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Spherical Functions on a family of quantum 3-spheres

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Abstract. A quantum space M which can be regarded as a total space of a family of quantum 3-spheres is introduced. By the action of the quantum group $SU_q(2)$, its algebra of functions is decomposed into irreducible components. Its spherical functions are explicitly described in terms of the big q-Jacobi polynomials and the q-Hahn polynomials.

0. Introduction

We introduce a new quantum space, denoted by M, which has the structure of a (G, K)-space over the quantum group $G = SU_q(2)$ and $K = U(1)$. Its algebra of functions $A(M)$ is the $\mathbb{C}$-algebra generated by four elements $\tilde{x}$, $\tilde{u}$, $\tilde{v}$, $\tilde{y}$ with the fundamental relations

$$\tilde{u}\tilde{x} = q\tilde{x}\tilde{u},\ \tilde{v}\tilde{x} = q\tilde{x}\tilde{v},\ \tilde{y}\tilde{u} = q\tilde{u}\tilde{y},\ \tilde{y}\tilde{v} = q\tilde{v}\tilde{y},\ \tilde{y}\tilde{x} - \tilde{x}\tilde{y} = q\tilde{v}\tilde{u} - q^{-1}\tilde{u}\tilde{v}.$$

This algebra has a $*$-operation such that $\tilde{x}^* = \tilde{y}$, $\tilde{u}^* = -q^{-1}\tilde{v}$. In this $*$-algebra we define the two self-adjoint elements d and c by

$$d := (q - q^{-1})^{-1}(\tilde{v}\tilde{u} - \tilde{u}\tilde{v}),\ c := \tilde{x}\tilde{y} - q^{-1}\tilde{u}\tilde{v} - d = \tilde{y}\tilde{x} - q\tilde{v}\tilde{u} - d.$$

The element c belongs to the center of $A(M)$ while d does not. With respect to the action of G and K, the elements c and d are invariant. In this sense the quantum space M is the total space of a "family of G-spaces" with parameters (c, d). Although the element d does not belong to the center of $A(M)$, we will regard M as a deformation family of quantum 3-spheres. In fact, by the specialization $(c, d) = (1, 0)$, $A(M)$ reduces to the algebra of functions on $SU_q(2)$, regarded as a quantum 3-sphere.

In this article, we study the spherical functions on this quantum (G, K)-space M. The algebra of functions of $A(M)$ contains a family of irreducible representations $\Phi_{j;n}$ of $SU_q(2)$ indexed by $j \in \frac{1}{2}\mathbb{N}$ and $n \in I_j = \{j, j-1, \ldots, -j\}$. Here the subspace $\Phi_{j;n}$ is a $(2j+1)$-dimensional irreducible $A(G)$-comodule consisting of right relative K-invariants of weight $2n$. Moreover the algebra $A(M)$ is decomposed into the direct sum

$$A(M) = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \bigoplus_{n \in I_j} \Phi_{j;n} \otimes \mathscr{R},$$

where $\mathscr{R} = \mathbb{C}[c, d]$ is the subring of (G, K)-invariants (Theorem 3.4). For each $j \in \frac{1}{2}\mathbb{N}$ and $n \in I_j$, the vector space $\Phi_{j;n}$ has a basis $(\varphi^j_{mn})_{m \in I_j}$ such that φ^j_{mn} are relatively invariant under the left action of the diagonal subgroup $K = U(1)$ of $\mathrm{SU}_q(2)$. We call the elements φ^j_{mn} the *spherical functions* on M.

Until now, some q-orthogonal polynomials are interpreted by quantum groups ([VS, K, M0, V, KK, KR etc]). In this article, we will give two expressions of the spherical functions φ^j_{mn} on the quantum space M in terms of the q-orthogonal polynomials; one by the big q-Jacobi polynomials (Theorem 3.5) and the other by the q-Hahn polynomials (Theorem 3.6). From the viewpoint of geometric interpretation, these are generalization of our previous work on the quantum 2-spheres [NM0]. More precisely, the big q-Jacobi polynomials of general type $P^{(\alpha,\beta)}_n(z; c, d{:}q)$ $(\alpha, \beta \in \mathbb{N})$ appear as the zonal parts of spherical functions φ^j_{mn} on the quantum space M, while on the quantum 2-spheres $S^2_q(c, d)$ appeared those of symmetric type $P^{(\alpha,\alpha)}_n(z; c, d{:}q)$. The quantum space M has a unique (G, K)-invariant $\mathscr{R}$-linear mapping h_M: $A(M) \to \mathscr{R}$ with $h_M(1) = 1$. This invariant measure h_M is represented by the Jackson integral on the q-interval $[-d, c]$ (Theorem 4.1). The spherical functions φ^j_{mn} are orthogonal under the hermitian form $\langle\ ,\ \rangle_L$ defined by h_M (Theorem 4.2). This gives the orthogonality relation for the big q-Jacobi polynomials. On the other hand, the matrix $\Phi^j = (\varphi^j_{mn})_{m,n}$ is unitary up to a diagonal matrix with entries in $\mathscr{R}$. This property of Φ^j leads to the orthogonality relation for the q-Hahn polynomials. Our interpretation of the q-Hahn polynomials is an extension of that of Koornwinder's q-Krawtchouk polynomials [K] on $\mathrm{SU}_q(2)$.

In the last section, we give a realization of the algebra $A(M)$ as a subalgebra of the tensor product of $A(G)$ and a non-commutative Laurent polynomial ring $\mathscr{C}[\lambda, \lambda^{-1}]$. In this construction, we give explicit formulas for the connection coefficients c^j_{mn} between the spherical functions φ^j_{mn} and the matrix elements w^j_{mn} of the irreducible unitary representations of $\mathrm{SU}_q(2)$. As the connection coefficients, Stanton's q-Krawtchouk polynomials are interpreted.

Throughout this article, we fix a non-zero real number q with $0 < q < 1$, and denote by G the quantum group $\mathrm{SU}_q(2)$.

The main results in this article are announced in [NM1].

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1. Preliminaries

Let $A(G) = \mathbb{C}[x, u, v, y]$ be the $*$-Hopf algebra on the quantum group

$G = \mathrm{SU}_q(2)$. The defining relations of the $\mathbb{C}$-algebra $A(G)$ are given by

$$\begin{cases} qxu = ux,\ qxv = vx,\ quy = yu,\ qvy = yv \\ uv = vu,\ xy - q^{-1}uv = yx - quv = 1. \end{cases} \tag{1.1}$$

The coproduct $\Delta: A(G) \to A(G) \otimes_{\mathbb{C}} A(G)$ is the $\mathbb{C}$-algebra homomorphism satisfying $\Delta(x) = x \otimes x + u \otimes v$, $\Delta(u) = x \otimes u + u \otimes y$, $\Delta(v) = v \otimes x + y \otimes v$ and $\Delta(y) = v \otimes u + y \otimes y$. The $*$-structure is defined by $x^* = y$, $u^* = -q^{-1}v$. The quantum group G has a diagonal subgroup K determined by the $\mathbb{C}$-algebra homomorphism π_K: $A(G) \to A(K) = \mathbb{C}[t, t^{-1}]$ such that

$$\pi_K \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \tag{1.2}$$

The quantum universal enveloping algebra $U = U_q(\mathrm{su}(2))$ is a $\mathbb{C}$-algebra generated by four elements k, k^{-1}, e and f with the defining relations

$$kk^{-1} = k^{-1}k = 1,\ kek^{-1} = qe,\ kfk^{-1} = q^{-1}f,\ ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}}. \tag{1.3}$$

Its $*$-structure is given by $e^* = f$, $f^* = e$ and $k^* = k$. The algebra U also has a structure of Hopf algebra. We take the coproduct $\Delta: U \to U \otimes U$ such that

$$\Delta(k) = k \otimes k,\ \Delta(e) = e \otimes k + k^{-1} \otimes e,\ \Delta(f) = f \otimes k + k^{-1} \otimes f. \tag{1.4}$$

The algebra U can be realized in the dual space $\mathrm{Hom}_{\mathbb{C}}(A(G), \mathbb{C})$ of $A(G)$ through the vector representation. The element k is a $\mathbb{C}$-algebra homomorphism $A(G) \to \mathbb{C}$ such that

$$k \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}. \tag{1.5}$$

The elements e and f are *twisted derivations of type* (k^{-1}, k) in the sense that

$$e(\varphi\psi) = k^{-1}(\varphi)e(\psi) + e(\varphi)k(\psi), \tag{1.6.a}$$

$$f(\varphi\psi) = k^{-1}(\varphi)f(\psi) + f(\varphi)k(\psi), \tag{1.6.b}$$

for all φ, $\psi \in A(G)$. At the generators of $A(G)$, they take the following values:

$$e \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.7}$$

The multiplication in U is connected with the coproduct of $A(G)$ by the formula

$$(a \cdot b)(\varphi) = (a \otimes b) \circ \Delta(\varphi) \quad \text{for } a, b \in U, \varphi \in A(G). \tag{1.8}$$

Any finite dimensional representation of G is completely reducible and unitarizable. The irreducible representations are parametrized by the half integers $j \in \frac{1}{2}\mathbb{N}$ and are realized as the left $A(G)$-subcomodules V_j of $A(G)$:

$$V_j = \bigoplus_{m \in I_j} \mathbb{C}\xi^j_m;\ \xi^j_m = \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2}_{q^2} x^{j+m} v^{j-m} (m \in I_j). \tag{1.9}$$

Here $I_j = \{-j, -j+1, \ldots, j\}$ and the symbol $\begin{bmatrix} N \\ k \end{bmatrix}_q$ stands for Gauss' binomial coefficient

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}},\ (a;q)_N = \prod_{k=0}^{N-1} (1 - aq^k) \quad (N \in \mathbb{N}). \tag{1.10}$$

Note that the left $A(G)$-comodule structure of V_j is induced by the coproduct Δ. The suffix m of ξ^j_m corresponds to its weight with respect to K. This convention of suffices is different from that of [$M0$, $M1$]: In the notation of [M1], $\xi^j_m = \xi^{(j)}_{-m}$. We define the matrix elements w^j_{mn} of V_j with respect to the basis ξ^j_m $(m \in I_j)$ by

$$\Delta(\xi^j_m) = \sum_{n \in I_j} w^j_{mn} \otimes \xi^j_n \quad \text{for } m \in I_j. \tag{1.11}$$

Then we have the direct sum decomposition of $A(G)$ as a two-sided $A(G)$-comodule (Peter-Weyl theorem):

$$A(G) = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} W_j,\ W_j = \bigoplus_{m,n \in I_j} \mathbb{C}w^j_{mn} \text{ and } W_j \xrightarrow{\sim} V_j \otimes V^*_j. \tag{1.12}$$

In [M1], the matrix elements w^j_{mn} are written as $w^{(j)}_{-m,-n}$.

The Haar measure h_G on $A(G)$ is represented in terms of the Jackson integral. The subalgebra of two-sided K-invariants in $A(G)$ is generated by $\zeta = -q^{-1}uv$. For any polynomial $f(\zeta) \in \mathbb{C}[\zeta]$, we have

$$h_G(f(\zeta)) = \int_0^1 f(\zeta) d_{q^2}\zeta = (1 - q^2) \sum_{k \geqslant 0} f(q^{2k}) q^{2k}. \tag{1.13}$$

The Haar measure h_G induces the positive definite hermitian forms $\langle\ ,\ \rangle_L$ and $\langle\ ,\ \rangle_R$:

$$\langle \varphi, \psi \rangle_L = h_G(\varphi^* \psi) \quad \text{and} \quad \langle \varphi, \psi \rangle_R = h_G(\varphi \psi^*) \quad \text{for } \varphi, \psi \in A(G). \tag{1.14}$$

Decomposition (1.12) is an orthogonal decomposition with respect to the hermitian form $\langle\ ,\ \rangle_L$ or $\langle\ ,\ \rangle_R$.

The matrix elements w^j_{mn} can be expressed by the little q-Jacobi polynomials defined by

$$p_n^{(\alpha,\beta)}(x;q) = {}_2\varphi_1\left(\begin{matrix} q^{-n},\ q^{\alpha+\beta+n+1} \\ q^{\alpha+1} \end{matrix};\ q,\ qx\right). \tag{1.15}$$

Here the symbol ${}_m\varphi_{m-1}$ stands for the basic hypergeometric series

$${}_m\varphi_{m-1}\left(\begin{matrix} a_1, a_2, \ldots, a_m \\ b_1, \ldots, b_{m-1} \end{matrix};\ q,\ x\right) = \sum_{k=0}^{\infty} \frac{(a_1;\, q)_k(a_2;\, q)_k \cdots (a_m;\, q)_k}{(q;\, q)_k(b_1;\, q)_k \cdots (b_{m-1};\, q)_k} x^k.$$

For details on the arguments in this section, we refer the reader to [M0, M1].

2. Quantum (G, K)-space M

A quantum space X will be called a *quantum* (G, K)*-space* if the algebra of functions $A(X)$ has a left $A(G)$-comodule structure $L_G\colon A(X) \to A(G) \otimes A(X)$ and a right $A(K)$-comodule structure $R_K\colon A(X) \to A(X) \otimes A(K)$ such that L_G and R_K are $\mathbb{C}$-algebra homomorphisms compatible in the sense $(L_G \otimes \mathrm{id}) \circ R_K = (\mathrm{id} \otimes R_K) \circ L_G$.

We now introduce a quantum (G, K)-space, which will be denoted by M throughout this paper. We define its algebra of functions $A(M)$ as the $\mathbb{C}$-algebra $\mathbb{C}[\tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}]$ generated by four elements $\tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}$ with the fundamental relations

$$\tilde{u}\tilde{x} = q\tilde{x}\tilde{u},\ \tilde{v}\tilde{x} = q\tilde{x}\tilde{v},\ \tilde{y}\tilde{u} = q\tilde{u}\tilde{y},\ \tilde{y}\tilde{v} = q\tilde{v}\tilde{y},\ \tilde{y}\tilde{x} - \tilde{x}\tilde{y} = q\tilde{v}\tilde{u} - q^{-1}\tilde{u}\tilde{v}. \tag{2.1}$$

We define the elements c and d in $A(M)$ by

$$d := (q - q^{-1})^{-1}(\tilde{v}\tilde{u} - \tilde{u}\tilde{v}),\ c := \tilde{x}\tilde{y} - q^{-1}\tilde{u}\tilde{v} - d = \tilde{y}\tilde{x} - q\tilde{v}\tilde{u} - d. \tag{2.21}$$

Then it is easily checked that d has the commutation relations

$$d\tilde{x} = q^2\tilde{x}d,\ d\tilde{v} = q^2\tilde{v}d,\ \tilde{u}d = q^2 d\tilde{u},\ \tilde{y}d = q^2 d\tilde{y} \tag{2.2b}$$

and that c belongs to the center of $A(M)$. The algebra $A(M)$ also has a $*$-structure determined by the condition

$$\tilde{x}^* = \tilde{y},\quad \tilde{u}^* = -q^{-1}\tilde{v},\quad c^* = c,\quad d^* = d. \tag{2.3}$$

This quantum space M has a structure of (G, K)-space with respect to $G = \mathrm{SU}_q(2)$ and its diagonal subgroup $K = U(1)$. By direct verification, we see that there exist two $\mathbb{C}$-algebra homomorphisms $L_G\colon A(M) \to A(G) \otimes A(M)$ and $R_K\colon A(M) \to A(M) \otimes A(K)$ such that

$$L_G\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix} = \begin{pmatrix} x & u \\ v & y \end{pmatrix} \otimes \begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}, \; L_G(c) = 1 \otimes c, \; L_G(d) = 1 \otimes d. \tag{2.4}$$

and

$$R_K\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix} = \begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \; R_K(c) = c \otimes 1, \; R_K(d) = d \otimes 1, \tag{2.5}$$

respectively. The above formulas mean $L_G(\tilde{x}) = x \otimes \tilde{x} + u \otimes \tilde{v}$ etc. It is easy to see that the quantum space M becomes a quantum (G, K)-space with these L_G and R_K. We remark that L_G and R_K are compatible with the $*$-structure.

The left $A(G)$-comodule structure L_G induces a right $U_q(\mathrm{su}(2))$-module structure of $A(M)$: $\varphi \cdot a = (a \otimes \mathrm{id}) \circ L_G(\varphi)$ for $a \in U_q(\mathrm{su}(2))$ and $\varphi \in A(M)$. The action of the generators of $U_q(\mathrm{su}(2))$ is explicitly described as follows:

$$\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}.k = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}, c.k = c, \; d.k = d, \tag{2.6}$$

$$\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}.e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}, \; c.e = d.e = 0, \tag{2.7}$$

and

$$\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}.f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix}, \; c.f = d.f = 0. \tag{2.8}$$

Here $.k$ is an algebra automorphism of $A(M)$ and both $.e$ and $.f$ are twisted derivations on $A(M)$:

$$(\varphi\psi).e = (\varphi.e)(\psi.k) + (\varphi.k^{-1})\psi.e), \tag{2.9}$$

$$(\varphi\psi).f = (\varphi.f)(\psi.k) + (\varphi.k^{-1})(\psi.f), \tag{2.10}$$

for any φ, $\psi \in A(M)$.

First we remark that the algebra of functions $A(M)$ contains the commutative subalgebra $\mathscr{R} = \mathbb{C}[c, d] \subset A(M)$. Note that c and d are (G, K)-invariant and that $c^* = c$ and $d^* = d$. We will see in Section 3 that this subalgebra $\mathscr{R}$ is isomorphic to the polynomial ring in two (commuting) indeterminates and that $\mathscr{R}$ coincides with the subalgebra of all (G, K)-invariants in $A(M)$. The algebra $\mathscr{R}$ is regarded as the algebra of functions on the euclidean space $\mathbb{R}^2$. Although the element d does not belong to the center of $A(M)$, we consider the quantum space M as the total space of a family $M \to \mathbb{R}^2$ of quantum (G, K)-spaces. In fact, the algebra $A(M)$ contains a quadratic relation which corresponds to a family of 3-spheres. To be precise, define the 'real coordinates' $(\xi_0, \xi_1, \xi_2, \xi_3)$ in $A(M)$ by

$$\xi_0 = \frac{1}{2}(\tilde{x} + \tilde{x}^*), \quad \xi_1 = \frac{1}{2i}(\tilde{x} - \tilde{x}^*), \quad \xi_2 = \frac{1}{2}(\tilde{u} + \tilde{u}^*), \quad \xi_3 = \frac{1}{2i}(\tilde{u} - \tilde{u}^*). \tag{2.11}$$

Then one sees that (2.1) and (2.2) imply the equation

$$\xi_0^2 + \xi_1^2 + \tfrac{1}{2}(1 + q^2)(\xi_2^2 + \xi_3^2) = c + \frac{(q + q^{-1})^2}{4} d. \tag{2.12}$$

In this sense, the quantum space M will be considered as the total space of a *family of quantum 3-spheres* with deformation parameters (c, d). It does not mean, however, that the parameters (c, d) can be freely specialized. We only remark that the algebra $A(M)$ reduces to the algebra of function on $G = \mathrm{SU}_q(2)$ by the specialization $(c, d) = (1, 0)$. In fact there exists a unique surjective homomorphism $\pi: A(M) \to A(G)$ of $*$-algebras such that $\pi(\tilde{x}) = x$, $\pi(\tilde{u}) = u$, $\pi(\tilde{v}) = v$, $\pi(\tilde{y}) = y$, $\pi(c) = 1$, $\pi(d) = 0$. This homomorphism π is also compatible with comodule structures over $A(G)$ and $A(K)$.

The algebra $A(M)$ has a left $A(K)$-comodule structure $L_K = (\pi_K \otimes \mathrm{id}) \circ L_G: A(M) \to A(K) \otimes_{\mathbb{C}} A(M)$ induced by the projection π_K in (1.2). In the rest of this section, we study the two-sided $A(K)$-comodule structure of $A(M)$.

The following lemma is directly proved by using the Diamond Lemma [B] (see also Lemma 1.4 in [M1]).

LEMMA 2.1. *The algebra $A(M)$ is a free left or right $\mathscr{R}$-module with basis $\{\tilde{x}^i \tilde{u}^j \tilde{v}^r \tilde{y}^s;\ i, j, r, s \in \mathbb{N},\ i = 0$ or $s = 0\}$.* $\square$

For each $m, n \in \mathbb{Z}$, we define the $\mathbb{C}$-vector subspace $A(M)[m, n]$ of $A(M)$ by

$$A(M)[m, n] = \{a \in A(M)\ L_K(a) = t^m \otimes a \quad \text{and} \quad R_K(a) = a \otimes t^n\} \tag{2.13}$$

Note that the $\mathbb{C}$-subspace $A(K\backslash M/K) = A[0, 0]$ of all two-sided K-invariants form a $\mathbb{C}$-subalgebra of $A(M)$ and each $A(M)[m, n]$ becomes a $A(M)[0, 0]$-bimodule.

PROPOSITION 2.2. (1) *The $\mathbb{C}$-algebra $A(M)$ is decomposed into the direct sum*

$$A(M) = \bigoplus_{m,n \in \mathbb{Z}} A(M)[m, n] \tag{2.14}$$

(2) The subalgebra $A(K\backslash M/K)$ of $A(M)$ is a polynomial ring $\mathscr{R}[z] = \mathbb{C}[c, d, z]$ where the element z is defined by

$$z = c - \tilde{x}\tilde{y} = -d - q^{-1}\tilde{u}\tilde{v}. \tag{2.15}$$

□

For each couple of integers $(m, n) \in \mathbb{Z}^2$ with $m \equiv n (\mathrm{mod}\, 2)$, we define an element e_{mn} of $A(M)[m, n]$ as follows:

$$\begin{aligned}
&\text{(I)} && e_{mn} = \tilde{x}^{\mu}\tilde{v}^{\nu} \quad \text{if } m + n \geqslant 0,\ m \leqslant n,\\
&\text{(II)} && e_{mn} = \tilde{x}^{\mu}\tilde{u}^{\nu} \quad \text{if } m + n \geqslant 0,\ m \geqslant n,\\
&\text{(III)} && e_{mn} = \tilde{u}^{\nu}\tilde{y}^{\mu} \text{ if } m + n \leqslant 0,\ m \geqslant n,\\
&\text{(IV)} && e_{mn} = \tilde{v}^{\nu}\tilde{y}^{\mu} \text{ if } m + n \leqslant 0,\ m \leqslant n,
\end{aligned} \tag{2.16}$$

where $\mu = \left|\dfrac{m + n}{2}\right|$, $\nu = \left|\dfrac{m - n}{2}\right|$.

PROPOSITION 2.3. *If* $m \equiv n (\mathrm{mod}\, 2)$, $A(M)[m, n]$ *is a free left or right* $\mathscr{R}[z]$-*module of rank one with basis* e_{mn}:

$$A(M)[m, n] = \mathscr{R}[z]e_{mn} = e_{mn}\mathscr{R}[z]. \tag{2.17}$$

Unless $m \equiv n$ (mod 2), $A(M)[m, n] = 0$. □

Propositions 2.2 and 2.3 follow immediately from Lemma 2.1.

LEMMA 2.4. (1) *The element* z *commutes with* $\tilde{u}$, $\tilde{v}$, c *and* d.

(2) $z\tilde{x} = q^2\tilde{x}z$, $\tilde{y}z = q^2 z\tilde{y}$.

(3) $\tilde{u}^n\tilde{v}^n = q^{n^2}(-d)^n(-z/d; q^{-2})_n$, $\tilde{v}^n\tilde{u}^n = q^{-n^2}(-d)^n(-q^2 z/d\, q^2)_n$,
$\tilde{x}^n\tilde{y}^n = c^n(z/c; q^{-2})_n$, $\tilde{y}^n\tilde{x}^n = c^n(q^2 z/c; q^2)_n$ *for* $n \in \mathbb{N}$. □

Note that the right hand sides of (3) are polynomials in z with coefficients in $\mathscr{R} = \mathbb{C}[c, d]$. Lemma 2.4 will be used in the calculation of spherical functions.

3. Spherical functions φ^j_{mn}

For each $j \in \frac{1}{2}\mathbb{N}$, $n \in I_j = \{-j, -j + 1, \ldots, j\}$, we define the element φ^j_{jn} of $A(M)$ by

$$\varphi^j_{jn} = \begin{bmatrix} 2j \\ j + n \end{bmatrix}_{q^2}^{1/2} \tilde{x}^{j+n}\tilde{u}^{j-n} \in A(M)[2j, 2n]. \tag{3.1}$$

Since $L_G(\tilde{x}) = x \otimes \tilde{x} + u \otimes \tilde{v}$ and $L_G(\tilde{u}) = x \otimes \tilde{u} + u \otimes \tilde{y}$, one can uniquely determine a family of elements $\varphi^j_{mn} \in A(M)$ $(m \in I_j)$ satisfying

$$L_G(\varphi^j_{jn}) = \sum_{m \in I_j} w^j_{jm} \otimes \varphi^j_{mn}, \tag{3.2}$$

where

$$w^j_{jm} = \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2}_{q^2} x^{j+m} u^{j-m}. \tag{3.3}$$

Here the elements w^j_{mn} are the matrix elements of irreducible representations of G, defined in (1.11).

LEMMA 3.1.

$$L_G(\varphi^j_{mn}) = \sum_k w^j_{mk} \otimes \varphi^j_{kn} \quad \textit{for } m, n \in I_j. \tag{3.4}$$

Proof. This follow, from the left $A(G)$-comodule structure of the algebra $A(M)$. Indeed we have

$$(\Delta \otimes \mathrm{id}) \circ L_G(\varphi^j_{jn}) = \sum_k \Delta(w^j_{jk}) \otimes \varphi^j_{kn}$$

$$= \sum_{m,k} w^j_{jm} \otimes w^j_{mk} \otimes \varphi^j_{kn} \quad \text{for } n \in I_j, \tag{3.5}$$

$$(\mathrm{id} \otimes L_G) \circ L_G(\varphi^j_{jn}) = \sum_m w^j_{jm} \otimes L_G(\varphi^j_{mn}) \quad \text{for } n \in I_j. \tag{3.6}$$

Since $(\Delta \otimes \mathrm{id}) \circ L_G = (\mathrm{id} \otimes L_G) \circ L_G$, linear independence of the elements w^j_{jm} implies (3.4). □

LEMMA 3.2. (1) $\varphi^j_{mn} \in A(M)[2m, 2n]$ for $m, n \in I_j$.

(2) *With the notation of Proposition* 2.3, *each* φ^j_{mn} $(j \in \frac{1}{2}\mathbb{N}, m, n \in I_j)$ *is uniquely written in the form*

$$\varphi^j_{mn} = e_{2m,2n} f^j_{mn}(z) = g^j_{mn}(z) e_{2m,2n}, \tag{3.7}$$

where $f^j_{mn}(z), g^j_{mn}(z) \in \mathscr{R}[z]$ $(z = c - \tilde{x}\tilde{y})$. *Both* $f^j_{mn}(z)$ *and* $g^j_{mn}(z)$ *have the following expression*:

$$a_0 z^k + \sum_{i=1}^{k} a_i z^{k-i} \text{ with } a_0 \in \mathbb{C}^*, a_i \in \mathscr{R}, \tag{3.8}$$

for $k = \min\{j+m, j-m, j+n, j-n\}$.

Proof. (1) By Lemma 3.1, we have

$$(\mathrm{id} \otimes R_K) \circ L_G(\varphi^j_{jn}) = \sum_m w^j_{jm} \otimes R_K(\varphi^j_{mn}). \tag{3.9}$$

By (3.1) and (3.2) we get

$$\begin{aligned}(L_G \otimes \mathrm{id}) \circ R_K(\varphi^j_{jn}) &= (L_G \otimes \mathrm{id})(\varphi^j_{jn} \otimes t^{2n}) \\ &= L_G(\varphi^j_{jn}) \otimes t^{2n} = \sum_m w^j_{jm} \otimes \varphi^j_{mn} \otimes t^{2n}.\end{aligned} \tag{3.10}$$

Since $(\mathrm{id} \otimes R_K) \circ L_G = (L_G \otimes \mathrm{id}) \circ R_K$, linear independence of w^j_{jm} implies the relation

$$R_K(\varphi^j_{mn}) = \varphi^j_{mn} \otimes t^{2n} \quad \text{for } m, n \in I_j. \tag{3.11}$$

Similar argument shows the relations

$$(\mathrm{id} \otimes L_K) \circ L_G(\varphi^j_{jn}) = \sum_k w^j_{jm} \otimes L_K(\varphi^j_{mn}) \tag{3.12}$$

and

$$\begin{aligned}(\mathrm{id} \otimes L_K) \circ L_G(\varphi^j_{jn}) &= (\mathrm{id} \otimes \pi_K \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ L_G(\varphi^j_{jn}) \\ &= (\mathrm{id} \otimes \pi_K \otimes \mathrm{id}) \left(\sum_k \Delta(w^j_{jm}) \otimes \varphi^j_{mn} \right) = \sum_m w^j_{jm} \otimes t^{2m} \otimes \varphi^j_{mn}.\end{aligned} \tag{3.13}$$

Here we used the property $(\mathrm{id} \otimes \pi_K) \circ \Delta(w^j_{jm}) = w^j_{jm} \otimes t^{2m}$. Therefore we have the relation

$$L_K(\varphi^j_{mn}) = t^{2m} \otimes \varphi^j_{mn} \quad \text{for } m, n \in I_j. \tag{3.14}$$

(2) This statement follows from the explicit description of the spherical elements φ^j_{mn} in Theorem 3.5 below. □

PROPOSITION 3.3. (1) *For each* $j \in \frac{1}{2}\mathbb{N}$, *the elements* φ^j_{mn} $(m, n \in I_j)$ *are linearly independent over* $\mathscr{R}$.

(2) *For each* $j \in \frac{1}{2}\mathbb{N}$, *define the* $\mathbb{C}$-*vector space* $\Phi_{j;n}$ *as follows:*

$$\Phi_{j;n} = \sum_{m \in I_j} \mathbb{C}\varphi^j_{mn} \quad \textit{for } N \in I_j. \tag{3.15}$$

Then each $\Phi_{j;n}$ is an irreducible left $A(G)$-*comodule*.

Proof. (1) Linear independence of the spherical functions φ^j_{mn} follows from Proposition 2.2 and Lemma 3.2.

(2) By Lemma 3.1, the $\mathbb{C}$-vector space $\Phi_{j;n}$ is a left $A(G)$-comodule with respresentation matrix $W^j = (w^j_{mn})_{m,n\in I_j}$. Irreducibility is implied by the classification of the irreducible left $A(G)$-comodule in [M1]. □

In view of Propositions 3.2 and 3.3, we call the elements φ^j_{mn} *spherical functions* on M. Combining Lemma 3.2 (2) with Proposition 2.3, one sees that, for each $m, n \in \frac{1}{2}\mathbb{N}$ with $m - n \in \mathbb{Z}$,

$$A(M)[2m, 2n] = \bigoplus_j \mathscr{R}\varphi^j_{mn} = \bigoplus_j \varphi^j_{mn}\mathscr{R}, \tag{3.16}$$

where the index j ranges over the set $\{j \in \frac{1}{2}\mathbb{N}; m, n \in I_j\}$. Hence, by Proposition 2.2 we see that the spherical functions φ^j_{mn} $(j \in \frac{1}{2}\mathbb{N}, m, n \in I_j)$ form a $\mathscr{R}$-basis for $A(M)$. Thus we have

THEOREM 3.4. *The algebra $A(M)$ is a free left or right $\mathscr{R}$-module with basis φ^j_{mn} $(m, n \in I_j, j \in \frac{1}{2}\mathbb{N})$. The algebra $A(M)$ is decomposed into the direct sum of left $A(G)$-comodules*

$$A(M) = \bigoplus_{j\in\frac{1}{2}\mathbb{N}} \bigoplus_{n\in I_j} \Phi_{j;n}\mathscr{R}. \tag{3.17}$$

□

The spherical functions φ^j_{mn} $(j \in \frac{1}{2}\mathbb{N}, m, n \in I_j)$ are described in terms of the *big q-Jacobi polynomials* in the variable $z = c - \tilde{x}\tilde{y}$. The big q-Jacobi polynomials are defined by

$$P^{(\alpha,\beta)}_n(z; c, d\colon q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\alpha+1}z/c \\ q^{\alpha+1}, -q^{\alpha+1}d/c \end{matrix}; q, q\right). \tag{3.18}$$

We remark here that our notation $P^{(\alpha,\beta)}_n$ is different from that of [AA] by a normalization constant.

THEOREM 3.5. *For each α, β, $n \in \mathbb{N}$, define the polynomial $F^{(\alpha,\beta)}_n(z; c, d\colon q)$ in $\mathscr{R}[z]$ by*

$$F^{(\alpha,\beta)}_n(z; c, d\colon q) = (-1)^n q^{-\frac{1}{2}n(2\alpha+\beta+n+1)} \begin{bmatrix} \alpha + n \\ \alpha \end{bmatrix}_q^{1/2} \begin{bmatrix} \alpha + \beta + n \\ \alpha \end{bmatrix}_q^{1/2}$$
$$\times c^n(-q^{\alpha+1}d/c; q)_n P^{(\alpha,\beta)}_n(z; c, d\colon q). \tag{3.19}$$

Then the spherical functions $\varphi^j_{mn}(m, n \in I_j, j \in \frac{1}{2}\mathbb{N})$ are expressed as follows:

Case (I)

$$\varphi^j_{mn} = \tilde{x}^{m+n} F^{(m+n,n-m)}_{j-n}(z;\ c,\ d\colon q^2)\tilde{v}^{n-m}, \quad \text{if } m+n \geqslant 0,\ m \leqslant n,$$

Case (II)

$$\varphi^j_{mn} = \tilde{x}^{m+n}\tilde{u}^{m-n} F^{(m+n,m-n)}_{j-m}(z;\ c,\ d\colon q^2), \quad \text{if } m+n \geqslant 0,\ m \geqslant n,$$

Case (III)

$$\varphi^j_{mn} = \tilde{u}^{m-n} F^{(-m-n,m-n)}_{j+n}(z;\ c,\ d\colon q^2)\tilde{y}^{-m-n}, \quad \text{if } m+n \leqslant 0,\ m \geqslant n,$$

Case (IV)

$$\varphi^j_{mn} = F^{(-m-n,n-m)}_{j+m}(z;\ c,\ d\colon q^2)\tilde{v}^{n-m}\tilde{y}^{-m-n}, \quad \text{if } m+n \leqslant 0,\ m \leqslant n.$$

Proof. By using the q-binomial theorem, one computes

$$\begin{aligned} L_G(\varphi^j_{jn}) &= L_G\left(\begin{bmatrix} 2j \\ j+n \end{bmatrix}_{q^2}^{1/2} q^{n^2-j^2}\tilde{u}^{j-n}\tilde{x}^{j+n}\right) \\ &= q^{n^2-j^2}\begin{bmatrix} 2j \\ j+n \end{bmatrix}_{q^2}^{1/2} (x\otimes\tilde{u} + u\otimes\tilde{y})^{j-n}(x\otimes\tilde{x} + u\otimes\tilde{v})^{j+n} \\ &= \sum_{m\in I_j}\begin{bmatrix} 2j \\ j+m \end{bmatrix}_{q^2}^{1/2} x^{j+m}u^{j-m}\otimes\varphi^j_{mn} = \sum_{m\in I_j} w^j_{jm}\otimes\varphi^j_{mn}, \end{aligned}$$

where

$$\begin{aligned} \varphi^j_{mn} &= q^{n^2-j^2}\left(\frac{(q^2;\,q^2)_{j+m}(q^2;\,q^2)_{j-m}}{(q^2;\,q^2)_{j+n}(q^2;\,q^2)_{j-n}}\right)^{1/2} \\ &\times \sum_{r+s=j+m}\begin{bmatrix} j+n \\ r \end{bmatrix}_{q^2}\begin{bmatrix} j-n \\ s \end{bmatrix}_{q^2} q^{r(j-n-s)}\tilde{u}^s\tilde{y}^{j-n-s}\tilde{x}^r\tilde{v}^{j+n-r}. \end{aligned} \tag{3.20}$$

By (3.20) it is easy to see that $(\varphi^j_{mn})^* = (-q)^{n-m}\varphi^j_{-m,-n}$(cf. (4.14) in [K]). Hence the cases (II) and (III) are equivalent to (IV) and (I), respectively. Since these two cases (IV) and (I) can be proved similarly, we study only the case (IV) hereafter. Under condition (IV): $m+n \leqslant 0$, $m \leqslant n$, by Lemma 2.4, one has

$$\begin{aligned} \tilde{u}^s\tilde{y}^{j-n-s}\tilde{x}^r\tilde{v}^{j+n-r} &= q^{s^2-(n-m+s)(m+n)}c^r(-d)^s \\ &\times(-z/d;\ q^{-2})_s(q^{2(1-m-n)}z/c;\ q^2)_r\tilde{v}^{n-m}\tilde{y}^{-m-n}. \end{aligned} \tag{3.21}$$

Thus we rewrite (3.20) into

$$\varphi^j_{mn} = q^{(m+j)(m-n)} \frac{(-d)^{j+m}}{(q^2;q^2)_{-m-n}} (-q^{2(1-j-m)}z/d;\, q^2)_{j+m}$$
$$\times \left(\frac{(q^2;q^2)_{j-m}(q^2;q^2)_{j-n}}{(q^2;q^2)_{j+n}(q^2;q^2)_{j+m}}\right)^{1/2}$$
$$\times {}_3\varphi_2\left(\begin{matrix} q^{2(-j-n)},\, q^{2(-j-m)},\, q^{2(1-m-n)}z/c \\ q^{2(-m-n+1)},\, -q^{2(-j-m+1)}z/d \end{matrix};\, q^2,\, -q^{2(n+1+j)}c/d\right) \tilde{v}^{n-m}\tilde{y}^{-m-n}.$$

By using the transformation formula ((1.30) in [AW])

$${}_3\varphi_2\left(\begin{matrix} q^{-n},\, a,\, d/b \\ aq^{-n+1}/c,\, d \end{matrix};\, q,\, bq/c\right) = \frac{(c;q)_n}{(c/a;q)_n} a^{-n}\, {}_3\varphi_2\left(\begin{matrix} q^{-n},\, a,\, b \\ c,\, d \end{matrix};\, q,\, q\right), \tag{3.23}$$

we get the desired result:

$$\varphi^j_{mn} = \frac{(-c)^{j+m}}{(q^2;q^2)_{-m-n}} (-q^{2(1-m-n)}d/c;\, q^2)_{j+m}$$
$$\times q^{(j+m)(2m+n-1-j)} \left(\frac{(q^2;q^2)_{j-m}(q^2;q^2)_{j-n}}{(q^2;q^2)_{j+m}(q^2;q^2)_{j+n}}\right)^{1/2}$$
$$\times {}_3\varphi_2\left(\begin{matrix} q^{2(-j-m)},\, q^{2(1+j-m)},\, q^{2(1-m-n)}z/c \\ q^{2(1-m-n)},\, -q^{2(1-m-n)}d/c \end{matrix};\, q^2,\, q^2\right) \tilde{v}^{n-m}\tilde{y}^{-m-n}. \tag{3.24}$$

□

REMARKS. (1) When $(c, d) = (1, 0)$, $F^{(\alpha,\beta)}_n(z; c, d\colon q)$ is expressed in terms of the little q-Jacobi polynomials:

$$F^{(\alpha,\beta)}_n(z;\, 1,\, 0;\, q) = q^{-n\beta/2} \begin{bmatrix} \beta+n \\ \beta \end{bmatrix}_q^{1/2} \begin{bmatrix} \alpha+\beta+n \\ \beta \end{bmatrix}_q^{1/2} p^{(\beta,\alpha)}_n(z;\, q). \tag{3.25}$$

(2) The big q-Jacobi polynomials have a symmetry with respect to the transformation $(\alpha, \beta, c, d) \mapsto (\beta, \alpha, -d, -c)$. The above $F^{(\alpha,\beta)}_n(z; c, d; q)$ have the symmetry

$$F^{(\alpha,\beta)}_n(z;\, c,\, d\colon q) = q^{n(\alpha-\beta)/2} F^{(\beta,\alpha)}_n(z;\, -d,\, -c\colon q). \tag{3.26}$$

This follows from the identity

$${}_3\varphi_2\left(\begin{matrix} q^{-n},\, a_1,\, a_2 \\ b_1,\, b_2 \end{matrix};\, q,\, q\right) = q^{n(n-1)} \left(\frac{b_1 b_2}{a_1}\right)^n \frac{(q^{1-n}a_1/b_1;q)_n (q^{1-n}a_1/b_2;q)_n}{(b_1;q)_n (b_2;q)_n}$$
$$\times {}_3\varphi_2\left(\begin{matrix} q^{-n},\, a_1,\, q^{1-n}a_1a_2/b_1b_2 \\ q^{1-n}a_1/b_1,\, q^{1-n}a_1/b_2 \end{matrix};\, q,\, q\right), \tag{3.27}$$

which is a special case of Sears' transformation formula (see (1.28) in [AW]).

The spherical functions φ^j_{mn} $(m, n \in I_j, j \in \frac{1}{2}\mathbb{N})$ are also rewritten in terms of the *q-Hahn polynomials* defined by

$$Q_n(x; a, b, N: q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{matrix}; q, q\right). \tag{3.28}$$

For the statement of the next theorem, we will use the notation $\varphi = \psi\tilde{y}^{-d}$ with $d \in \mathbb{N}$ to refer an element φ in $A(M)$ such that $\varphi\tilde{y}^d = \varphi$. By Proposition 2.3, one can easily check that the right multiplication $\varphi \mapsto \varphi\tilde{y}^d$: $A(M) \to A(M)$ is injective. This justifies the notation $\psi\tilde{y}^{-d}$ in the sense that, if there exists an element $\varphi \in A(M)$ with $\varphi\tilde{y}^d = \psi$, then such a φ is uniquely determined.

THEOREM 3.6. *For* $j \in \frac{1}{2}\mathbb{N}$ *and* $m, n \in I_j$, *we have the expression*

$$\varphi^j_{mn} = q^{-2(j+n)j}\begin{bmatrix} 2j \\ j+n \end{bmatrix}^{1/2}_{q^2}\begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2}_{q^2} \tag{3.29}$$

$$\times\, \tilde{u}^{j+m}\tilde{y}^{j-m}Q_{j+n}(q^{-2(j+m)}; -z/d, q^{-2}c/z; 2j: q^2)\tilde{v}^{j+n}\tilde{y}^{-j-n}. \tag{3.29}$$

Proof. To get the above formula, we use the transformation formula

$$\begin{aligned} {}_3\varphi_2\left(\begin{matrix} q^{-n}, a_1, a_2 \\ b_1, b_2 \end{matrix}; q, q\right) &= \left(\frac{a_1a_2}{b_1}\right)^n \frac{(b_1b_2/a_1a_2; q)_n}{(b_2; q)_n} \\ &\times {}_3\varphi_2\left(\begin{matrix} q^{-n}, b_1/a_1, b_1/a_2 \\ b_1, b_1b_2/a_1a_2 \end{matrix}; q, q\right), \end{aligned} \tag{3.30}$$

which is also a special case of the Sears' transformation formula. Recall that

$$\begin{aligned} F^{(\alpha,\beta)}_k(z; c, d: q) &= (-1)^k q^{-k(2\alpha+\beta+k+1)/2}\begin{bmatrix} \alpha+k \\ \alpha \end{bmatrix}^{1/2}_q\begin{bmatrix} \alpha+\beta+k \\ \alpha \end{bmatrix}^{1/2}_q \\ &\times c^k(-q^{\alpha+1}d/c\ q)_k\, {}_3\varphi_2\left(\begin{matrix} q^{-k}, q^{k+\alpha+\beta+1}, q^{\alpha+1}z/c \\ q^{\alpha+1}, -q^{\alpha+1}d/c \end{matrix}\ q, q\right]. \end{aligned} \tag{3.31}$$

Applying (3.30) to (3.31) with $n = k$ and $b_1 = q^{\alpha+1}$, we obtain

$$\begin{aligned} F^{(\alpha,\beta)}_k(z; c, d: q) &= (-1)^k q^{k(\beta+k+1)/2}\begin{bmatrix} \alpha+k \\ \alpha \end{bmatrix}^{1/2}_q\begin{bmatrix} \alpha+\beta+k \\ \alpha \end{bmatrix}^{1/2}_q \\ &\times z^k(-q^{-(\beta+k)}d/z; q)_k\, {}_3\varphi_2\left(\begin{matrix} q^{-k}, q^{-(\beta+k)}, c/z \\ q^{\alpha+1}, -q^{-(\beta+k)}d/z \end{matrix}; q, q\right). \end{aligned} \tag{3.32}$$

Next we apply (3.27) to (3.32) with $n = k$, $a_1 = q^{-(\beta+k)}$, so that

$$F_k^{(\alpha,\beta)}(z;\, c,\, d\colon q) = q^{-k\beta/2}\begin{bmatrix}\alpha+\beta+2k\\ k\end{bmatrix}_q^{1/2}\begin{bmatrix}\alpha+\beta+2k\\ \beta+k\end{bmatrix}_q^{1/2}$$
$$\times(-d)^k(-z/d;\, q^{-1})_k\, {}_3\varphi_2\left(\begin{matrix}q^{-k},\, q^{-(\beta+k)},\, -q^{-(\alpha+k)}c/d\\ q^{-(\alpha+\beta+2k)},\, -q^{1-k}z/d\end{matrix};\, q,\, q\right). \tag{3.33}$$

First we consider the matrix element φ^j_{mn} in the cases (III) with $\alpha = -m-n$, $\beta = m-n$, $k = j+n$ and (IV) with $\alpha = -m-n$, $\beta = n-m$, $k = j+m$. In either case, we have $\{k, \beta+k\} = \{j+m, j+n\}$. Then one can easily check that

$$\varphi^j_{mn}\tilde{y}^{j+n} = q^{-2(n+j)j}\begin{bmatrix}2j\\ j+n\end{bmatrix}_{q^2}^{1/2}\begin{bmatrix}2j\\ j+m\end{bmatrix}_{q^2}^{1/2}$$
$$\times\tilde{u}^{j+m}\tilde{y}^{j-m}\, {}_3\varphi_2\left(\begin{matrix}q^{-2(j+m)},\, q^{-2(j+n)},\, -q^{2(j+n)}c/d\\ q^{-4j},\, -q^2z/d\end{matrix};\, q^2,\, q^2\right)\tilde{v}^{j+n}, \tag{3.34}$$

by using (3.33) and Lemma 2.4. This gives the expression (3.29) for cases (III) and (IV). Using the symmetry (3.26), we can rewrite (3.33) into the form

$$F_k^{(\alpha,\beta)}(z;\, c,\, d\colon q) = (-1)^k q^{-k\beta/2}\begin{bmatrix}\alpha+\beta+2k\\ \alpha+\beta+k\end{bmatrix}_q^{1/2}\begin{bmatrix}\alpha+\beta+2k\\ \alpha+k\end{bmatrix}_q^{1/2}$$
$$\times c^k(z/c;\, q^{-1})_k\, {}_3\varphi_2\left(\begin{matrix}q^{-k},\, q^{-(\alpha+k)},\, -q^{-(\beta+k)}d/c\\ q^{-(\alpha+\beta+2k)},\, q^{1-k}z/c\end{matrix};\, q,\, q\right). \tag{3.35}$$

Applying (3.30) to (3.35) with $n = \alpha+k$, $b_1 = q^{-(\alpha+\beta+2k)}$, we obtain

$$F_k^{(\alpha,\beta)}(z;\, c,\, d\colon q) = q^{-k\beta/2+\alpha(\alpha+k)}\begin{bmatrix}\alpha+\beta+2k\\ \alpha+\beta+k\end{bmatrix}_q^{1/2}\begin{bmatrix}\alpha+\beta+2k\\ \alpha+k\end{bmatrix}_q^{1/2}$$
$$\times\frac{(-d)^{\alpha+k}(-z/d;\, q^{-1})_{\alpha+k}}{c^\alpha(qz/c;\, q)_\alpha}\, {}_3\varphi_2\left(\begin{matrix}q^{-(\alpha+k)},\, q^{-(\alpha+\beta+k)},\, -q^{-(\alpha+k)}c/d\\ q^{-(\alpha+\beta+2k)},\, -q^{1-(\alpha+k)}z/d\end{matrix};\, q,\, q\right). \tag{3.36}$$

We now consider the matrix element φ^j_{mn} in the cases (I) with $\alpha = m+n$, $\beta = n-m$, $k = j-n$ and (II) with $\alpha = m+n$, $\beta = m-n$, $k = j-m$. In either case, we have $\{\alpha+k, \alpha+\beta+k\} = \{j+m, j+n\}$. Then one can derive (3.29) for these cases by (3.36). □

We emphasize here that each spherical function φ^j_{mn} ($j \in \frac{1}{2}\mathbb{N}, m, m \in I_j$) is the eigenfunction for the right action of the Casimir element $C = (qk^2 + q^{-1}k^{-2} - 2)/(q - q^{-1})^2 + fe$ of the quantum universal enveloping algebra $U_q(\mathrm{su}(2))$. Through the expression in Theorem 3.5, this property of φ^j_{mn} gives an interpretation of the q-difference equation for the big q-Jacobi polynomials:

$$[(c - q^{\alpha+1}z)(d + q^{\beta+1}z)T_q - (1 + q)cd - q\{c(1 + q^{\beta}) - d(1 + q^{\alpha})\}z \\ + q^{-n+1}(1 + q^{\alpha+\beta+2n+1})z^2 + q(c - z)(d + z)T_q^{-1}]P_n^{(\alpha,\beta)}(z; c, d: q) = 0, \tag{3.37}$$

where T_q is the q-shift operator defined by $(T_q f)(z) = f(qz)$. This equation is proved by the same argument as in [M1]. We also get the Rodrigues formula stated in Appendix of [NM0] by analysing the action of $U_q(\mathrm{su}(2))$ on $A(M)$.

4. Invariant measure and orthogonality

Let h_M be the projection $A(M) \to \mathscr{R} = \Phi_{0;0}\mathscr{R}$ in the decomposition (3.17) of $A(M)$. It is clear that h_M is a homomorphism of two-sided $\mathscr{R}$-modules with $h_M(1) = 1$ and is (G, K)-invariant: For any $\varphi \in A(M)$,

$$(\mathrm{id} \otimes h_M) \circ L_G(\varphi) = 1 \cdot h_M(\varphi) \quad \text{in } A(G) \otimes \mathscr{R}, \tag{4.1}$$

$$(h_M \otimes \mathrm{id}) \circ R_K(\varphi) = h_M(\varphi) \cdot 1 \quad \text{in } \mathscr{R} \otimes A(K). \tag{4.2}$$

We call this $\mathscr{R}$-homomorphism $h_M: A(M) \to \mathscr{R} = \mathbb{C}[c, d]$ the *invariant measure* on M. Note that the above property characterizes h_M. Moreover, this measure h_M is compatible with the Haar measure $h_G: A(G) \to \mathbb{C}$ on G in the sense that

$$(h_G \otimes \mathrm{id}) \circ L_G(\varphi) = h_M(\varphi) \cdot 1 \quad \text{in } A(M) \tag{4.3}$$

for any $\varphi \in A(M)$.

The invariant measure h_M can be represented by the Jackson integral on the q-interval $[-d, c]$. Recall that Jackson integral on the q-interval $[-d, c]$ is defined by

$$\int_{-d}^{c} F(z)d_qz = \int_0^c F(z)d_qz - \int_0^{-d} F(z)d_qz, \tag{4.4}$$

where

$$\int_0^c F(z)d_qz = c(1 - q) \sum_{k \geq 0} F(cq^k)q^k. \tag{4.5}$$

THEOREM 4.1. *The invariant measure h_M is factored through the projection $A(M) \to A(M)[0,0] = \mathscr{R}[z]$ in decomposition (2.14). Furthermore, the values of h_M on $\mathscr{R}[z]$ are represented by the Jackson integral:*

$$h_M(F(z)) = \frac{1}{c+d} \int_{-d}^{c} F(z) d_{q^2} z \quad \text{for } F(z) \in \mathscr{R}[z]. \tag{4.6}$$

Proof. The former half follows from (4.1) and (4.2). To prove (4.6), we use the action of $U_q(\mathrm{su}(2))$ on $A(M)$. Direct calculation shows that

$$\begin{aligned}(\tilde{x}\tilde{u}z^n).e \\ = q^{-1/2-2n} \left\{ \frac{1-q^{2n}}{1-q^2} cdz^{n-1} + \frac{1-q^{2(n+1)}}{1-q^2} (c-d)z^n - \frac{1-q^{2(n+2)}}{1-q^2} z^{n+1} \right\}.\end{aligned}$$

By (4.1), we have $h_M(\varphi.a) = h_M(\varphi)a(1)1$ for any $\varphi \in A(M)$ and $a \in U_q(\mathrm{su}(2))$. This implies $h_M((\tilde{x}\tilde{u}z^n).e) = 0$. Hence, by setting $a_n = \dfrac{1-q^{2(n+1)}}{1-q^2} h_M(z^n)$, we obtain a recurrence relation

$$a_{n+1} - (c-d)a_n - cda_{n-1} = 0 \quad \text{for } n \in \mathbb{N}. \tag{4.7}$$

Then we have

$$a_n = \frac{c^{n+1} - (-d)^{n+1}}{c+d}, \text{ i.e. } h_M(z^n) = \frac{c^{n+1} - (-d)^{n+1}}{c+d} \frac{1-q^2}{1-q^{2(n+1)}}. \tag{4.8}$$

□

The invariant measure $h_M \colon A(M) \to \mathscr{R}$ gives rise to the two hermitian forms $\langle\ ,\ \rangle_L$ and $\langle\ ,\ \rangle_R$ on $A(M)$ with values in $\mathscr{R}$:

$$\langle \varphi, \psi \rangle_L = h_M(\varphi^* \psi), \ \langle \varphi, \psi \rangle_R = h_M(\varphi \psi^*) \quad \text{for } \varphi, \psi \in A(M). \tag{4.9}$$

Note that the hermitian form $\langle\ ,\ \rangle_L$ is conjugate linear in the left argument and $\langle\ ,\ \rangle_R$ in the right.

THEOREM 4.2. (1) *The spherical functions φ^j_{mn} $(m, n \in I_j, j \in \frac{1}{2}\mathbb{N})$ are orthogonal with respect to the hermitian forms $\langle\ ,\ \rangle_L$ and $\langle\ ,\ \rangle_R$.*

(2) *The square lengths of φ^j_{mn} with respect to the Hermitian forms $\langle\ ,\ \rangle_L$ and*

$\langle\ ,\ \rangle_R$ *are given by*

$$\langle \varphi^j_{mn}, \varphi^j_{mn}\rangle_L = q^{2(j-m)} \frac{1-q^2}{1-q^{2(2j+1)}} \Pi_{-j+n\leqslant k\leqslant j+n, k\neq 0}(c+q^{2k}d), \tag{4.10}$$

$$\langle \varphi^j_{mn}, \varphi^j_{mn}\rangle_R = q^{2(j+n)} \frac{1-q^2}{1-q^{2(2j+1)}} \Pi_{-j-n\leqslant k\leqslant j-n, k\neq 0}(c+q^{2k}d). \tag{4.11}$$

Proof. The left $A(G)$-comodule structure gives the relation

$$L_G(\varphi^{j_1 *}_{m_1 n_1}\varphi^{j_2}_{m_2 n_2}) = \sum_{k_1,k_2} w^{j_1 *}_{m_1 k_1} w^{j_2}_{m_2 k_2} \otimes \varphi^{j_1 *}_{k_1 n_1}\varphi^{j_2}_{k_2 n_2}. \tag{4.12}$$

By the compatibility (4.3), we have the equality

$$\langle \varphi^{j_1}_{m_1 n_1}, \varphi^{j_2}_{m_2 n_2}\rangle_L = \sum_{k_1,k_2} \langle w^{j_1}_{m_1 k_1}, w^{j_2}_{m_2 k_2}\rangle_L \varphi^{j_1 *}_{k_1 n_1}\varphi^{j_2}_{k_2 n_2}. \tag{4.13}$$

The right hand side of (4.13) is calculated as follows by use of Theorem 3.7 in [M1]:

$$\langle \varphi^{j_1}_{m_1 n_1}, \varphi^{j_2}_{m_2 n_2}\rangle_L = \delta_{j_1 j_2}\delta_{m_1 m_2} \frac{1-q^2}{1-q^{2(j_1+1)}} q^{2(j_1-m_1)} \sum_{k_1} \varphi^{j_1 *}_{k_1 n_1}\varphi^{j_1}_{k_1 n_2}. \tag{4.14}$$

Moreover, the orthogonality

$$\langle \varphi^{j_1}_{m_1 n_1}, \varphi^{j_1}_{m_1 n_2}\rangle_L = 0 \quad \text{if } n_1 \neq n_2 \tag{4.15}$$

follows from the fact $\varphi^{j_1 *}_{m_1 n_1}\varphi^{j_1}_{m_1 n_1} \in A(M)[0, 2(n_2-n_1)]$ and Theorem 4.1. Equalities (4.14) and (4.15) prove the assertion (1) for $\langle\ ,\ \rangle_L$. The square length $\langle \varphi^j_{mn}, \varphi^j_{mn}\rangle_L$ is calculated as follows. First, we calculate the square length $\langle \varphi^j_{jn}, \varphi^j_{jn}\rangle_L$. Lemma 2.4 gives

$$\begin{aligned}\varphi^{j *}_{jn}\varphi^j_{jn} &= \begin{bmatrix} 2j \\ j+n \end{bmatrix}_{q^2} (-q^{-1})^{j-n}\tilde{y}^{j+n}\tilde{x}^{j+n}\tilde{v}^{j-n}\tilde{u}^{j-n} \\ &= \begin{bmatrix} 2j \\ j+n \end{bmatrix}_{q^2} q^{-(j-n)(j-n+1)} c^{j+n}(q^2 z/c\, q^2)_{j+n} d^{j-n}(-q^2 z/d; q^2)_{j-n}.\end{aligned} \tag{4.16}$$

Theorem 4.1 and Theorem 1 in [AA1] lead the formula

$$h_M(c^{j+n}(q^2z/c;\, q^2)_{j+n}d^{j-n}(-q^2z/d;\, q^2)_{j-n})$$
$$= \frac{(q^2;\, q^2)_1(q^2;\, q^2)_{j+n}(q^2;\, q^2)_{j-n}}{(q^2;\, q^2)_{2j+1}}\, c^{j+n}(-q^2d/c;\, q^2)_{j+n}d^{j-n}(-q^2c/d;\, q^2)_{j-n}. \tag{4.17}$$

Thus, by (4.16) and (4.17), we have

$$\langle \varphi^j_{jn},\, \varphi^j_{jn}\rangle_L = \frac{1-q^2}{1-q^{2(2j+1)}}\, \Pi_{-j+n\leqslant k\leqslant j+n,k\neq 0}(c+q^{2k}d). \tag{4.18}$$

On the other hand, we have the equality

$$\langle \varphi^j_{mn},\, \varphi^j_{m,n}\rangle_L = q^{2(j-m)}\frac{1-q^2}{1-q^{2(j+1)}}\sum_{k\in I_j}\varphi^{j*}_{kn}\varphi^j_{kn} \tag{4.19}$$

by (4.14). Hence,

$$\langle \varphi^j_{mn},\, \varphi^j_{mn}\rangle_L = q^{2(j-m)}\langle \varphi^j_{jn},\, \varphi^j_{jn}\rangle_L. \tag{4.20}$$

Therefore, by combining (4.18) and (4.20), we deduce the desired result. The statements for $\langle\ ,\ \rangle_R$ are proved by an argument similar to that in Proposition 4 of [NM0]. □

Combining Theorem 3.5, Theorem 4.1 and Theorem 4.2, we get the orthogonality relation for the big q-Jacobi polynomials.

$$\int_{-d}^{c} P^{(\alpha,\beta)}_m(z;\, c,\, d\colon q)P^{(\alpha,\beta)}_n(z;\, c,\, d\colon q)(qz/c;\, q)_\alpha \times (-qz/d;\, q)_\beta\, d_qz = 0, \quad \text{if } m\neq n.$$

As a corollary to the proof of Theorem 4.2, we have

PROPOSITION 4.3. *For any $j\in\frac{1}{2}\mathbb{N}$ and $m,\, n\in I_j$, one has*

$$\sum_k \varphi^{j*}_{km}\varphi^j_{kn} = \delta_{mn}\Pi_{-j+n\leqslant k\leqslant j+n,k\neq 0}(c+q^{2k}d). \tag{4.21}$$

□

Theorem 3.6 and Proposition 4.3 imply the orthogonality relation for the q-Hahn polynomials:

$$\sum_{x=0}^{N} Q_m(q^{-x}; a, b, N; q)Q_n(q^{-x}; a, b, N; q)\frac{(aq; q)_x(bq; q)_{N-x}}{(q; q)_x(q; q)_{N-x}}(aq)^{-x} = 0$$

for integers m, n and N such that $m \neq n$ and $0 \leqslant m, n \leqslant N$. This generalizes the interpretation of a q-analogue of Krawtchouk polynomials, due to Koornwinder [K], to the q-Hahn polynomials.

REMARK. By means of the realization of $A(M)$ in Section 5, it is also possible to give an interpretation of the dual q-Hahn polynomials. It seems difficult to carry this out within this algebra $A(M)$.

5. Realization of $A(M)$ on $\mathrm{SU}_q(2)$

The algebra $A(M)$ can be constructed from the algebra of functions $A(G)$ on $G = \mathrm{SU}_q(2)$ by extending the coefficient ring.

We consider the polynomial ring $\mathscr{C} = \mathbb{C}[\alpha, \beta, \gamma, \delta]$ with four commuting indeterminates α, β, γ and δ and define the $*$-structure of $\mathscr{C}$ by $\alpha^* = \delta$, $\beta^* = -\gamma$. Let $\lambda: \mathscr{C} \to \mathscr{C}$ be the $\mathbb{C}$-algebra automorphism of $\mathscr{C}$ such that

$$\lambda\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & q\beta \\ q\gamma & \delta \end{pmatrix}. \tag{5.1}$$

By adjoining this automorphism λ to $\mathscr{C}$, we construct a noncommutative Laurent polynomial ring $\mathscr{C}[\lambda, \lambda^{-1}] = \mathscr{C} \otimes_{\mathbb{C}} \mathbb{C}[\lambda, \lambda^{-1}]$ with the commutation relation $\lambda.a = \lambda(a)a.\lambda$ for $a \in \mathscr{C}$. Then the algebra $\mathscr{C}[\lambda, \lambda^{-1}]$ has a natural $*$-structure characterized by the condition $\lambda^* = \lambda^{-1}$.

Next we consider the extension $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ of the $\mathbb{C}$-algebra $A(G) = A(\mathrm{SU}_q(2))$. Here we define the algebra structure of $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ so that the elements of $A(G)$ commute with those of $\mathscr{C}[\lambda, \lambda^{-1}]$. Note that the algebra $A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}]$ has a natural $*$-structure induced from those of $A(G)$ and $\mathscr{C}[\lambda, \lambda^{-1}]$. In what follows, we consider $A(G)$ and $\mathscr{C}[\lambda, \lambda^{-1}]$ as subalgebras of $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ and use frequently the abbreviation $\varphi.a$ instead of $\varphi \otimes a$ for $\varphi \in A(G)$ and $a \in \mathscr{C}[\lambda, \lambda^{-1}]$.

It should be emphasized that some operations on $A(G)$ are naturally extended to $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$. For example, we have the following three $\mathscr{C}[\lambda, \lambda^{-1}]$-homomorphisms:

$$\Delta \otimes \mathrm{id}\colon A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}] \to A(G) \otimes A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}], \tag{5.2}$$

$$\varepsilon \otimes \mathrm{id}\colon A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}] \to \mathscr{C}[\lambda, \lambda^{-1}], \tag{5.3}$$

$$h_G \otimes \mathrm{id}\colon A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}] \to \mathscr{C}[\lambda, \lambda^{-1}]. \tag{5.4}$$

Note that the first two are algebra homomorphisms. The algebra $A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}]$ becomes a left $A(G)$-comodule endowed with the homomorphism $\Delta \otimes \mathrm{id}$ of (5.2).

We can realize the algebra $A(M)$ defined in Section 2 as a $\mathbb{C}$-subalgebra of $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$. First, define the four elements X, U, V and Y of $A(G) \otimes \mathscr{C}$ by

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} = \begin{pmatrix} x & u \\ v & y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{5.5}$$

THEOREM 5.1. *There exists a unique $\mathbb{C}$-algebra homomorphism $\rho\colon A(M) \to A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ such that*

$$\rho(\tilde{x}) = X\lambda^{-1}, \ \rho(\tilde{u}) = U\lambda, \quad \rho(\tilde{v}) = V\lambda^{-1}, \quad \rho(\tilde{y}) = Y\lambda, \quad \rho(c) = \alpha\delta, \quad \rho(d) = -\beta\gamma. \tag{5.6}$$

Moreover ρ is compatible with the $*$-structure.

Proof. Let $\tau\colon A(G) \to A(G)$ be the $\mathbb{C}$-algebra automorphism defined by $\tau(x) = q^{-1}x$, $\tau(u) = u$, $\tau(v) = v$, $\tau(y) = qy$. Then one can directly show that

$$\varphi\tau(\psi) - \psi\tau(\varphi) \in \mathbb{C} \tag{5.7}$$

if φ and ψ are linear combinations of the generators x, u, v, y. Moreover one has

$$\lambda(X) = q\tau(X), \ \lambda^{-1}(U) = \tau(U), \ \lambda(V) = \tau(V), \ \lambda^{-1}(Y) = q^{-1}\tau(Y) \tag{5.8}$$

where λ and τ are regarded as automorphisms of $A(G) \otimes \mathscr{C}$. By using formulas (5.7) and (5.8), one can directly check that the set of elements $X\lambda^{-1}$, $U\lambda$, $V\lambda^{-1}$, $Y\lambda$ in $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ satisfies the defining relations of $A(M)$. Compatibility with the $*$-structure is also directly verified. □

By the definition (2.4), it is clear that ρ is a homomorphism of left $A(G)$-comodules. Note also that the commutative subalgebra $\mathscr{R} = \mathbb{C}[c, d]$ of $A(M)$ is identified with the subalgebra $\rho(\mathscr{R}) = \mathbb{C}[\alpha\delta, -\beta\gamma]$ of $\mathscr{C}$, so that ρ is a $\mathscr{R}$-homomorphism of two-sided $\mathscr{R}$-modules.

PROPOSITION 5.2. (1) *The invariant measure h_M on $A(M)$ is compatible with $h_G \otimes \mathrm{id}$ of (5.4). Namely one has*

$$h_M(\varphi) = (h_G \otimes \mathrm{id}) \circ \rho(\varphi) \quad \textit{for all } \varphi \in A(M). \tag{5.9}$$

(2) *The homomorphism* $\rho: A(M) \to A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ *defined by* (5.6) *is injective.*

Proof. (1) Since $\rho: A(M) \to A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}]$ is a homomorphism of left $A(G)$-comodules, we have $\rho(\Phi_{j;n}) \subset w_j \otimes \mathscr{C}[\lambda, \lambda^{-1}]$ for all $j \in \frac{1}{2}\mathbb{N}$, $n \in I_j$. Hence $(h_G \otimes \mathrm{id}) \circ \rho(\varphi^j_{mn}) = 0$ for $j > 0$ and $m, n \in I_j$. This shows that equality (5.9) holds for any $\varphi \in A(M)$ since $(h_G \otimes \mathrm{id}) \circ \rho$ is an $\mathscr{R}$-homomorphism and $(h_G \otimes \mathrm{id}) \circ \rho(1) = 1$.
Equality (5.9) implies that

$$\langle \varphi, \psi \rangle_L = (h_G \otimes \mathrm{id})(\rho(\varphi)^* \rho(\psi)) \tag{5.10}$$

for any $\varphi, \psi \in A(M)$. Hereafter, we regard $\mathscr{R}$ as the subalgebra $\mathbb{C}[\alpha\delta, -\beta\gamma]$ of $\mathscr{C}[\lambda, \lambda^{-1}]$. Suppose that φ is a finite sum

$$\varphi = \sum_{j,m,n} \varphi^j_{mn} a^j_{mn} \quad \text{with } a^j_{mn} \in \mathscr{R}. \tag{5.11}$$

Then by using Theorem 4.2 and (5.10), we have

$$\langle \varphi^j_{mn}, \varphi^j_{mn} \rangle_L a^j_{mn} = (h_G \otimes \mathrm{id})(\rho(\varphi^j_{mn})^* \rho(\varphi)). \tag{5.12}$$

Thus we have $a^j_{mn} = 0$ for all j, m, n if $\rho(\varphi) = 0$, which shows that ρ is injective. □

Proposition 5.2.(2) means that the subalgebra of $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$ generated by the six elements in (5.6) is isomorphic to $A(M)$. In what follows, we identify the algebra $A(M)$ with this $*$-subalgebra of $A(G) \otimes_{\mathbb{C}} \mathscr{C}[\lambda, \lambda^{-1}]$. Under this identification, we have $L_G = \Delta \otimes \mathrm{id}$ and $h_M = h_G \otimes \mathrm{id}$. We remark that the $\mathbb{C}$-algebra homomorphism $\varepsilon \otimes \mathrm{id}: A(M) \to \mathscr{C}[\lambda, \lambda^{-1}]$ induced from (5.3) is described as

$$(\varepsilon \otimes \mathrm{id}) \begin{pmatrix} \tilde{x} & \tilde{u} \\ \tilde{v} & \tilde{y} \end{pmatrix} = \begin{pmatrix} \alpha\lambda^{-1} & \beta\lambda \\ \gamma\lambda^{-1} & \delta\lambda \end{pmatrix}, \ (\varepsilon \otimes \mathrm{id})(c) = \alpha\delta, \ (\varepsilon \otimes \mathrm{id})(d) = -\beta\gamma. \tag{5.13}$$

In this realization of $A(M)$, we can consider the connection formula between φ^j_{mn} and w^j_{mn}. The connection coefficients are expressed by Stanton's q-Krawtchouk polynomials [S] defined by

$$K_k(q^{-x}; c, N: q) = {}_3\varphi_2 \left(\begin{matrix} q^{-k}, q^{-x}, -cq^{k+1} \\ 0, q^{-N} \end{matrix} ; q, q \right), \tag{5.14}$$

where k and N are integers such that $0 \leqslant k \leqslant N$.

THEOREM 5.3. *For each $j \in \frac{1}{2}\mathbb{N}$ and $m, n \in I_j$, we have*

$$\varphi^j_{mn} = \sum_{k \in I_j} w^j_{mk} \otimes c^j_{kn} \quad \text{in } A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}], \tag{5.15}$$

where

$$c^j_{mn} = q^{(m-n)j + m(m-1)/2 - n(n-1)/2} \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2}_{q^2} \begin{bmatrix} 2j \\ j+n \end{bmatrix}^{1/2}_{q^2} \tag{5.16}$$
$$\times \beta^{j+m}\gamma^{j+n}\delta^{-m-n} K_{j+n}(q^{-2(j+m)}; q^{-4j-2}\alpha\delta/\beta\gamma, q^{-4j}: q^2)\lambda^{-2n}.$$

Proof. Since $L_G = \Delta \otimes \mathrm{id}$, we have

$$(\Delta \otimes \mathrm{id})(\varphi^j_{mn}) = \sum_k w^j_{mk} \otimes \varphi^j_{kn} \quad \text{in } A(G) \otimes A(G) \otimes \mathscr{C}[\lambda, \lambda^{-1}] \tag{5.17}$$

by (3.4). Applying the operator $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}$ on the both sides, we have

$$\varphi^j_{mn} = \sum_k w^j_{mk} \otimes c^j_{kn}, \quad \text{where } c^j_{mn} = (\varepsilon \otimes \mathrm{id})(\varphi^j_{mn}). \tag{5.18}$$

The above expression (5.16) of c^j_{mn} is thus obtained from Theorem 3.6 by using (5.13). □

Combining Theorem 5.3 and Proposition 4.3, we get the orthogonality relation for the Stanton's q-Krawtchouk polynomials.

PROPOSITION 5.4. *Let k, j and N be integers such that $0 \leqslant k, j \leqslant N$. Then we have*

$$\sum_{x=0}^{N} K_k(q^{-x}; c, N; q) K_j(q^{-x}; c, N; q) \frac{(q^{-N}; q)_x}{(q; q)_x} (-cq)^{-x} = 0 \quad \text{if } k \neq j. \tag{5.19}$$

□

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