(n-1)/n} s
\]

(resp. \( \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto \omega^{-1}(\det a_2)|a_1|^{(n-1)/n}|\det a_2|^{-s/n} \)).

We define standard sections, holomorphic sections, and meromorphic sections as usual. We define the intertwining operator \( M_w: I_Q(\omega, s) \mapsto I_Q(\omega^{-1}, -s) \)
(resp. \( M_w : I_Q(\omega, s) \to I_Q(\omega^{-1}, -s) \)). Here

\[
\begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & 1
\end{pmatrix}.
\]

**Sublemma 2.** \( L \left( s - \frac{n-2}{2}, \omega \right)^{-1} M(w, s) \) and \( L \left( s - \frac{n-2}{2}, \omega \right)^{-1} M(\omega', s) \) are holomorphic.

**Proof.** This can be proved in the same way as [22, §4]. (See also [12 §5].)

**Sublemma 3.**

\[
M(w', \omega^{-1}) \circ M(w, \omega) = \omega(-1)^{n+1} e'(s - \frac{n-2}{2}, \omega, \psi)^{-1} e'(s - \frac{n-2}{2}, \omega^{-1}, \psi)^{-1} \cdot \text{id}.
\]

**Proof.** This can be proved in the same way as the proof of Lemma 1.1.

We now return to the proof of Lemma 1.3. Let \( w = w_f \) be an element of \( \Omega_n \). We prove that

\[
[d(\omega, s) c_{\omega}(\omega, s)]^{-1} M_w f(s)
\]

is holomorphic. \( M_w \) can be considered as an intertwining operator of \( I(\omega, s + \frac{i_1 - 1}{2}) \) on \( \text{Sp}_{n-i_1+1} \). We may assume \( i_1 = 1 \) by replacing \( n \) by \( n-i_1+1 \) and \( I \) by \( \{i_r-i_1+1 | 1 \leq r \leq k\} \). We proceed by the induction on \( \delta(w) = n-k \). When \( n=k \), this is Sublemma 1. Assume \( n-k \geq 1 \). Put

\[
m = \max\{r | i_r < n-k+r\},
\]

\[
I' = I \cup \{n-k+m\},
\]

\[
w' = w_{I'}.
\]

Then \( \#I' = k+1, \ l(w') = l(w)+k-m+1 \) and

\[
w' = w_{i_1} w_{i_2} \cdots w_{i_k+m}.
\]
Put
\[ w_{(0)} = w, \]
\[ w_{(r)} = w' = w_{\alpha - k + m + r - 1} \cdots w_{\alpha - k + m + 1} w_{\alpha - k + m}, \quad 1 \leq r \leq k - m + 1. \]

Then
\[
M_{w_{(r)}} = M(w_{\alpha - k + m + r - 1}, \chi_{\alpha}^{\omega(r-1)}) \circ M_{w_{(0)}}, \quad 1 \leq r \leq k - m + 1
\]
\[
c_{w_{(r)}}(s) = c_{w_{(r-1)}}(s) \times \begin{cases} 
\frac{L(2s + n - k - m - r, \omega^2)}{L(2s + n - k - m - r + 1, \omega^2)}, & 1 \leq r \leq k - m \\
L\left(s + \frac{n - 1}{2} - k, \omega\right), & r = k - m + 1
\end{cases}
\]

We have
\[
c_{w}(s) = \frac{L(2s + n - 2k, \omega^2)}{L(2s + n - k - m, \omega^2)} \frac{L\left(s + \frac{n + 1}{2} - k, \omega\right)}{L\left(s + \frac{n + 1}{2} - k, \omega\right)} c_{w}(s).
\]

It is easy to see that
\[
M(w_{\alpha - 1}, \chi_{\alpha}^{\omega(\alpha - m - 1)}) \circ \cdots \circ M(w_{\alpha - k + m}, \chi_{\alpha}^{\omega})
\]
is an intertwining operator on $GL_{k-m}$. By (1.2.3) and Sublemma 3,
\[
M(w_{\alpha - k + m}, \chi_{\alpha}^{\omega(\alpha - m)}) \circ \cdots \circ M(w_{\alpha - 1}, \chi_{\alpha}^{\omega}) \circ M(w_{\alpha}, \chi_{\alpha}^{\omega}) \circ M_{w'}
\]
\[
= \omega(-1) \varepsilon'\left(s + \frac{n - 1}{2} - k, \omega, \psi\right)^{-1} \varepsilon'\left(-s - \frac{n - 1}{2} + k, \omega^{-1}, \psi\right)^{-1}
\]
\[
\times \varepsilon'(2s + n - 2k, \omega^2, \psi)^{-1} \varepsilon'(-2s - n + k + m + 1, \omega^{-2}, \psi)^{-1} M_{w}
\]

By (1.2.2), Sublemma 2, and the induction assumption,
\[
L\left(-s - \frac{n - 1}{2} + k, \omega^{-1}\right)^{-1} M(w_{\alpha}, \chi_{\alpha}^{\omega}),
\]
\[
L(-2s - n + k + m + 1, \omega^{-2})^{-1} M(w_{\alpha - k + m}, \chi_{\alpha}^{\omega(\alpha - m)})
\]
and
\[ [d(\omega, s)c_w(\omega, s)]^{-1}M_w. \]

are holomorphic. Thus we have
\[
L\left( -s - \frac{n-3}{2} + k, \omega^{-1} \right)^{-1}L(-2s-n+2k+1, \omega^{-2})^{-1}[d(\omega, s)c_w(\omega, s)]^{-1}M_w
\]
is holomorphic.

On the other hand, put
\[
w_k = \begin{pmatrix} 1_{n-k} & 0 \\ 0 & 1_k \\ 0 & 1_{n-k} \end{pmatrix},
\]
\[ w = w'w_k. \]

Then \( M_w = M_w' \circ M_{w_k} \). Here, as in \([22 \S 4]\), \( M_{w'} \) is an intertwining operator on certain induced representation of \( \text{GL}_n \). As in \([22 \S 4]\), we can prove
\[
\prod_{r=1}^{k} L(2s + i_r - 2r + 1, \omega^2)^{-1}M_w
\]
is holomorphic (cf. \([22, \text{Remark 4.1}]\)). As for \( M_{w_k} \), by Sublemma 1,
\[
L\left( s + \frac{n+1}{2} - k, \omega \right)^{-1} \prod_{r=1}^{\lfloor k/2 \rfloor} L(2s+n-2k+2r, \omega^2)^{-1}M_{w_k}
\]
is holomorphic. Putting together, we can easily deduce
\[
\prod_{r=\lfloor k+1/2 \rfloor}^{k} L(2s+n-2r, \omega^2)^{-1}[d(\omega, s)c_w(\omega, s)]^{-1}M_w
\]
is holomorphic. Since
\[
L\left( -s - \frac{n-3}{2} + k, \omega^{-1} \right)L(-2s-n+2k+1, \omega^{-2})
\]
has no poles in \( \text{Re}(s) < -\frac{n}{2} + k + \frac{1}{2} \), and
\[
\prod_{r=[k+1/2]}^{k} L(2s+n-2r, \omega^2)
\]
has no poles in \( \text{Re}(s) > -\frac{n}{2} + k \), it follows that
\[
[d(\omega, s)c_w(\omega, s)]^{-1} M_w
\]
is holomorphic. Thus Lemma 1.3 is proved.

**REMARK.** Our definition of good section is different from that of [22]. But we can prove that “germs” of good section of \( I(\omega, s) \) at \( s=s_0 \) are generated by the following two families:

1. germs of holomorphic sections of \( I(\omega, s) \) at \( s=s_0 \),
2. \( \{ M_{w_0}^* f^{(\alpha)} | f^{(\alpha)} \text{ is a germ of holomorphic section of } I(\omega^{-1}, -s) \text{ at } s=s_0 \} \).

In fact, we may assume \( \omega \) is unitary and \( \text{Re}(s_0) \geq 0 \), by Lemma 1.2. Since \( d(\omega, s) \) does not have zero at \( s=s_0 \), any good section of \( I(\omega, s) \) is holomorphic at \( s=s_0 \). It is easy to see that when \( k \) is non-archimedean, our definition agrees to that of [22] because there are essentially finite number of singularities.

**Appendix 2. An interpretation of the normalizing factor**

We give an interpretation of the normalizing factor \( d(\omega, s) \) in terms of Arthur's conjecture [1]. Let \( G \) be a reductive group, \( P \) be a maximal parabolic subgroup of \( G \), \( M \) be a Levi factor of \( P \), \( N \) be the unipotent radical of \( P \), and \( A \) be the maximal split torus of the center of \( M \). Let \( \pi \) be an irreducible discrete automorphic representation of \( M \). Then, according to Arthur's conjecture, \( \pi \) is associated to a homomorphism
\[
\varphi_\pi: \mathcal{L} \times \text{SL}_2(\mathbb{C}) \to ^L M.
\]
Here \( \mathcal{L} \) is the conjectual Langlands group. Let \( ^L N \) be the Lie algebra of \( ^L N \). Decompose \( ^L N \) as in Shahidi [24].
\[
^L N = \prod_{i=1}^r \mathcal{L} N_i.
\]
Consider the induced representation \( \text{Ind}_{\mathcal{M}}^M \pi \mathcal{L} \). Here \( \mathcal{L} \) is as in [24]. Let \( \text{Ad}_{\mathcal{L} \times A} \) be
the adjoint action of $LM$ on $\mathcal{L}\mathcal{N}_i$. If $\pi$ is cuspidal and $\varphi_\pi$ is trivial on $SL_2(C)$, then the normalizing factor should be given by

$$\prod_{i=1}^{r} L(1 + is, \varphi_\pi \circ \text{Ad}_{\mathcal{L}\mathcal{N}_i}).$$

(cf. Shahidi [24], Langlands [15].) Consider the general case where $\varphi_\pi \circ \text{Ad}_{\mathcal{L}\mathcal{N}_i}$ is not trivial on $SL_2(C)$. In this case, decompose $\varphi_\pi \circ \text{Ad}_{\mathcal{L}\mathcal{N}_i}$ into irreducible representation:

$$\varphi_\pi \circ \text{Ad}_{\mathcal{L}\mathcal{N}_i} = \bigoplus_{j=1}^{m_i} \varphi_{ij} \otimes \text{sym}^{r_{ij}},$$

where $\varphi_{ij}$ is an irreducible representation of $\mathcal{L}$, and $\text{sym}^{r_{ij}}$ is the $r_{ij}$th symmetric power of the standard representation of $SL_2(C)$. Then we claim the normalizing factor should be

$$\prod_{i=1}^{r} \prod_{j=1}^{m_i} L(is + \frac{r_{ij}}{2} + 1, \varphi_{ij}).$$

In fact, the $c$-function $c_{w_0}(\pi, s)$ for the longest element $w_0$ of the Weyl group is given by

$$c_{w_0}(\pi, s) = \prod_{i=1}^{r} \frac{L(is, \varphi_\pi \circ \text{Ad}_{\mathcal{L}\mathcal{N}_i})}{L(1 + is, \varphi_\pi \circ \text{Ad}_{\mathcal{L}\mathcal{N}_i})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \frac{L(is, \varphi_{ij} \otimes \text{sym}^{r_{ij}})}{L(1 + is, \varphi_{ij} \otimes \text{sym}^{r_{ij}})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \prod_{a=0}^{r_{ij}} \frac{L(is - \frac{r_{ij}}{2} + a, \varphi_{ij})}{L(is - \frac{r_{ij}}{2} + a + 1, \varphi_{ij})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{m_i} \frac{L(is - \frac{r_{ij}}{2}, \varphi_{ij})}{L(is + \frac{r_{ij}}{2} + 1, \varphi_{ij})},$$

at least up to bad primes. If $\pi$ is cuspidal, this is the only non-trivial $c$-function. This means at least when $\pi$ is cuspidal, our claim is justified, since the normalizing factor should be the least common denominator of the $c$-functions. One can expect that the least common denominator of the $c$-functions is equal to
the denominator of the c-function for the longest Weyl element even when \( \pi \) is not cuspidal.

Observe that in our case, \( G = \text{Sp}_n \), \( M = \text{GL}_n \), \( \pi = \omega \), \( \varphi_\pi = \omega \otimes \text{sym}^{n-1} \), \( \text{Ad}_{\mathfrak{c}_1} = \rho \), \( \text{Ad}_{\mathfrak{c}_2} = \Lambda^2 \rho \). Here \( \rho \) is the standard representation of \( \text{GL}_n \). Therefore,

\[
\varphi_\pi \circ \text{Ad}_{\mathfrak{c}_1} = \omega \otimes \text{sym}^{n-1}
\]

gives \( L \left( s + \frac{n+1}{2}, \omega \right) \), and

\[
\varphi_\pi \circ \text{Ad}_{\mathfrak{c}_2} = \bigotimes_{j=1}^{\lfloor n/2 \rfloor} (\omega^2 \otimes \text{sym}^{2n-4j})
\]

gives \( \Pi_{r=1}^{\lfloor n/2 \rfloor} L(2s + n + 1 - 2r, \omega^2) \).

1.3. Eisenstein series

In this subsection, we assume \( k \) to be a global field. We will investigate the poles of Eisenstein series associated to good sections.

Let \( \omega \) be a quasi-character of \( \mathbb{A}^\times / k^\times \). Put \( K_n = \Pi_v K_{n,v} \). Let \( I(\omega, s) \) be the space of functions \( f(h) \) on \( H_n(A) \) which satisfy (1) and (2):

1. \( f \) is right \( K_n \)-finite.
2. For any \( p = \begin{pmatrix} A & * \\ 0 & A^{-1} \end{pmatrix} \in P_n(A) \),

\[
f(ph) = \omega(\det A)|\det A|^{(s+(n+1)/2)} f(h).
\]

Clearly, \( I(\omega, s) = \bigotimes_v I(\omega_v, s) \). We also define holomorphic sections and meromorphic sections similarly. We say that a meromorphic section of \( I(\omega, s) \) is a good section if it is a finite sum of decomposable elements \( f^{(s)} = \Pi_v f_v^{(s)} \) satisfying following (i) and (ii).

(i) For almost all unramified \( v \), \( f_v^{(s)} = d(\omega_v, s) \phi_{\omega_v, s} \).
(ii) \( f_v^{(s)} \) is a good section of \( I(\omega_v, s) \) for all \( v \).

In other words, the space of global good sections is the restricted tensor product of the local good sections with respect to \( d(\omega_v, s) \phi_{\omega_v, s} \). Note that the product \( f^{(s)} = \Pi_v f_v^{(s)} \) is absolutely convergent for \( \text{Re}(s) > \frac{n+1}{2} \), and can be meromorphically continued to \( \mathbb{C} \).
We define the Eisenstein series $E(h; f^{(s)})$ associated to $f^{(s)}$ by

$$E(h; f^{(s)}) = \sum_{\gamma \in \Gamma \setminus \mathbb{H}} f^{(s)}(\gamma h).$$

This is absolutely convergent for $\Re(s) \gg 0$, and can be meromorphically continued to $\mathbb{C}$. The functional equation of $E(h; f^{(s)})$ is given by

$$E(h; f^{(s)}) = E(h; M_{w_0}f^{(s)}).$$

Here $M_{w_0}$ is the global intertwining operator:

$$M_{w_0} = \bigotimes_v (M_{w_0})_v.$$

The global intertwining operator $M_{w_0}$ does not depend on the choice of representative of $w_0 \in W_{H_n}$ in $\text{Norm}(T_n)$.

**Lemma 1.4.** If $f^{(s)}$ is a good section of $I(\omega, s)$, then $M_{w_0}f^{(s)}$ is a good section of $I(\omega^{-1}, -s)$.

**Proof.** Let $S$ be a finite set of places of $k$ such that if $v \notin S$, then $\omega_v$ is unramified, $\psi_v$ is of order 0, and $f_v^{(s)} = d(\omega_v, s)\phi_{\omega_v^{-1}, -s}$. Then

$$M_{w_0}f^{(s)} = \prod_{v \notin S} d(\omega_v, s)\phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0}f_v^{(s)}$$

$$= \prod_{v \notin S} a_{w_0}(\omega_v, s)\phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0}f_v^{(s)}$$

$$= \prod_{v \notin S} d(\omega_v^{-1}, -s)\phi_{\omega_v^{-1}, -s} \times \prod_{v \in S} M_{w_0}f_v^{(s)}.$$

By Lemma 1.2, the lemma follows.

**Lemma 1.5.** Suppose that $n = 1$, and $\omega = 1$. Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the global intertwining operator $M_w : I(1, s) \to I(1, -s)$ is holomorphic at $s = 0$, and is equal to the scalar multiplication by $-1$ at $s = 0$.

**Proof.** Put $f^{(s)} = \prod_v \phi_{1, s}$, and $\zeta(s) = |D|^{s/2}\zeta(s)$. Here $D$ is the discriminant of $k$ (resp. $D = q^{2g-2}$, $g$ is the genus of $k$) if $k$ is a number field (resp. if $k$ is a function field). Then

$$M_wf^{(s)} = \frac{\zeta(s)}{\zeta(s+1)} \prod_v \phi_{1, -s}.$$ (1.3.1)

Since $\zeta(1-s) = \zeta(s)$ and $\zeta(s)$ has a simple pole at $s = 0, 1$, the right-hand side of...
(1.3.1) is holomorphic at \( s = 0 \), and
\[
M_w f^{(0)} = -f^{(0)}.
\]

Since \( I(1, s) \) is irreducible on some neighbourhood of \( s = 0 \), the lemma follows.

**PROPOSITION 1.6.** Suppose that \( k \) is a number field. If \( f^{(s)} \) is a good section of \( I(w, s) \), then the pole of \( E(h; f^{(s)}) \) are at most simple. The set of possible poles is as follows.

1. When \( \omega \) is principal: we may assume \( \omega = 1 \). Then the set of possible poles is:
\[
\left\{ \frac{n + 1}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n + 1, m \neq \frac{n + 1}{2} \right\}
\]

2. When \( \omega \) is not principal, and \( \omega^2 \) is principal: we may assume \( \omega^2 = 1 \). Then the set of possible poles is:
\[
\left\{ \frac{n - 1}{2} - m \mid m \in \mathbb{Z}, 0 \leq m \leq n - 1, m \neq \frac{n - 1}{2} \right\}
\]

3. If \( \omega^2 \) is not principal, then \( E(h; f^{(s)}) \) is entire.

**Proof.** As in [22], the constant term \( E^0(h; f^{(s)}) \) of \( E(h; f^{(s)}) \) along \( U_n(A) \) is given by
\[
E^0(h; f^{(s)}) = \int_{U_d(k) \backslash U_d(A)} E(uh; f^{(s)}) du
= \sum_{w \in \Omega_n} M_w f^{(s)}.
\]

Let \( S \) be as in the proof of Lemma 1.4. Then
\[
M_w f^{(s)} = \prod_{v \notin S} d(\omega_v, s) c_{w(\omega_v, s)} \phi_{w, v}^{w} \times \prod_{v \in S} M_w f^{(s)}
= d(\omega, s) c_{w(\omega, s)} \prod_{v \notin S} \phi_{w, v}^{w}
\times \prod_{v \in S} [d(\omega_v, s) c_{w(\omega_v, s)}]^{-1} M_w f^{(s)}.
\]

Therefore the poles of \( E(h; f^{(s)}) \) comes from the poles of \( d(\omega, s) c_{w(\omega, s)} \). In particular, if \( \omega^2 \) is not principal, \( E(h; f^{(s)}) \) is entire.

We may assume \( \omega^2 = 1 \), without loss of generality. When \( \omega = 1 \), (resp. \( \omega^2 = 1 \),
\( \omega \neq 1 \), the possible poles of \( d(\omega, s)c_w(\omega, s) \) are integral or half-integral points in 
\[
\left[ \frac{-n+1}{2}, \frac{n+1}{2} \right] \quad \left( \text{resp.} \quad \left[ \frac{-n-1}{2}, \frac{n-1}{2} \right] \right).
\]

We first prove the proposition for the case \( n = 1 \) or \( n = 2 \). If \( n = 1 \), \( \omega \neq 1 \), then (2) is obvious since \( d(\omega, s)c_w(\omega, s) \) are entire. If \( n = 1 \), \( \omega = 1 \), then we have to show that \( s = 0 \) is not a pole of \( E^0(h; f^{(s)}) \). Note that \( f^{(s)} \) may have a simple pole at \( s = 0 \). Let \( w \) be as in Lemma 1.5. Then by Lemma 1.5,

\[
\lim_{s \to 0} sE^0(h; f^{(s)}) = (1 + M_w) \left[ \lim_{s \to 0} sf^{(s)} \right] = 0.
\]

Thus \( E^0(h; f^{(s)}) \) is holomorphic at \( s = 0 \).

If \( n = 2 \), the possible poles of \( d(\omega, s)c_w(\omega, s) \) are as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>( l(w) )</th>
<th>( d(\omega, s)c_w(\omega, s) )</th>
<th>poles ( (\omega = 1) )</th>
<th>poles ( (\omega^2 = 1, \omega \neq 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( \emptyset )</td>
<td>0 ( L(s + \frac{1}{2})L(2s + 1) )</td>
<td>( { -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0 } )</td>
<td>( { -\frac{1}{2}, 0 } )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>{2}</td>
<td>1 ( L(s + \frac{1}{2})L(2s + 1) )</td>
<td>( { -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2} } )</td>
<td>( { -\frac{1}{2}, 0 } )</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>{1}</td>
<td>2 ( L(s + \frac{1}{2})L(2s) )</td>
<td>( { -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} } )</td>
<td>( { 0, \frac{1}{2} } )</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>{1, 2}</td>
<td>3 ( L(s - \frac{1}{2})L(2s) )</td>
<td>( { 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} } )</td>
<td>( { 0, \frac{1}{2} } )</td>
</tr>
</tbody>
</table>

Here, \( L(s) = L(s, \omega) \). By functional equation, we may assume \( \text{Re}(s) \geq 0 \), so what we have to prove are reduced to the following two statements.

(1.3.2) If \( \omega = 1 \),

\[
\lim_{s \to 1/2} (s - \frac{1}{2})^2(M_{w_3} + M_{w_4})f^{(s)} = 0.
\]

(1.3.3) If \( \omega^2 = 1 \),

\[
\lim_{s \to 0} s(1 + M_{w_2} + M_{w_3} + M_{w_4})f^{(s)} = 0.
\]

Proof of (1.3.2)

\[
\lim_{s \to 1/2} (s - \frac{1}{2})^2M_{w_4}f^{(s)} = \lim_{s \to 1/2} M_{w_2}L^{w_3} \circ [(s - \frac{1}{2})^2M_{w_3}f^{(s)}].
\]

We know that \( (s - \frac{1}{2})^2M_{w_3}f^{(s)} \) is holomorphic at \( s = \frac{1}{2} \). Moreover, by (1.2.1) and
Lemma 1.5, $M(w_{a_2}, \chi_s^{w_3})$ is holomorphic and is equal to the scalar multiplication by $-1$ at $s = \frac{1}{2}$. Hence (1.3.2).

**Proof of** (1.3.3). By the same way as above, we can prove

$$\lim_{s \to 0} s(M(w_2 + M(w_3)) f^{(s)} = 0.$$ 

But the proof that

$$\lim_{s \to 0} s(1 + M(w_2)) f^{(s)} = 0$$

is more delicate. We have

$$M_{w_4} f^{(s)} = M(w_{a_2}, \chi_s^{w_5}) \circ M(w_{a_1}, \chi_s^{w_2}) \circ M(w_{a_2}, \chi_s) f^{(s)}.$$ 

By (1.2.1) and Lemma 1.5, $M(w_{a_1}, \chi_s^{w_2})$ is holomorphic and is equal to the scalar multiplication by $-1$ at $s = 0$. Moreover, by (1.2.1), $M(w_{a_2}, \chi_s)$ (resp. $M(w_{a_3}, \chi_s^{w_3})$) is essentially the intertwining operator

$$M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s + \frac{1}{2}\right) : I\left(\omega, s + \frac{1}{2}\right) \rightarrow I\left(\omega, -s - \frac{1}{2}\right)$$

(resp. $M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -s - \frac{1}{2}\right) : I\left(\omega, -s - \frac{1}{2}\right) \rightarrow I\left(\omega, s + \frac{1}{2}\right)$)

on $SL_2$. Moreover, these two are mutually the inverse of the other except for their singular points. Since the representations $I(\omega, s + \frac{1}{2})$ and $I(\omega, -s - \frac{1}{2})$ of $SL_2(A)$ are irreducible on some neighbourhood of $s = 0$, there is an integer $\alpha$ such that

$$s^{-\alpha}M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s + \frac{1}{2}\right) \quad \text{and} \quad s^\alpha M\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -s - \frac{1}{2}\right)$$

are holomorphic, and are mutually the inverse of each other at $s = 0$. In fact, it is easy to see that $\alpha = \text{ord}_s = \frac{1}{2} L(s, \omega)$. We have

$$\lim_{s \to 0} s M_{w_4} f^{(s)} = \lim_{s \to 0} [s^\alpha M(w_{a_2}, \chi_s^{w_5})] \circ [M(w_{a_1}, \chi_s^{w_2})] \circ [s^{-\alpha} M(w_{a_2}, \chi_s)] [sf^{(s)}].$$

Each term is holomorphic at $s = 0$, so the exchange of limit and the composition is possible. Hence (1.3.3).

Now we assume $n \geq 3$. By the functional equation, it is enough to investigate
the integral or half-integral points in $\left[0, \frac{n+1}{2}\right]$. Note that $f^{(s)}$ is holomorphic on the right half plane $\text{Re}(s) \geq 0$ except for the case $n$ is even and $s = 0$. In particular, if $n$ is odd, $s = 0$ is not a pole of $E(h; f^{(s)})$, by [16].

We recall the theory of degenerate Eisenstein series on $\text{GL}_n$ (see [12, §5]). Let $Q$ be the maximal parabolic subgroup of $\text{GL}_n$ given by

$$Q = \left\{ \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mid a_1 \in \text{GL}_{n-1}, \ a_2 \in k^\times \right\}.$$ 

Let $I_Q(s)$ be the representation of $\text{GL}_n$ induced from the character of $Q$ given by

$$\begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} \mapsto |\det a_1|^{s/n} |a_2|^{-(n-1)s/n}.$$ 

We define standard sections, holomorphic sections etc. as usual. For each prime $v$ of $k$, let $F_{0,v}^{(s)}$ be the meromorphic section of $I_{Q,v}(s)$ which takes value $\zeta_v(s + \frac{n}{2})$ on the standard maximal compact subgroup of $\text{GL}_{n,v}$.

Taking any finite set $S$ of primes of $k$, put

$$F^{(s)} = \prod_{v \notin S} F_{0,v}^{(s)} \times \prod_{v \in S} F_v^{(s)}$$

where $F_v^{(s)}$, $v \in S$ are arbitrary holomorphic sections of $I_{Q,v}(s)$. Define degenerate Eisenstein series on $\text{GL}_n$ by

$$E(g; F^{(s)}) = \sum_{\gamma \in Q \setminus \text{GL}_n} F^{(s)}(\gamma g).$$

Then the possible poles of $E(g; F^{(s)})$ are $s = \pm \frac{n}{2}$. Moreover, each pole is at most simple and the residue is a constant function. The functional equation is given by

$$E(g; F^{(s)}) = E(g; M_w F^{(s)}).$$

Here

$$w = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$
$M_w F^{(s)}$ is a meromorphic section of the representation induced from the character

\[
\begin{pmatrix}
a_1 & * \\
0 & a_2
\end{pmatrix} \mapsto |a_1|^{-(n-1)s/n} |\det a_2|^{s/n}
\]

of the parabolic subgroup

\[
Q' = \left\{ \begin{pmatrix} a_1 & * \\
0 & a_2 \end{pmatrix} \mid a_1 \in k^*, a_2 \in \text{GL}_{n-1} \right\}.
\]

$M_w F^{(s)}$ has at most simple poles at $s = \frac{n}{2}, \frac{n}{2} - 1$.

We return to the proof of Proposition 1.6. Let

\[
f^{(s)} = \prod_{v \in S} d(\omega_v, s) \phi_{\omega_v, s} \times \prod_{v \in S} f_v^{(s)}
\]

be a good section. We may assume each $f_v^{(s)}, v \in S$ is a standard section, since $d(\omega_v, s)$ has no pole in $\text{Re}(s) \geq 0$.

Let $P^*_1$ be the parabolic subgroups of $H_n$ given by

\[
P^*_1 = \left\{ \begin{pmatrix} a & * & * \\
0 & A & * \\
n & a^{-1} & 0 \\
0 & * & tA^{-1} \end{pmatrix} \in H_n \mid a \in k^*, A \in \text{GL}_{n-1} \right\}.
\]

Let $t = (t_1, t_2) \in \mathbb{C}^2$. Let $I_{P^*_1(\omega_v, t)}^t$, be the space of right $K_v$-finite function $f^{(t)}_{P^*_1}$ on $H_{n,v}$ such that

\[
f^{(t)}_{P^*_1}(p, h) = \omega(a \det A)|a|^{t_1+n}|\det A|^{t_2+n/2} f^{(t)}_{P^*_1}(h),
\]

where

\[
p_1 = \begin{pmatrix}
a & * & * \\
0 & A & * \\
n & a^{-1} & 0 \\
0 & * & tA^{-1} \end{pmatrix} \in P^*_1.
\]

For each $v \in S$, let $\tilde{f}^{(t)}_v$ be a standard section (of two variables) of $I_{P^*_1(\omega_v, t)}$ defined by

\[
\tilde{f}^{(t)}_v(p, k) = |a|^{n-1} |\det A|^{-1}|t_1-t_2/n + 1/2 f^{(t)}_v(k),
\]
where \( p_1 \) is as above, \( k \in K_v \), and

\[
s = \frac{t_1 + (n - 1)t_2}{n}.
\]

When \( v \not\in S \), let \( \phi_{P^v, \omega_v} \) be the standard section of \( I_{P^v}(\omega_v, t) \) which is identically 1 on \( K_v \). Put

\[
\tilde{f}^{(0)}(\vec{e}) = \prod_{v \not\in S} L_v(t_1 + 1) \zeta_v \left( t_1 - t_2 + \frac{n}{2} \right) \zeta_v \left( t_1 + t_2 + \frac{n}{2} \right) L_v \left( t_2 + \frac{n}{2} \right) \prod_{r=1}^{l(n-1)/2} \zeta_v(2t_2 + n - 2r)
\]

\[
\times \prod_{v \not\in S} \phi_{P^v, \omega_v} \times \prod_{v \not\in S} \tilde{f}^{(0)}(\vec{e}).
\]

Here \( L_v(s) \) stands for \( L(\omega_v, s) \). Put

\[
E(h; \tilde{f}^{(0)}) = \sum_{\gamma \in P^v \setminus \Gamma_n} \tilde{f}^{(0)}(\gamma h)
\]

\[
= \sum_{\gamma \in P^v \setminus \Gamma_n} \sum_{\gamma_1 \in P^v \setminus \Gamma_n} \tilde{f}^{(0)}(\gamma_1 \gamma h).
\]  

(1.3.4)

The inner sum in the last expression is a degenerate Eisenstein series on \( GL_n \). In particular, the residue of this inner Eisenstein series along \( t_1 - t_2 = \frac{n}{2} \) is, up to non-zero constant, equal to

\[
L_S \left( s + \frac{n+1}{2} \right) \zeta_S(s+n-1)L_S \left( s + \frac{n-1}{2} \right) \prod_{r=1}^{l(n-1)/2} \zeta_S(2s+n+1-2r)
\]

\[
\times \prod_{v \not\in S} \phi_{\omega_v, s} \times \prod_{v \not\in S} f_v^{(s)}(\gamma h).
\]

Here \( s = t_2 + \frac{1}{2} \). So, the residue of \( E(h; \tilde{f}^{(0)}) \) along \( t_1 - t_2 = \frac{n}{2} \) is, up to non-zero constant, equal to

\[
\begin{aligned}
L_S \left( s + \frac{n}{2} \right) \zeta_S(2s)E(h; f^{(s)}), & \text{ if } n \text{ is even} \\
L_S \left( s + \frac{n-1}{2} \right) E(h; f^{(s)}), & \text{ if } n \text{ is odd}.
\end{aligned}
\]  

(1.3.5)

Put

\[
D_1 = \left\{ (t_1, t_2) | \text{Re}(t_1) > \text{Re}(t_2) + \frac{n}{2}, \text{Re}(t_2) > \frac{n}{2} \right\}.
\]
Then $\tilde{f}^{(\omega)}$ is holomorphic on $D_1$, and the summation (1.3.4) is absolutely convergent on $D_1$, so $E(h; \tilde{f}^{(\omega)})$ is holomorphic on $D_1$. Put

$$P^*_2 = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & A & * & B \\ 0 & 0 & a^{-1} & 0 \\ 0 & C & * & D \end{pmatrix} \in H_n \mid a \in k^\times, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H_{n-1} \right\}.$$ 

Then

$$E(h; \tilde{f}^{(\omega)}) = \sum_{\gamma \in P^*_1 \setminus H_1} \sum_{\gamma_1 \in P^*_1 \setminus P^*_2} \tilde{f}^{(\omega)}(\gamma_1; \gamma h).$$

(1.3.6)

The inner sum of (1.3.6) is

$$L_S(t_1 + 1) \zeta_S \left( t_1 - t_2 + \frac{n}{2} \right) \zeta_S \left( t_1 + t_2 + \frac{n}{2} \right)$$

times an Eisenstein series on $H_{n-1}$ associated to a good section of $I(\omega, t_2)$. By the induction assumption, the poles of this Eisenstein series is

$$\begin{cases} t_2 = \frac{n}{2} - m & m \in \mathbb{Z}, 0 \leq m \leq n, n \neq \frac{n}{2} \} & \text{if } \omega = 1 \\ t_2 = \frac{n-2}{2} - m & m \in \mathbb{Z}, 0 \leq m \leq n - 2, n \neq \frac{n-2}{2} \} & \text{if } \omega \neq 1 \}$$

(1.3.7)

By the functional equation of the inner Eisenstein series, $E(h; \tilde{f}^{(\omega)})$ is holomorphic on the domain

$$D_2 = \left\{ (t_1, t_2) \mid \text{Re}(t_1) > \text{Re}(t_2) + \frac{n}{2}, \text{Re}(t_1) > -\text{Re}(t_2) + \frac{n}{2}, \text{Re}(t_2) > \frac{n}{2} \right\}.$$ 

Therefore $E(h; \tilde{f}^{(\omega)})$ can be meromorphically continued to the convex closure of $D_1 \cup D_2$, and the singularities in this domain are given by (1.3.7).

Similarly, by the functional equation of degenerate Eisenstein series on $GL_n$, $E(h; \tilde{f}^{(\omega)})$ is holomorphic on the domain

$$D_3 = \left\{ (t_1, t_2) \mid \text{Re}(t_1) > 1, \text{Re}(t_2) > \text{Re}(t_1) + \frac{n}{2} \right\}$$

and can be meromorphically continued to the convex closure of $D_1 \cup D_3$. The
singularities in this domain are given by

\[ \left\{ t_1 - t_2 = \pm \frac{n}{2} \right\}. \tag{1.3.8} \]

By the same reason, \( E(h; \mathbf{f}^{(\omega)}) \) is holomorphic on

and can be meromorphically continued to the convex closure of \( D_2 \cup D_4 \). The singularity in this domain is

\[ \left\{ t_1 + t_2 = \pm \frac{n}{2} \right\}. \tag{1.3.9} \]

Thus \( E(h; \mathbf{f}^{(\omega)}) \) can be meromorphically continued to the convex closure of \( D_1 \cup D_2 \cup D_3 \cup D_4 \) and the singularity in this domain is the union of (1.3.7), (1.3.8) and (1.3.9). Therefore (1.3.5) has at most simple poles at

\[
\begin{align*}
    &s = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{n+1}{2}, \quad \text{if } n \text{ is even} \\
    &s = \frac{1}{2}, 1, 2, \ldots, \frac{n+1}{2}, \quad \text{if } n \text{ is odd}
\end{align*}
\]

for \( \text{Re}(s) \geq 0 \). Here \( \frac{n+1}{2} \) is a pole only if \( \omega = 1 \). If \( n \) is even, \( L_S \left( s + \frac{n-1}{2} \right) \) has neither poles nor zeros for \( \text{Re}(s) \geq 0 \). If \( n \) is odd, \( L_S \left( s + \frac{n-1}{2} \right) \zeta_S(2s) \) has a simple pole at \( s = \frac{1}{2} \) and has no zero at positive integral or half-integral points. Note that we already know that \( s = 0 \) is not a pole if \( n \) is odd. Thus we have proved Proposition 1.6.

COROLLARY. Let \( f^{(\omega)} \) be a global holomorphic section of \( I(\omega, s) \). Let \( S \) be a finite set of places of \( k \) such that \( f^{(\omega)} \) is invariant under \( K_v, v \notin S \). Then the set of poles of

\[ d_S(\omega, s)E(h; f^{(\omega)}) \]

is given by Proposition 1.6.

This result is also proved in [14].

If \( k \) is a function field, we can prove the following proposition similarly.
PROPOSITION 1.7. Suppose \( k \) is a function field. If \( f(s) \) is a good section of \( I(\omega, s) \), then the poles of \( E(h; f(s)) \) are at most simple. The set of possible poles is as follows.

1. When \( \omega \) is principal: we may assume \( \omega = 1 \). The set of possible poles is:

\[
\left\{ \pm \frac{n+1}{2} + \frac{2\pi\sqrt{-1}}{\log q} \mathbb{Z} \right\} \\
\cup \left\{ \frac{n-1}{2} - m + \frac{\pi\sqrt{-1}}{\log q} \mathbb{Z} | m \in \mathbb{Z}, 0 \leq m \leq n-1, m \neq \frac{n-1}{2} \right\}
\]

2. When \( \omega \) is not principal, and \( \omega^2 \) is principal: we may assume \( \omega^2 = 1 \). Then the set of possible poles is:

\[
\left\{ \frac{n-1}{2} - m + \frac{\pi\sqrt{-1}}{\log q} \mathbb{Z} | m \in \mathbb{Z}, 0 \leq m \leq n-1, m \neq \frac{n-1}{2} \right\}
\]

3. If \( \omega^2 \) is not principal, then \( E(h; f(s)) \) is entire.

REMARK. Proposition 1.6 or 1.7 implies that the possible poles of Langlands L-function of irreducible cuspidal automorphic representations of \( \text{Sp}_n \) attached to the standard representation of the L-group \( {}^\text{L} \text{Sp}_n \simeq \text{SO}(2n+1) \) are

\[
\{-n+1, -n+2, \ldots, n-1, n\}
\]

or

\[
\left\{ -n+1 + \frac{\pi\sqrt{-1}}{\log q} \mathbb{Z}, -n+2 + \frac{\pi\sqrt{-1}}{\log q} \mathbb{Z}, \ldots, n-1 + \frac{\pi\sqrt{-1}}{\log q} \mathbb{Z}, n + \frac{\pi\sqrt{-1}}{\log q} \mathbb{Z} \right\}
\]

and all of them are at most simple (cf. [14], [20], [21]).

1.4. Calculation of the residue at \( s = \frac{n-1}{2} \)

In this subsection, we assume \( \omega = 1 \). Then there exists a class 1 element of \( I(\omega, s) \).

Take \( \phi_s \in I(\omega, s) \) such that \( \phi_s|_{\kappa_s} \equiv 1 \). Put

\[
E(h, s) = E(h; \phi_s), \\
\tilde{E}(h, s) = \zeta \left( s + \frac{n+1}{2} \right) \prod_{r=1}^{[n/2]} \zeta(2s+n+1-2r)E(h, s).
\]
\( \tilde{E}(h, s) \) satisfies the following functional equation:

\[
\tilde{E}(h, s) = \tilde{E}(h, -s).
\]

We will determine the residue of \( E(h; s) \) at \( s = \frac{n-1}{2} \). Let \( P_{n,r} \) be a parabolic subgroup of \( H_n \) given by

\[
P_{n,r} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & A & * & B \\ 0 & 0 & t_a^{-1} & 0 \\ 0 & C & * & D \end{pmatrix} \in H_n \left| a \in \text{GL}_{n-r}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_r \right. \right\}.
\]

Let \( s \in \mathbb{C} \) and \( t = (t_1, t_2, \ldots, t_n) \in \mathbb{C}^n = X^* (T_n) \otimes \mathbb{Z} \). Let \( \phi(h; P_{n,r}; s), \phi(h; B_{n}; t) = \phi(h; B_{n}; t_1, t_2, \ldots, t_n) \) be the functions on \( H_n(A) \) given by

\[
\begin{align*}
\phi(pk; P_{n,r}; s) &= |a|^{s+(n+r+1)/2} \\
\phi(bk; B_{n}; s) &= \prod_{i=1}^{n} |b_i|^{t_i+n+1-i},
\end{align*}
\]

where \( k \in K_n \).

\[
p = \begin{pmatrix} a & * & * & * \\ 0 & A & * & B \\ 0 & 0 & t_a^{-1} & 0 \\ 0 & C & * & D \end{pmatrix} \in P_{n,r}(A),
\]

\[
b = \begin{pmatrix} b_1 & * & & \\ b_2 & \ddots & & \\ & \ddots & \ddots & \\ 0 & \cdots & b_n^{-1} & \end{pmatrix} \in B_n(A).
\]
Put
\[ E_{P_n}(h, s) = \sum_{\gamma \in \mathbb{P}_n \setminus H_n} \phi(\gamma h; P_{n,r}; s), \]
\[ E_{B_n}(h, t) = \sum_{\gamma \in \mathbb{B}_n \setminus H_n} \phi(\gamma h; B_n; t). \]

For any \( z \in \Phi_{H_n}^+ \), let \( l^+_z(t) \) and \( \mathcal{F}^+_z \) be linear forms and hyperplanes of \( \mathbb{C}^n \) given by
\[ l^+_z(t) = \langle \tilde{x}, t \rangle - 1, \quad l^-_z(t) = \langle \tilde{x}, t \rangle + 1, \]
\[ \mathcal{F}^+_z = \{ t \in \mathbb{C}^n | l^+_z(t) = 0 \}, \quad \mathcal{F}^-_z = \{ t \in \mathbb{C}^n | l^-_z(t) = 0 \}. \]

It is easy to see that the residue along \( \mathcal{F}^+_x, \ldots, \mathcal{F}^+_{x_{n-r}}, \mathcal{F}^-_{x_{n-r+1}}, \ldots, \mathcal{F}^+_{x_n} \) in the sense of \([9, p. 195] \) is
\[ R^{n-1} \prod_{i=2}^{n-r} \zeta(i)^{-1} \prod_{i=1}^{r} \zeta(2i)^{-1} E_{P_n}(h, t_{n-r} + \frac{n-r-1}{2}), \]
where \( R = \text{Res}_{s=1} \zeta(s) \). Put
\[ \tilde{E}_{B_n}(h, t) = \prod_{z \in \Phi_{H_n}^+} \zeta(\langle \tilde{x}, t \rangle + 1) E_{B_n}(h, t) \]
\[ = \prod_{1 \leq i < j \leq n} \zeta(t_i + t_j + 1) \zeta(t_i - t_j + 1) \prod_{i=1}^{n} \zeta(t_i + 1) E_{B_n}(h, t). \]

Then it is known that
\[ \prod_{z \in \Phi_{H_n}^+} l^+_z(t) l^-_z(t) E_{B_n}(h, t) \]
(1.4.6)
is entire and invariant under \( t \to wtw^{-1} \) for any \( w \in W_{H_n} \).

The value of (1.4.6) at \( t = \left( s + \frac{n-1}{2}, s + \frac{n-3}{2}, \ldots, s - \frac{n-1}{2} \right) \) is
\[ (2R)^{n-1} \prod_{i=2}^{n-1} (i-1)(i+1) \zeta(i)^{-n-i} \]
\[ \times \prod_{i=1}^{n} \left( s + \frac{n+3}{2} - i \right) \left( s + \frac{n-1}{2} - i \right) \zeta \left( s + \frac{n+3}{2} - i \right) \]
\[ \times \prod_{1 \leq i < j \leq n} (2s+n+2-i-j)(2s+n-i-j) \zeta(2s+n+2-i-j) \]
\[ \times E_{P_{n,0}}(h, s).]
So the value of (1.4.6) at \( t=(n-1, n-2, \ldots, 1, 0) \) is

\[
(2R)^{n-1} \prod_{i=2}^{n-1} \{ (i-1)(i+1)\zeta(i) \}^{n-i} \times (-R)!(n-2)! \prod_{i=2}^{n} \zeta(i) \times 2\zeta(2) \prod_{i=2}^{n-1} \prod_{j=1}^{i} \zeta(i+j) \times 2 \text{ Res}_{s=(n-1)/2} E_{P_{n,n-1}}(h, s).
\]

On the other hand, the value of (1.4.6) at \( t=(s, n-1, n-2, \ldots, 1) \) is

\[
(2R)^{n-1} \prod_{i=2}^{n-1} \{ (i-1)(i+1)\zeta(i) \}^{n-i} \times \prod_{1 \leq i < j \leq n-1} (i+j+1)(i+j-1)\zeta(i+j) \times 2^{n-1} \prod_{i=1}^{n} (s-n+i+1)(s-n+i-1)\zeta(s-n+i+1) \times E_{P_{n,n-1}}(h, s).
\]

It follows that \( E_{P_{n,n-1}}(h, s) \) is holomorphic at \( s=0 \), and the value of (1.4.6) at \( t=(0, n-1, n-2, \ldots, 1) \) is

\[
(2R)^{n-1} \prod_{i=2}^{n-1} \{ (i-1)(i+1)\zeta(i) \}^{n-i} \times \prod_{1 \leq i < j \leq n-1} (i+j+1)(i+j-1)\zeta(i+j) \times (-R^2)(n!)^2 \prod_{i=2}^{n} \zeta(i) \prod_{i=2}^{n-1} \zeta(i) \times E_{P_{n,n-1}}(h, 0).
\]

Thus we get the following proposition.
PROPOSITION 1.8.

\[
\text{Res}_{s=(n-1)/2} E_{\rho_{n,0}}(h, s) = \frac{1}{2} R \prod_{i=1}^{[n/2]-1} \xi(2i+1) \prod_{i=1}^{[n/2]} \xi(2n-2i)^{-1} E_{\rho_{n-1}}(h, 0),
\]

or, equivalently

\[
\text{Res}_{s=(n-1)/2} \tilde{E}_{\rho_{n,0}}(h, s) = \frac{1}{2} R \xi(n) \prod_{i=1}^{[n/2]-1} \xi(2i+1) E_{\rho_{n-1}}(h, 0).
\]

LEMMA 1.9. \( I \left( 1, \frac{n-1}{2} \right) \) is generated by class 1 vectors.

Proof. Let \( \chi \) be a character of \( T_n \) given by

\[
\chi(t) = \prod_{i=1}^{n} |t_i|^{n-i}.
\]

Then \( I \left( 1, \frac{n-1}{2} \right) \) is a quotient of \( \text{Ind}_{H_n}^{H_n} \chi \). It is sufficient to prove that \( \text{Ind}_{H_n}^{H_n} \chi \) is generated by class 1 vectors. Let \( P \) be the standard parabolic subgroup of \( H_n \) corresponding to \( \alpha_n \). Then

\[
\text{Ind}_{H_n}^{H_n} \chi = \text{Ind}_{P}^{H_n}(\text{Ind}_{B_n}^{P} \chi).
\]

The restriction of \( \text{Ind}_{B_n}^{P} \chi \) to \( \iota_{2n}(\text{SL}_2) \) is an irreducible tempered representation. Let \( M \) be the standard Levi factor of \( P \) and \( w \) be the longest element of \( W_{M \backslash W_{H_n}} \), i.e.,

\[
w = \begin{pmatrix}
1 & \begin{pmatrix} -1 \end{pmatrix}_{n-1} \\
1_{n-1} & 1
\end{pmatrix}.
\]

By the well-known theory of Langlands quotient, \( \text{Ind}_{H_n}^{H_n}(\text{Ind}_{B_n}^{P} \chi) \) is generated by any element \( f \) such that \( M_w f \neq 0 \). It is easy to check that a non-zero class 1 vector satisfies this condition.
Let $f^{(s)}$ be any good section of $I(1, s)$. Put

$$w = w_{[2, \ldots, n]}$$

$$= \begin{pmatrix}
1 & 0 & \cdots & -1 \\
0 & 1 & \cdots & 0 \\
& 0 & \ddots & 1 \\
& & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (1.4.7)

It is easy to check that $M_w f^{(s)}$ has at most a simple pole at $s = \frac{n-1}{2}$ and

$$\text{Res}_{s = (n-1)/2} M_w f^{(s)}$$

is in $\text{Ind}_{\sigma, n-1}^H$. An easy calculation shows

$$\text{Res}_{s = (n-1)/2} M_w \phi(h; P_{n,0}; s)$$

$$= R \prod_{i=1}^{[n/2]-1} \xi(2i+1) \prod_{i=1}^{[n/2]} \xi(2n-2i)^{-1} \phi(h; P_{n,n-1}; 0).$$

Thus by Proposition 1.8,

$$\text{Res}_{s = (n-1)/2} E_{P_{n,0}}(h, \phi(h; P_{n,0}; s))$$

$$= \frac{1}{2} E_{P_{n,n-1}}(h, \text{Res}_{s = (n-1)/2} M_w \phi(h; P_{n,0}; s)).$$

**Proposition 1.10.**

$$\text{Res}_{s = (n-1)/2} E_{P_{n,0}}(h; f^{(s)}) = \frac{1}{2} E_{P_{n,n-1}}(h; \text{Res}_{s = (n-1)/2} M_w f^{(s)}).$$

**Proof.** By Proposition 1.8, this equation holds for a non-zero class 1 vector. Since both sides are $H_n$-equivariant, it holds for any $f^{(s)}$. 
2. Triple L-functions

Let \( k \) be a global field. Let \( K \) be a semi-simple abelian algebra of degree 3 over \( k \). There are three cases:

Case (1) \( K = k \oplus k \oplus k \).
Case (2) \( K = k \oplus k', k' \) is a quadratic extension of \( k \).
Case (3) \( K = k'', k'' \) is a cubic extension of \( k \).

Let \( G \) be an algebraic group defined over \( k \) given by

\[
G = \{ g \in \text{GL}_2(K) \mid \det g \in k^\times \}.
\]

Thus \( G \) is

Case (1) \( \{(g^{(1)}, g^{(2)}, g^{(3)}) \in (\text{GL}_2)^3 \mid \det g^{(1)} = \det g^{(2)} = \det g^{(3)}\} \),
Case (2) \( \{(g^{(1)}, g^{(2)}) \in \text{GL}_2 \times R_{k'/k} \text{GL}_2 \mid \det g^{(1)} = \det g^{(2)}\} \),
Case (3) \( \{g \in R_{k''/k} \text{GL}_2 \mid \det g \in k^\times \} \).

As in [22, §0], we take an 8-dimensional representation \( \sigma \) of the L-group of \( \text{GL}_2(K) \). The L-group is the semi-direct product of \( \text{GL}_2(C) \times \text{GL}_2(C) \times \text{GL}_2(C) \) and \( W_k \). \( W_k \) acts by permuting the three \( \text{GL}_2(C) \) factors. The restriction of \( \sigma \) to \( \text{GL}_2(C) \times \text{GL}_2(C) \times \text{GL}_2(C) \) is \( \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \), where \( \sigma_2 \) is the standard 2-dimensional representation of \( \text{GL}_2(C) \). The restriction of \( \sigma \) to \( W_k \) is the permutation of the three factors.

We denote by \( Z \) the connected component of the center of \( G \). \( Z \) is naturally isomorphic to \( \text{GL}_1 \). We embed \( G \) into

\[
\text{GSp}_3 = \left\{ h \in \text{GL}_6 \mid h \begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix} ; h = m(h) \begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix} , m(h) \in k^\times \right\}
\]

as in [22, §1]. We denote this embedding by \( i \).

Let \( \Pi \) be an irreducible cuspidal automorphic representation of \( \text{GL}_2(A \otimes K) \), i.e.,

Case (1) \( \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \), where \( \pi_1, \pi_2, \) and \( \pi_3 \) are irreducible cuspidal automorphic representation of \( \text{GL}_2(A_k) \),
Case (2) \( \Pi = \pi_1 \otimes \pi_2 \), where \( \pi_1 \) (resp. \( \pi_2 \)) is an irreducible cuspidal automorphic representation of \( \text{GL}_2(A_k) \) (resp. \( \text{GL}_2(A_k) \)),
Case (3) \( \Pi \) is an irreducible cuspidal automorphic representation of \( \text{GL}_2(A_{k''}) \).

Let \( \Omega_{\Pi} \) be the central quasi-character of \( \Pi \), and \( \omega_{\Pi} \) be the restriction of \( \Omega_{\Pi} \) to
Let $\omega = \omega_{\Pi}$. Let $\mathcal{W}(\Pi, \psi)$ be the Whittaker model of $\Pi$, i.e.,

Case (1) $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi) \otimes \mathcal{W}(\pi_3, \psi),$

Case (2) $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \psi | \operatorname{tr}_{k/k})$, 

Case (3) $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\Pi, \psi | \operatorname{tr}_{k/k})$.

If $\varphi$ is a cusp form belonging to $\Pi$, then there exists $W \in \mathcal{W}(\Pi, \psi)$ such that

$$
\varphi(g) = \sum_{v \in K} W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right).
$$

We assume that $W$ is decomposable: $W = \Pi_v W_v$. Here, $v$ runs over all places of $k$. Put

$$
P = \left\{ \begin{pmatrix} mA & * \\ 0 & t_A^{-1} \end{pmatrix} \in \operatorname{GSp}_3 \right\}.
$$

By [22, §1], the double cosets $P \backslash \operatorname{GSp}_3 / \iota(G)$ contains one open coset and the other cosets are all negligible in the terminology of [20]. We choose a representative $\eta_0$ of the open double coset and put

$$
R_0 = \{ g \in G \mid \eta_0 \iota(g) \eta_0^{-1} \in P \}.
$$

We can choose $\eta_0$ so that

$$
R_0 = \left\{ \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \in \operatorname{GL}_2(K) \mid a \in k^\times, \operatorname{tr}_{k/k} n = 0 \right\}.
$$

Let $v$ be a place of $k$. Let $J(\omega_v, s)$ be the space of functions $f_v(h)$ on $\operatorname{GSp}_3(k_v)$ which satisfy the following (i) and (ii):

(i) $f_v$ is right finite by the standard maximal compact subgroup of $\operatorname{GSp}_3(k_v)$.

(ii) For $p = \begin{pmatrix} mA & * \\ 0 & t_A^{-1} \end{pmatrix} \in P(k_v)$,

$$
f_v(ph) = \omega_v(m) |m|^{3s + (3/2) \omega_v(\det A)} |\det A|^{2s} f_v(h).
$$

Observe that if $f_v \in J(\omega_v, s)$, then $f_v|_{\operatorname{Sp}_3(k_v)} \in I(\omega_v, 2s - 1)$. We define holomorphic sections and meromorphic sections of $J(\omega_v, s)$ in the same way as in Section 1. The intertwining operator $M_w$ can be defined similarly. We define a meromorphic section $f_v(\omega)$ is good if

$$
[d(\omega_v, 2s - 1) c_w(\omega_v, 2s - 1)]^{-1} M_w f_v(\omega)
$$
is holomorphic for all \( w \in \Omega_3 \). Obviously this condition is equivalent to say that 
\( \rho(\phi)f_v^{(s)}|_{\text{Sp}_3(k_v)} \) is a good section of \( I(\omega_v, 2s - 1) \) for each Hecke operator \( \phi \) on \( \text{Sp}_3(k_v) \). By Lemma 1.2, \( f_v^{(s)}(h) \) is a good section of \( J(\omega_v, s) \) if and only if 
\( \omega_v(m(h))M^*_{w_0}f_v^{(s)}(h) \) is a good section of \( J(\omega_v^{-1}, 1 - s) \), where \( m(h) \) is the multiplier of \( h \), and by Lemma 1.3, any holomorphic section of \( J(\omega_v, s) \) is a good section. 

For each meromorphic section \( f_v^{(s)} \in J(\omega_v, s) \), and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \), put

\[
\Psi_s(f_v^{(s)}, W_v) = \int_{R_{w_0}\backslash G_v} f_v^{(s)}(\eta_0 \iota(g)) W_v(g) \, dg.
\]

In [7], [22], it is proved that \( \Psi_s(f_v^{(s)}, W_v) \) is absolutely convergent for \( \Re(s) \gg 0 \), and has meromorphic continuation to \( \mathbb{C} \), and if \( v \) is non-archimedean, \( \Psi_s(f_v^{(s)}, W_v) \) is a rational function of \( q_v^{-s} \). By [22, Proposition 3.3], for each \( s_0 \in \mathbb{C} \), there exists a holomorphic section \( f_v^{(s)} \) of \( J(\omega_v, s) \), and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \) such that

\[
\Psi_{s_0}(f_v^{(s_0)}, W_v) \neq 0.
\]

Put \( \tilde{W}_v(g) = \Omega_v(\det g)^{-1} W_v(g) \), where \( \Omega_v \) is the central quasi-character of \( \Pi_v \). Then \( \tilde{W}_v \in \mathcal{W}(\tilde{\Pi}_v, \psi_v) \). It is proved in [7], [22], that there exists a meromorphic function \( \varepsilon(s, \Pi_v, \sigma, \psi_v) \) such that

\[
\Psi_{1-s}(\omega_v(m(h))M^*_{w_0}f_v^{(s)}; \tilde{W}_v) = \varepsilon(s, \Pi_v, \sigma, \psi_v) \Psi_s(f_v^{(s)}; W_v).
\]

For a non-archimedean place \( v \), we consider the fractional ideal \( I_v \) of \( \mathbb{R}_v = \mathbb{C}[q_v^{-s}, q_v^s] \), generated by \( \Psi_s(f_v^{(s)}; W_v) \) attached to good sections \( f_v^{(s)} \) of \( J(\omega_v, s) \) and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \). Then by [22, Appendix 3 to §3], \( I_v \) admits a common denominator and \( 1 \in I_v \). Thus \( I_v \) has a generator of the form \( P(q_v^{-s})^{-1} \), \( P(X) \in \mathbb{C}[X] \), \( P(0) = 1 \). We let

\[
L(s, \Pi_v, \sigma) = P(q_v^{-s})^{-1},
\]

\[
\varepsilon(s, \Pi_v, \sigma, \psi_v) = \varepsilon(s, \Pi_v, \sigma, \psi_v) L(s, \Pi_v, \sigma) L(1-s, \tilde{\Pi}_v, \sigma)^{-1},
\]

then \( \varepsilon(s, \Pi_v, \sigma, \psi_v) \) is of the form \( a q_v^{bs} \), \( a \in \mathbb{C}, b \in \mathbb{Z} \), and

\[
\frac{\Psi_{1-s}(\omega_v(m(h))M^*_{w_0}f_v^{(s)}; \tilde{W}_v)}{L(1-s, \tilde{\Pi}_v, \sigma)} = \varepsilon(s, \Pi_v, \sigma, \psi_v) \frac{\Psi_s(f_v^{(s)}; W_v)}{L(s, \Pi_v, \sigma)}.
\]

When \( v \) is unramified, this definition agrees to usual definition \( \det(1_g - \sigma(g_v, \text{Fr})q_v^{-s})^{-1} \), where \( g_v \) is the Langlands class of \( \Pi_v \). For a holomorphic section \( f_v^{(s)} \) and \( W_v \in \mathcal{W}(\Pi_v, \psi_v) \), a careful calculation of denominator of
$\Psi_s(f_v^{(n)}; W_v)$ shows that the denominator divides $\det(1 - \sigma(g_v, Fr)q_v^{-s})$ (cf. [22, Appendix 3 to §3]). It follows that $L(s, \Pi_v, \sigma)^{-1}$ is a divisor of $d(\omega_v, 2s - 1)^{-1} \det(1 - \sigma(g_v, Fr)q_v^{-s})$. On the other hand, there are a good section $f_v^{(n)}$ of $J(\omega_v, s)$ and $W_v \in \mathcal{W}(\Pi_v, \psi_v)$ such that $\Psi_s(f_v^{(n)}; W_v)^{-1} = \det(1 - \sigma(g_v, Fr)q_v^{-s})^{-1}$. This shows that $L(s, \Pi_v, \sigma)^{-1}$ is a multiple of $\det(1 - \sigma(g_v, Fr)q_v^{-s})$. Moreover we know

$$e'(s, \Pi_v, \sigma, \psi_v) = \frac{\det(1 - \sigma(g_v, Fr)q_v^{-s})}{\det(1 - \sigma(g_v, Fr)^{-1}q_v^{s-1})}$$

Since $d(\omega_v, 2s - 1)^{-1}$ and $d(\omega_v^{-1}, 1 - 2s)^{-1}$ have no common divisor, we have $L(s, \Pi_v, \sigma) = \det(1 - \sigma(g_v, Fr)q_v^{-s})^{-1}$, as we expected.

When $k_v$ is archimedean, we define $L$-factor $L(s, \Pi_v, \sigma)$ as follows. The proof of [7, Proposition 5.1] shows that there is a meromorphic function $\alpha(s) \neq 0$ such that

$$\alpha(s)^{-1} \Psi_s(f_v^{(n)}; W_v)$$

is holomorphic for any holomorphic section $f_v^{(n)}$ and $W_v \in \mathcal{W}(\Pi_v, \psi_v)$. Though [7] has dealt with only case (1), it is not difficult to generalize the result to the case $k_v = \mathbb{R}, K_v = \mathbb{R} \oplus \mathbb{C}$. We have only to use the local functional equation of Asai-type $L$-functions instead of the results of [8]. By Weierstrass theorem, there is a meromorphic function $\lambda(s)$ such that

$$\lambda(s)^{-1} \Psi_s(f_v^{(n)}; W_v)$$

is holomorphic for any good section $f_v^{(n)}$ and $W_v \in \mathcal{W}(\Pi_v, \psi_v)$ and if $\lambda'(s)$ is another function with this property, then $\lambda(s)\lambda'(s)^{-1}$ is holomorphic. Obviously, for each $s_0 \in \mathbb{C}$, there exists a good section $f_v^{(n)}$ and $W_v \in \mathcal{W}(\Pi_v, \psi_v)$ such that (2.2) does not have a zero at $s = s_0$. By Lemma 1.3 and [22, Proposition 3.3], $\lambda(s)$ has no zeros. We define $L(s, \Pi_v, \sigma) = \lambda(s)$. Then (2.1) holds with some entire function $\varepsilon(s, \Pi_v, \sigma, \psi_v)$ which have no zeros. Note that $L(s, \Pi_v, \sigma)$ and $\varepsilon(s, \Pi_v, \sigma, \psi_v)$ is determined only up to entire functions which have no zeros.

Let $v$ be any place of $k$. Assume $\Pi_v$ is unitary. We define a non-negative real number $\lambda(\Pi_v)$ as follows.

Case (1) $\Pi_v = \pi_1 \otimes \pi_2 \otimes \pi_3$: When $\pi_i$ is tempered, put $\lambda(\pi_i) = 0$. When $\pi_i$ is the complementary series $\pi(\mu^0, \mu^0)$, ($\mu$ is a unitary character of $k_v^*$), put $\lambda(\pi_i) = |\mu|$. Put $\lambda(\Pi_v) = \lambda(\pi_1) + \lambda(\pi_2) + \lambda(\pi_3)$.

Case (2) $\Pi_v = \pi_1 \otimes \pi_2$: let $\lambda(\pi_i)$ be as above, and put $\lambda(\Pi_v) = \lambda(\pi_1) + 2\lambda(\pi_2)$.

Case (3) $\Pi_v = \pi_1$: let $\lambda(\pi_1)$ be as above, and put $\lambda(\Pi_v) = 3\lambda(\pi_1)$.
LEMMA 2.1. If $\Pi_v$ is unitary, then $L(s, \Pi_v, \sigma)$ has no poles on the domain $\text{Re}(s) > \lambda(\Pi_v)$.

Proof. By an argument similar to [7, Theorem 1], [22, Proposition 3.2], we can show that if $f_v^{(s)}$ is a holomorphic section of $J(\omega_v, s)$ and $W_v \in \mathcal{W}(\Pi_v, \psi_v)$, then $\Psi_s(f_v^{(s)}, W_v)$ is absolutely convergent for $\text{Re}(s) > \lambda(\Pi_v)$. Since $d(\omega_v, s)$ has no poles for $\text{Re}(s) > 0$, a good section $f_v^{(s)}$ is holomorphic for $\text{Re}(s) > 0$. This proves the lemma.

LEMMA 2.2. Assume $K$ is not a cubic extension of $k$. Assume $\Pi_v$ is unitary. Assume each component is a subquotient of a principal series, and $\lambda(\Pi_v) < 1/2$. Then $L(s, \Pi_v, \sigma)$ (resp. $\varepsilon(s, \Pi_v, \sigma, \psi_v)$) agrees to $L$-factor (resp. $\varepsilon$-factor) associated to the 8-dimensional representation of the Weil group $W_{k_v}$ determined by $\Pi_v$ and $\sigma$.

Proof. By [7, Proposition 5.1], $\varepsilon'(s, \Pi_v, \sigma, \psi_v)$ coincides $\varepsilon'$-factor determined by the Weil group. The proof of [7] Proposition 5.1 works for case (2). By the assumption, $L(s, \Pi_v, \sigma)$ has no poles on the domain $\text{Re}(s) > \lambda(\Pi_v)$ and $L(1-s, \Pi_v, \sigma)$ has no poles on the domain $\text{Re}(s) < 1 - \lambda(\Pi_v)$. This proves the lemma.

REMARK. By Lemma 2.2, we can identify the archimedean L-factors and usual $\Gamma$-factors if $\Pi$ is generated by Hilbert modular forms over a totally real field.

COROLLARY. Assume $K$ is not a cubic extension of $k$. Assume $\Pi_v$ is unitary. Assume no component is extraordinary, and $\lambda(\Pi_v) < 1/2$. Then the conclusion of Lemma 2.2 holds.

Proof. For simplicity, we assume $K = k \oplus k \oplus k$, $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$, and all of $\pi_{1,v}, \pi_{2,v}$ and $\pi_{3,v}$ are supercuspidal. $\pi_{i,v} = \pi(\chi_{i,v})$ ($i = 1, 2, 3$) for some quasi-character $\chi_{i,v}$ of some quadratic extension $K_{i,v}$ of $k_v$. Choose global quadratic extension $K_i$ of $k$ such that $K_{i,v} = K_{i,v}$. It is easy to check that there exists global quasi-character $\chi_i$ of $A_{k_v}$ such that $v$-part of $\chi_i$ is $\chi_{i,v}$ and $\pi(\chi_i)$ is principal series outside of $v$ and all archimedean place. Put $\Pi = \pi(\chi_1) \otimes \pi(\chi_2) \otimes \pi(\chi_3)$. Then $L(s, \Pi, \sigma)$ is L-function associated to 8-dimensional representation of global Weil group. The conclusion of Lemma 2.2 holds outside $v$, so does at $v$.

We now consider the global theory. We say that a meromorphic section of $J(\omega, s)$ is a good section if it is a finite sum of decomposable elements $f_v^{(s)} = \Pi_v f_v^{(s)}$, satisfying the following two conditions:

(i) For almost all unramified places $v$, $f_v^{(s)}|_{k_v} = d(\omega_v, 2s - 1)$.
(ii) $f_v^{(s)}$ is a good section of $J(\omega_v, s)$ for all $v$.

Note that the infinite product $\Pi_v f_v^{(s)}$ is absolutely convergent for $\text{Re}(s) > 0$, and can be meromorphically continued to $C$. 

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For each good section \( f^{(s)} \) of \( J(\omega, s) \), put

\[
E(h; f^{(s)}) = \sum_{\gamma \in \Gamma(Sp_3)} f^{(s)}(\gamma h).
\]

Then the restriction of \( E(h; f^{(s)}) \) to \( Sp_3(A) \) is an Eisenstein series on \( Sp_3(A) \) investigated in Section 1.3. In [7], [22], it is proved that if \( f^{(s)} = \Pi_v f_v^{(s)} \) is decomposable, then

\[
\int_{Z(A)G(k) \backslash G(A)} E(t(g); f^{(s)}) \varphi(g) \, dg = \prod_v \Psi_s(f_v^{(s)}, W_v),
\]

for \( \Re(s) \gg 0 \). Set

\[
L(s, \Pi, \sigma) = \prod_v L(s, \Pi_v, \sigma)
\]

and

\[
\varepsilon(s, \Pi, \sigma) = \prod_v \varepsilon(s, \Pi_v, \sigma, \psi_v).
\]

Then by Proposition 1.6, (2.1), and (2.3), we have the following propositions.

**Proposition 2.3.** \( L(s, \Pi, \sigma) \) can be meromorphically continued to \( \mathbb{C} \). It is entire if \( \omega^2 = 0 \) is not a principal quasi-character. If \( \omega^2 = 1 \), and \( k \) is a number field, then \( L(s, \Pi, \sigma) \) has possible poles at \( s = 0, 1 \). If \( \omega^2 = 1 \), and \( k \) is a function field with constant field \( F_q \), then \( L(s, \Pi, \sigma) \) has possible poles at \( s \in \frac{\pi \sqrt{-1}}{2 \log q} \mathbb{Z}, 1 + \frac{\pi \sqrt{-1}}{2 \log q} \mathbb{Z} \).

All the possible poles are at most simple.

**Proposition 2.4.** \( L(s, \Pi, \sigma) \) satisfies the following functional equation:

\[
L(s, \Pi, \sigma) = \varepsilon(s, \Pi, \sigma)L(1 - s, \tilde{\Pi}, \sigma).
\]

Now we investigate the poles of \( L(s, \Pi, \sigma) \). By Proposition 2.3, we may assume \( \omega^2 = 1 \) and \( s = 0 \) or 1. By the functional equation, \( s = 0 \) is reduced to \( s = 1 \). If \( L(s, \Pi, \sigma) \) has a pole at \( s = 1 \), then there exists a good section \( f^{(s)} \) of \( J(\omega, s) \) and a cusp form \( \varphi \) belonging to \( \Pi \) such that

\[
\int_{Z(A)G(k) \backslash G(A)} [\text{Res}_{s=1} E(t(g); f^{(s)})] \varphi(g) \, dg \neq 0.
\]

**Proposition 2.5.** If \( \omega = 1 \), then \( L(s, \Pi, \sigma) \) is holomorphic at \( s = 1 \). In
particular, if \( k \) is a number field, \( L(s, \Pi, \sigma) \) is entire (cf. [22, Theorem 5.1]).

Proof. By Proposition 1.10, the restriction of \( \text{Res}_{s=1} E(h; f^{(s)}) \) to \( \text{Sp}_3 \) is an Eisenstein series associated to a function in the representation induced from the trivial character of the maximal parabolic subgroup \( P_{3,2} \). It is easy to see that each coset in \( (i(G) \cap \text{Sp}_3) \backslash \text{Sp}_3 / P_{3,2} \) is negligible. It follows that (2.4) is identically zero.

We now assume that \( \omega^2 = 1, \omega \neq 1 \) and \( L(s, \Pi, \sigma) \) has a pole at \( s = 1 \). Let \( K \) be the quadratic extension of \( k \) corresponding to \( \omega \) by class field theory, and \( \theta \) be the non-trivial element of \( \text{Gal}(K/k) \).

Suppose that \( K = k'', k'' \) is a cubic extension of \( k \). Let \( \Pi_K \) be the base change of \( \Pi \) to \( \text{GL}_2(A_{k''}) \) (cf. [18]). Consider the triple \( L \)-function \( L(s, \Pi_K, \sigma_K) \) of \( \Pi_K \) over \( K \). Here, \( \sigma_K \) is the restriction of \( \sigma \) to the semi-direct product of \( \text{GL}_2(C) \times \text{GL}_2(C) \times \text{GL}_2(C) \) and \( W_K \). Then an easy calculation shows

\[
L(s, \Pi_K, \sigma_K) = L(s, \Pi \otimes \tilde{\omega}, \sigma)L(s, \Pi, \sigma).
\]

Here, \( \tilde{\omega} \) is any extension of \( \omega \) to \( \mathbb{A}_{k''} \). Note that \( G \) is a Levi subgroup of the quasi-split simply connected group \( \text{Spin}(8) \) of either type \( 3D_4 \) or \( 6D_4 \) according as \( k''/k \) is cyclic or not (see Shahidi [23]). Then [23, Theorem 5.1] implies

\[
L(1 + 2s, \omega)L(1 + s, \Pi \otimes \tilde{\omega}, \sigma) \neq 0
\]

for \( \text{Re}(s) = 0 \). Since \( \omega \) is a non-trivial unitary character of \( A_{k''}^\times \), this implies the non-vanishing of \( L(s, \Pi, \sigma) \) at \( s = 1 \). So, \( L(s, \Pi_K, \sigma_K) \) has a pole at \( s = 1 \). But since \( \omega_{\Pi_K} = 1 \), \( \Pi_K \) cannot be cuspidal by Proposition 2.5. It follows that there is a quasi-character \( \chi \) of \( A_{k''}^\times \) such that \( \Pi = \pi(\chi) \). By a simple calculation, the triple \( L \)-function \( L(s, \pi(\chi), \sigma) \) is given by

\[
L(s, \pi(\chi), \sigma) = L_k(s, \chi|_{A_k^\times})L_{k''}(s, (\chi \circ N_{k''/k})\chi^{-1}\chi^\theta). \tag{2.5}
\]

Here, \( \theta \) is regarded as an element of \( \text{Gal}(k''/k''') \), by the natural isomorphism \( \text{Gal}(k''/k') \simeq \text{Gal}(K/k) \). This equality holds up to bad prime factors. But in fact, (2.5) is an equality of global \( L \)-functions. To see this, observe that

\[
\prod_{v \in S} e'(s, \Pi_v, \sigma, \psi_v)
\]

has no zero on \( \text{Re}(s) > 0 \), and has no poles on \( \text{Re}(s) < 1 \), by comparing the functional equation as a triple \( L \)-function and that as a \( L \)-function associated to 8-dimensional representation of the Weil group. By Lemma 2.1,
coincides with the product of L-factors of the right-hand side, since \( \lambda(\Pi) = 0 \) for \( \Pi = \pi(\chi) \). It follows that (2.5) is an equality of global L-functions.

Let us prove \( \chi|_{A^*_k} = 1 \). First observe that \( \chi|_{A^*_k} = 1 \), since \( \omega_{\pi(x)} = \omega \cdot \chi|_{A^*_k} \). Suppose \( \chi|_{A^*_k} \neq 1 \). Then \( L_{k,K}(s, (\chi \circ N_{k,K/K})\chi^{-1}) \) has a pole at \( s = 1 \), therefore we have

\[
\chi \circ N_{k,K/K} = \chi(\theta)^{-1}.
\]

Put \( I = \text{Im}(N_{k,K/K} : A^*_{k,K} \to A^*_k) \). Then the index \([A^*_k : I \cdot K^*] \) is 1 or 3, by the class fields theory. Let \( y \in A^*_k, x = N_{k,K/K}(y) \). Then

\[
\chi(\theta)(x) = \chi(y^0)(y^{-1}) = \chi(x)^{-1}.
\]

It follows that

\[
\chi(x^3) = \chi(N_{k,K/K}(x)) = \chi(x)^{\theta}(x)^{-1} = \chi(x^2).
\]

So \( \chi \) is trivial on \( I \cdot K^* \). It follows that \( \chi|_{A^*_k} = 1 \), since \( I \cdot K^* \cdot A^*_k = A^*_k \). Thus we have proved the following theorem.

**THEOREM 2.6.** Suppose that \( K = k'' \), \( k'' \) is a cubic extension of \( k \), and \( L(s, \Pi, \sigma) \) has a pole somewhere. Then

(a) Let \( \Pi', \omega' \) be the objects obtained by twisting \( \pi_1 \) by \( \omega^o, s_0 \in \mathbb{C} \). Then \( \omega'^2 = 1, \omega' \neq 1 \), and \( L(s, \Pi', \sigma) \) has a simple pole at \( s = 1 \), for some \( s_0 \in \mathbb{C} \).

(b) Assume that \( \omega^2 = 1, \omega \neq 1 \), and \( L(s, \Pi, \sigma) \) has a pole at \( s = 1 \). Let \( K \) be the quadratic extension of \( k \) corresponding to \( \omega \) by class field theory. Let \( \theta \) be the non-trivial element of \( \text{Gal}(k''/k'') \). Then there exists a quasi-character \( \chi \) of \( A^*_{k,K}/k''K^* \) such that \( \Pi = \pi(\chi) \) and \( \chi|_{A^*_k} = 1 \). Moreover the triple L-function is given by

\[
L(s, \pi(\chi), \sigma) = \zeta_K(s)L_{k,K}(s, \chi^{-1}\theta^0).
\]

Next, suppose that \( K = k \oplus k \oplus k \), \( \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \). By the assumption, \( \omega_1 \omega_2 \omega_3 = \omega \). Let \( \pi_i, K \) \((i = 1, 2, 3)\) be the base change of \( \pi_i \) to \( GL_2(A_k) \). Put \( \Pi_K = \pi_{1,K} \otimes \pi_{2,K} \otimes \pi_{3,K} \). Then,

\[
L(s, \Pi_K, \sigma_K) = L(s, \Pi \otimes \omega, \sigma)L(s, \Pi, \sigma).
\]
Here, $\Pi \otimes \omega$ means $(\pi_1 \otimes \omega) \otimes \pi_2 \otimes \pi_3$. As in case (3), the left-hand side has a pole at $s = 1$, and $\omega_{\Pi_1} = 1$. This time, we can deduce that one of $\pi_{1,k}, (i = 1, 2, 3)$, say $\pi_{1,k}$, is not cuspidal. So there is a quasi-character $\chi$ of $A_k^x/K^x$ such that $\chi_{|A_k}$ is cuspidal. Observe that $\chi_{|A_k} = \omega_2^{-1}\omega_3^{-1}$, since the central quasi-character of $\pi(\chi)$ is $\omega \cdot \chi_{|A_k}$. The triple L-function $L(s, \Pi, \sigma)$ is given by

$$L(s, \Pi, \sigma) = L_k(s, (\pi_{2,k} \otimes \chi) \times \pi_{3,k}).$$

Let us now prove that neither $\pi_{2,k}$ nor $\pi_{3,k}$ are cuspidal. Suppose that $\pi_{2,k}$ or $\pi_{3,k}$, say $\pi_{2,k}$, is cuspidal. Then

$$\pi_{2,k} \otimes \chi \simeq \tilde{\pi}_{3,k}.$$  \hspace{1cm} (2.6)

In particular, $\pi_{3,k}$ is cuspidal, too. Since $\pi_{2,k}$ and $\pi_{3,k}$ are $\theta$-invariant,

$$\pi_{2,k} \otimes \chi^\theta \simeq \tilde{\pi}_{3,k}.$$  \hspace{1cm} (2.7)

Put $\varepsilon = \chi(\chi^\theta)^{-1}$. Since $\pi(\chi)$ is cuspidal, $\varepsilon \neq 1$. By (2.6) and (2.7), we have $\pi_{2,k} \otimes \varepsilon \simeq \pi_{2,k}$. It follows that $\varepsilon^2 = 1$. Since $\varepsilon_\theta = \varepsilon^{-1} = \varepsilon$, there is a character $\varepsilon'$ of $A_k^{x}/k^{x}$ such that $\varepsilon = \varepsilon' \circ N_{k/k}$. Taking the central quasi-character of (2.6), we have

$$(\omega_2 \circ N_{k/k})\chi^2 = (\omega_3 \circ N_{k/k})^{-1}.$$

Put $I = \text{Im}(N_{k/k} : A_k^{x} \to A_k^{x})$. Let $y \in A_k^{x}$, $x = N_{k/k}(y)$. Then

$$\omega_2(x) = \omega_3(x)^{-1}\chi(y)^{-2}$$
$$= \omega_3(x)^{-1}\chi(y)^{-1} \omega_3(y)^{-1} \varepsilon(y)$$
$$= \omega_3(x)^{-1} \chi(x)^{-1} \varepsilon'(x).$$

It follows that

$$\omega_1(x)\omega_2(x)\omega_3(x) = \chi(x)\omega(x) \omega_3(x)^{-1} \chi(x)^{-1} \varepsilon'(x) \omega_3(x)$$
$$= \omega(x)\varepsilon'(x).$$

This contradicts the assumption $\omega_1 \omega_2 \omega_3 = \omega$, since $\varepsilon'$ is not trivial on $I$.

We have proved that there are quasi-characters $\chi_i (i = 1, 2, 3)$ of $A_k^{x}$ such that $\pi_i = \pi(\chi_i)$. The triple L-function is given by

$$L(s, \Pi, \sigma) = L_k(s, \chi_1 \chi_2 \chi_3)L_k(s, \chi'^{\theta}_1 \chi_2 \chi_3)L_k(s, \chi_1 \chi'^{\theta}_2 \chi_3)L_k(s, \chi_1 \chi_2 \chi'^{\theta}_3).$$
In this case, this equality holds for every local L-factor, by Lemma 2.2. Replacing \( \chi_i \) by \( \chi_i^\theta \) if necessary, we have \( \chi_1\chi_2\chi_3 = 1 \). We have proved the following theorem.

**THEOREM 2.7.** Suppose that \( K = k \oplus k \oplus k \), and \( L(s, \Pi, \sigma) \) has a pole somewhere. Then the following two assertions hold:

(a) Let \( \Pi', \omega' \) be the objects obtained by twisting \( \pi_1 \) by \( x^{s_0}, s_0 \in \mathbb{C} \). Then \( \omega'^2 = 1 \), \( \omega' \neq 1 \), and \( L(s, \Pi', \sigma) \) has a simple pole at \( s = 1 \), for some \( s_0 \in \mathbb{C} \).

(b) Assume that \( \omega^2 = 1 \), \( \omega \neq 1 \), and \( L(s, \Pi, \sigma) \) has a pole at \( s = 1 \). Let \( K \) be the quadratic extension of \( k \) corresponding to \( \omega \) by class field theory. Let \( \theta \) be the generator of \( \text{Gal}(K/k) \). Then there exist quasi-characters \( \chi_1, \chi_2, \) and \( \chi_3 \) of \( \mathbb{A}_k^* / K^* \) such that \( \pi_1 = \pi(\chi_1), \pi_2 = \pi(\chi_2), \pi_3 = \pi(\chi_3), \) and \( \chi_1\chi_2\chi_3 = 1 \). Moreover, the triple L-function is equal to

\[
\zeta_K(s)L_K(s, \chi_1^{-1}\chi_1^\theta)L_K(s, \chi_2^{-1}\chi_2^\theta)L_K(s, \chi_3^{-1}\chi_3^\theta).
\]

Now, suppose that \( K = k \oplus k', k' \) is a quadratic extension of \( k \), \( \Pi = \pi_1 \otimes \pi_2 \). Let \( \omega_1 \) and \( \omega_2 \) be the central quasi-characters of \( \pi_1 \) and \( \pi_2 \), respectively. By the assumption, \( \omega_1^{-1}(\omega_2|_{\mathbb{A}_k^*}) = \omega \).

We first prove \( K \neq k' \). Assume that \( K = k' \). In this case we have, as in case (3),

\[
L(s, \Pi \otimes \omega, \sigma)L(s, \Pi, \sigma) = L_K(s, \pi_{1,K} \times \pi_2 \times \pi_2^\theta),
\]

and this has a pole at \( s = 1 \). Here, \( \Pi \otimes \omega \) means \( (\pi_1 \otimes \omega) \otimes \pi_2 \). As in case (3), we can prove that \( \pi_{1,K} \) is not cuspidal. It follows that there is a quasi-character \( \chi \) of \( K \) such that \( \pi_1 = \pi(\chi) \). Then

\[
L(s, \Pi, \sigma) = L_K(s, (\pi_2 \otimes \chi) \times \pi_2^\theta).
\]

Therefore we have \( \pi_2 \otimes \chi \cong \tilde{\pi}_2^\theta \). Then \( \pi_2 \otimes \varepsilon \cong \pi_2 \), where \( \varepsilon = \chi(\theta)^{-1} \). As in case (1), we can prove that \( \varepsilon^2 = 1, \varepsilon \neq 1, \varepsilon^\theta = \varepsilon \) and that there is a character \( \varepsilon' \) of \( \mathbb{A}_k^* / k^* \) such that \( \varepsilon = \varepsilon' \circ N_{K/k} \). Taking the central character of \( \pi_2 \otimes \chi \cong \tilde{\pi}_2^\theta \), we have

\[
\omega_2 \chi^2 = (\omega_2^\theta)^{-1}.
\]

Let \( I, x \) and \( y \) be as in the case (1). Then

\[
\omega_2(y) = \omega_2(\chi^\theta)^{-1} \chi(y)^{-2} = \omega_2(\chi^\theta)^{-1} \chi(\chi^\theta)^{-1} \sigma(y) = \omega_2(\chi^\theta)^{-1} \chi(x)^{-1} \varepsilon'(x).
\]
It follows that
\[
\omega_1(x)\omega_2(x) = \chi(x)\omega(x)\omega_2(yy')
\]
\[
= \chi(x)\omega(x)\chi(x)^{-1}e'(x)
\]
\[
= \omega(x)e'(x).
\]
This contradicts to the assumption \(\omega_1 \cdot \omega_2|_{\Lambda_1^*} = \omega\), since \(e'\) is non-trivial on \(I\). Thus we have proved \(K \neq k'\).

Suppose \(K \neq k'\). Let \(\pi_{1,K}\) (resp. \(\pi_{2,K}\)) be the base change of \(\pi_1\) (resp. \(\pi_2\)) to \(\text{GL}_2(\mathbb{A}_k)\) (resp. \(\text{GL}_2(\mathbb{A}_{k'K})\)). In this case we can prove that at least one of \(\pi_{1,K}\) and \(\pi_{2,K}\) is not cuspidal as in case (1). We first prove that actually \(\pi_{2,K}\) is not cuspidal. Suppose that \(\pi_{2,K}\) is cuspidal. Then \(\pi_{1,K}\) is not cuspidal, so there is a quasi-character \(\chi\) of \(\mathbb{A}_K^\times\) such that \(\pi_1 = \pi(\chi)\). Then the triple L-function is given by the Asai-L-function of \(\pi_{2,K}\) twisted by \(\chi\):

\[
L(s, \Pi, \sigma) = L(\pi_{2,K}, \chi)_\text{Asai}.
\]

Let \(\eta\) be the character of \(\mathbb{A}_k^\times/K^\times\) corresponding to \(k'/K/K\) by class field theory. Then

\[
L_K(s, (\pi_{2,K} \otimes \chi) \times \pi_{2,K}^\theta) = L_K(s, \pi_{2,K}, \chi)_\text{Asai}L_K(s, \pi_{2,K}, \chi\eta)_\text{Asai}.
\]

Since \(L_K(s, \pi_{2,K}, \chi\eta)_\text{Asai}\) is the triple L-function for \(\pi(\chi\eta) \times \pi_2\), it does not have a zero at \(s = 1\), so \(L_K(s, (\pi_{2,K} \otimes \chi) \times \pi_{2,K}^\theta)\) has a pole at \(s = 1\). As in the case \(K = k'\), this is impossible.

Thus \(\pi_{2,K}\) is not cuspidal, so \(\pi_2 = \pi(\chi)\) for some quasi-character \(\chi\) of \(\mathbb{A}_{k'K}^\times\). The triple L-function is given by

\[
L(s, \Pi, \sigma) = L(s, \pi_1 \times \pi(\chi|_{\Lambda_1^*}))L(s, \pi_1 \times \pi(\chi|_{\Lambda_2^*})),
\]

up to finite number of Euler factors. Here, \(K'\) is the quadratic extension of \(k\), contained in \(k'K\) different from \(K\) and \(k'\).

It follows that \(\pi_1 \simeq \pi(\chi^{-1}|_{\Lambda_1^*})\) or \(\pi_1 \simeq \pi(\chi^{-1}|_{\Lambda_2^*})\), but the latter is impossible for the following reason. First we observe the central quasi-character of \(\pi(\chi)\), \(\pi(\chi^{-1}|_{\Lambda_1^*})\), and \(\pi(\chi^{-1}|_{\Lambda_2^*})\) are \(\chi|_{\Lambda_1^*}: \omega_{k'/k}', \chi^{-1}|_{\Lambda_1^*}: \omega, \) and \(\chi^{-1}|_{\Lambda_1^*}: \omega_{k'/k}\), respectively. Here, \(\omega_{k'/k}'\) (resp. \(\omega_{k'/k}\)) is the character of \(\mathbb{A}_{k'/k}^\times\) (resp. \(\mathbb{A}_k^\times/k^\times\)) of order 2 corresponding to \(k'/K/k\) (resp. \(K'/k\)) by class field theory. If \(\pi_1 \simeq \pi(\chi^{-1}|_{\Lambda_1^*})\), we have

\[
\omega_1(x)\omega_2(x) = \chi^{-1}(x)\omega_{k'/k}(x)\chi(x)\omega_{k'/k}(x)
\]
\[
= \omega_{k'/k}(x).
\]
This contradicts to the assumption $\omega_1 \cdot (\omega_2|_{A^*_k}) = \omega$, so one cannot have $\pi_1 \simeq \pi(\chi^{-1}|_{A^*_k})$.

Suppose $\pi_1 \simeq \pi(\chi^{-1}|_{A^*_k})$, and $\pi_2 \simeq \pi(\chi)$. Then an easy calculation shows that the triple $L$-function is equal to

$$\zeta_K(s) L_K(s, (\chi^{-1} \chi^\theta)|_{A^*_k}) L_{K', K}(s, \chi^{-1} \chi^\theta).$$

Here, $\theta$ is regarded as an element of $\text{Gal}(k'K/k')$, by the natural isomorphism $\text{Gal}(k'K/k') \simeq \text{Gal}(K/k)$. As in case (1), this equation holds for all place $v$.

Thus we have proved the following theorem.

**THEOREM 2.8.** Suppose that $K = k \oplus k'$, $k'$ is a quadratic extension of $k$, and $L(s, \pi, \sigma)$ has a pole somewhere. Then the following two assertions hold:

(a) Let $\Pi'$, $\omega'$ be the objects obtained by twisting $\Pi$ by $\alpha^s$, $s_0 \in \mathbb{C}$. Then $\omega'^2 = 1$, $\omega' \neq 1$, $\omega'$ does not correspond to $k'/k$ by class field theory, and $L(s, \Pi, \sigma)$ has a simple pole at $s = 1$, for some $s_0 \in \mathbb{C}$.

(b) Assume that $\omega^2 = 1$, $\omega \neq 1$, $\omega$ does not correspond to $k'/k$ by class field theory, and $L(s, \Pi, \sigma)$ has a simple pole at $s = 1$. Let $K$ be the quadratic extension of $k$ corresponding to $\omega$ by class field theory. Let $\theta$ be the generator of $\text{Gal}(k'K/k')$. Then there exists a quasi-character $\chi$ of $A^*_k/k'K^*$ such that $\pi_1 \simeq \pi(\chi^{-1}|_{A^*_k})$, and $\pi_2 = \pi(\chi)$. Moreover, the triple $L$-function is equal to

$$\zeta_K(s) L_K(s, (\chi^{-1} \chi^\theta)|_{A^*_k}) L_{K', K}(s, \chi^{-1} \chi^\theta).$$

**References**

15. R. P. Langlands: *Euler products*, Yale University, New Haven.