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## Non-integrability of the equations of heavy gyrostат

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**Abstract.** We prove that the equations of heavy gyrostат possess an additional algebraic first integral only in the cases of Zhukovsky, Lagrange, and Yehia.

### 1. Introduction

Consider a rigid body moving about a fixed point under the action of uniform gravity and gyroscopic forces, due to symmetric rotors or holes filled with an ideal incompressible fluid. Such a body is usually called heavy gyrostат. Its equations of motion can be written in the form [8, 15]

$$\begin{cases} J \frac{d}{dt} \omega = (J\omega + \lambda) \times \omega + \varepsilon e \times r \\ \frac{d}{dt} e = e \times \omega \end{cases} \quad (1.1)$$

where  $\omega = (\omega_1, \omega_2, \omega_3)$  is the angular velocity,  $e = (e_1, e_2, e_3)$  is the unit vector along the direction of the gravitational field,  $J\omega = (A\omega_1, B\omega_2, C\omega_3)$  is the kinetic momentum,  $r = (x_0, y_0, z_0)$  is the center of mass (the components of these vectors are referred to the fixed in the body frame, formed by the principal axes of inertia of the body at the fixed point),  $\varepsilon$  is the mass of the body,  $A, B, C$  are the principal moments of inertia, and  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  is the gyrostatic moment (due to the gyroscopic forces). Let us denote  $M = (M_1, M_2, M_3) = J\omega$ . The system (1.1) may be represented in a Hamiltonian form

$$\frac{d}{dt} z = P(z) \nabla H$$

where  $z = (M, e) \in \mathbb{R}^6$ ,  $\nabla H = \left( \frac{\partial H}{\partial M}, \frac{\partial H}{\partial e} \right) \in \mathbb{R}^6$ ,  $H$  is the energy of the body

$$H = \frac{1}{2} \langle \omega, J\omega \rangle + \varepsilon \langle r, e \rangle \quad (1.2)$$

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$\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^3$ ,

$$P(z) = \begin{bmatrix} G & F \\ F & 0 \end{bmatrix}; \quad G = \begin{bmatrix} 0 & -M_3 - \lambda_3 & M_2 + \lambda_2 \\ M_3 + \lambda_3 & 0 & -M_1 - \lambda_1 \\ -M_2 - \lambda_2 & M_1 + \lambda_1 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}.$$

The matrix  $P(z)$  defines a Poisson structure on  $\mathbb{R}^6$ . The two geometric first integrals of (1.1)

$$H_1 = \langle J\omega + \lambda, e \rangle \quad (1.3)$$

$$H_2 = \langle e, e \rangle \quad (1.4)$$

are Casimirs of this structure. The vector field (1.1) restricted on the four-dimensional level set

$$I_{h_1, h_2} = \{(M, e) \in \mathbb{R}^6 : H_1 = h_1, H_2 = h_2, h_2 > 0\} \cong T^*S^2$$

is a two degrees of freedom Hamiltonian system. Thus for Liouville complete integrability of the system (1.1) we need, except of the Casimirs  $H_1, H_2$ , and the Hamiltonian  $H$ , an additional integral of motion  $H_4$ . Such integral does exist in the following three cases

- (i)  $x_0 = y_0 = z_0 = 0$  (Zhukovsky [8]),

$$H_4 = \langle M + \lambda, M + \lambda \rangle$$

- (ii)  $A = B, x_0 = y_0 = 0, \lambda_1 = \lambda_2 = 0$  (Lagrange [8]),

$$H_4 = \omega_3$$

- (iii)  $A = B = 2C, y_0 = z_0 = 0, \lambda_1 = \lambda_2 = 0$  (Yehia [16]),

$$H_4 = (C(\omega_1^2 - \omega_2^2) - \varepsilon x_0 e_1)^2 + (2C\omega_1\omega_2 - \varepsilon x_0 e_2)^2 - 4\varepsilon x_0 \lambda_3 \omega_1 e_3 + 2\lambda_3(\omega_1^2 + \omega_2^2)(C\omega_3 - \lambda_3).$$

REMARK. Note that the integrable cases  $A = B = C, r \times \lambda = 0$ , and  $A = B = 2C, z_0 = 0, \lambda_1 = \lambda_2 = 0$ , are equivalent to (ii) and (iii) respectively (after a suitable

rotation of the inertial frame). For that reason we do not consider these cases separately.

If  $\lambda = 0$  the system (1.1) turns into the customary Euler-Poisson equations

$$\begin{cases} J \frac{d}{dt} \omega = J \omega \times \omega + \varepsilon e \times r \\ \frac{d}{dt} e = e \times \omega \end{cases} \quad (1.1^0)$$

describing motion of a rigid body about a fixed point in a presence of gravity. The first integrals (i), (ii), and (iii) turn into the well known Euler, Lagrange and Kovalevskaya first integrals of the system (1.1<sup>0</sup>). We recall here the following classical result.

**THEOREM** (Husson [11]). *The Euler-Poisson equations (1.1<sup>0</sup>) possess an additional algebraic first integral only in the three cases of Euler, Lagrange, and Kovalevskaya.*

In the present paper we generalize the Husson's result.

*Theorem 1.1. The equations of heavy gyrostat (1.1) possess an additional algebraic first integral only in the three cases of Zhukovsky, Lagrange, and Yehia.*

**REMARK.** It was believed until recently that the equations of heavy gyrostat (1.1) possess an additional algebraic first integral only in the three cases of Zhukovsky, Lagrange, and Kovalevskaya ( $\lambda_1 = \lambda_2 = \lambda_3 = 0$  in the last case) as claimed a result of Keis [12]. In spite of that assertion Yehia [16] found a new additional first integral in case (iii) above.

Theorem 1.1 will be proved in two steps. First we show that if the system (1.1) possesses an additional rational first integral, then the corresponding system (1.1<sup>0</sup>) which is obtained from (1.1) after substituting  $\lambda = (0, 0, 0)$ , also possesses an additional rational first integral (Section 2). Using Husson's theorem we conclude that if the system (1.1) possesses an additional rational first integral then the parameters  $A, B, C, r = (x_0, y_0, z_0)$ , have to satisfy either Euler, or Lagrange, or Kovalevskaya condition, i.e. the condition (i), or (ii), or (iii) (note that there are no restrictions on  $\lambda$  at this step). At the second step we prove that for these values of the parameters  $A, B, C, r$ , and an arbitrary value of  $\lambda$  the system (1.1) is not of Painlevé type, except in the known integrable cases of Zhukovsky, Lagrange, and Yehia. More precisely, using Horn's theorem [10] (see Section 3) we conclude that the system (1.1) is not of Painlevé type because a five-parameter (the dimension of the phase space minus one) family of its solutions possess logarithmic branch points. The basic observation of the paper is that this result can be improved as it is explained in Lemma 3.1. This lemma

implies that if the system (1.1) possesses an additional rational first integral, but it is not of Painlevé type, then the pole divisor  $\Gamma$  (see Section 3) of the corresponding Euler-Poisson equations (1.1<sup>0</sup>) contains straight lines (genus zero curves). As the pole divisor can be easily studied (Lemma 3.3) we arrive at the desirable contradiction. To complete the proof of Theorem 1.1 we note that if the system (1.1) possesses an additional algebraic first integral then it also possesses an additional rational first integral.

## 2. Similarity and non-similarity invariant systems of ordinary differential equations

Consider the complex system of ODE

$$\frac{d}{dt} x_i = F_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n \quad (2.1)$$

where  $F_1, F_2, \dots, F_n$  are rational functions in the variables  $x_1, x_2, \dots, x_n$ . As Yoshida [17] has noted, there is a connection between the first integrals of the system (2.1) and the first integrals of a simpler ‘reduced’ system of ODE. Below we describe this connection in detail. One may easily prove, following Bruns [3], that if the system (2.1) possesses  $k$  algebraic functionally independent first integrals, then this system also possesses  $k$  rational functionally independent first integrals. For that reason further we shall consider only rational first integrals.

For an arbitrary monomial  $y = \prod_{i=1}^n x_i^{k_i}$  let the weighed degree  $\deg(y)$  of  $y$  be  $\sum_{i=1}^n k_i \cdot g_i$ , where  $g_1, g_2, \dots, g_n$  are fixed rational numbers. Any polynomial  $\Phi$  can be represented as a sum of weight-homogeneous polynomials  $\Phi = \sum_{i=s_1}^{s_2} \Phi_i$  where  $\deg(\Phi_i) = i$ . Let us denote  $\Phi^0 = \Phi_{s_2}$ . If  $\Phi = \Phi_1/\Phi_2$  is a rational function, where  $\Phi_1$  and  $\Phi_2$  are polynomials, then we denote  $\Phi^0 = \Phi_1^0/\Phi_2^0$  and  $\deg(\Phi^0) = \deg(\Phi_1^0) - \deg(\Phi_2^0)$ . Consider the following system of ODE

$$\frac{d}{dt} x_i = F_i^0(x_1, x_2, \dots, x_n), \quad (i = 1, 2, \dots, n). \quad (2.1^0)$$

**DEFINITION.** If the system (2.1<sup>0</sup>) is invariant under the similarity transformation

$$t \rightarrow \alpha^{-1} \cdot t, \quad x_i \rightarrow \alpha^{g_i} \cdot x_i, \quad i = 1, 2, \dots, n \quad (2.2)$$

then it is called reduction of the system (2.1) with respect to the transformation (2.2).

EXAMPLE. The system (1.1<sup>0</sup>) is reduction of the system (1.1) with respect to the similarity transformation

$$t \rightarrow \alpha^{-1} \cdot t, \quad \omega_i \rightarrow \alpha \cdot \omega_i, \quad e_i \rightarrow \alpha^2 \cdot e_i, \quad i = 1, 2, \dots, n. \quad (2.3)$$

The following lemma holds

LEMMA 2.1. *If the system (2.1) possesses  $k$  rational functionally independent first integrals, then any reduction of the system (2.1) also possesses  $k$  rational functionally independent first integrals.*

Some simple generalizations of Lemma 2.1 are given in [4]. Applying this lemma to system (1.1) and making use of Husson's theorem we obtain the following

COROLLARY. *If the system (1.1) possesses an additional rational first integral, then either*

- (i)  $x_0 = y_0 = z_0 = 0$  (Euler case) or
- (ii)  $A = B, x_0 = y_0 = 0$  (Lagrange case) or
- (iii)  $A = B = 2C, y_0 = z_0 = 0$  (Kovalevskaya case).

The above corollary will be used in Section 3. To prove Lemma 2.1 we shall need the following

LEMMA 2.2. *If the  $k$  rational functions  $\Phi_1, \Phi_2, \dots, \Phi_k$  in the variables  $x_1, x_2, \dots, x_n$  are functionally independent, and the functions  $\Phi_1^0, \Phi_2^0, \dots, \Phi_{k-1}^0$  are also functionally independent, then there exists a polynomial  $\Phi$  in the variables  $\Phi_1, \Phi_2, \dots, \Phi_k$  and such that  $\Phi_1^0, \Phi_2^0, \dots, \Phi_{k-1}^0, \Phi^0$  are functionally independent.*

*Proof of Lemma 2.2.* The proof repeats the arguments of the algebraic lemma proved by Ziglin [18], part I, p. 184. Namely, let

$$M = \det \left( \frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_k)}{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})} \right)$$

be a minor of the matrix

$$\frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_k)}{\partial(x_1, x_2, \dots, x_n)}.$$

As  $\Phi_1, \Phi_2, \dots, \Phi_k$  are functionally independent then we may suppose that  $M \neq 0$  and let  $M$  be chosen in such a way that it maximizes the sum  $\deg(M^0) + \sum_{j=1}^k g_{i_j}$ .

Put

$$\mu = \mu(\Phi_1, \Phi_2, \dots, \Phi_k) = \deg(M^0) + \sum_{j=1}^k g_{i_j} - \sum_{i=1}^k \deg(\Phi_i^0). \tag{2.4}$$

Obviously  $\mu \leq 0$ , and  $\mu = 0$  iff  $\Phi_1^0, \Phi_2^0, \dots, \Phi_k^0$  are functionally independent. We shall prove that if  $\mu < 0$  then there exists a polynomial  $P(z_1, z_2, \dots, z_k)$  such that the functions

$$\Phi_1, \Phi_2, \dots, \Phi_{k-1}, \Phi = P(\Phi_1, \Phi_2, \dots, \Phi_k)$$

are functionally independent, and

$$\mu(\Phi_1, \Phi_2, \dots, \Phi_{k-1}, \Phi) > \mu(\Phi_1, \Phi_2, \dots, \Phi_k).$$

Then after a finite number of steps we can make  $\mu = 0$  which will imply the proof of Lemma 2.2.

Suppose that  $\mu < 0$ . As  $\Phi_1^0, \Phi_2^0, \dots, \Phi_k^0$  are functionally dependent then there exists a polynomial  $P(z_1, z_2, \dots, z_k)$  with real coefficients such that  $P(\Phi_1^0, \Phi_2^0, \dots, \Phi_k^0) \equiv 0$ . We shall also suppose without loss of generality that  $P(z_1, z_2, \dots, z_k)$  is weight-homogeneous with respect to  $z_1, z_2, \dots, z_k$  where  $\deg(z_i) = \deg(\Phi_i)$ , and that  $P(z_1, z_2, \dots, z_k)$  is irreducible in the ring of the polynomials with real coefficients. As  $\Phi_1^0, \Phi_2^0, \dots, \Phi_{k-1}^0$  are functionally independent then  $P(z_1, z_2, \dots, z_k)$  depends upon  $z_k$  and hence  $\frac{\partial P}{\partial z_k}(z_1, z_2, \dots, z_k) \neq 0$ .

If  $\frac{\partial P}{\partial z_k}(\Phi_1^0, \Phi_2^0, \dots, \Phi_k^0) \equiv 0$  then this together with  $P(\Phi_1^0, \Phi_2^0, \dots, \Phi_k^0) \equiv 0$  implies that  $\Phi_1, \Phi_2, \dots, \Phi_{k-1}$  are functionally dependent which is a contradiction. Hence  $\frac{\partial P}{\partial z_k}(\Phi_1^0, \Phi_2^0, \dots, \Phi_k^0) \neq 0$ . The upshot is that the following inequality holds

$$\deg(\tilde{\Phi}^0) + \deg(\Phi_k^0) > \deg(\Phi^0) \tag{2.5}$$

where we have put  $\tilde{\Phi} = \frac{\partial P}{\partial z_k}(\Phi_1, \Phi_2, \dots, \Phi_k)$ ,  $\Phi = P(\Phi_1, \Phi_2, \dots, \Phi_k)$ .

The inequality (2.5) implies immediately  $\mu(\Phi_1, \Phi_2, \dots, \Phi_{k-1}, \Phi) > \mu(\Phi_1, \Phi_2, \dots, \Phi_k)$ . Indeed, let  $\mu(\Phi_1, \Phi_2, \dots, \Phi_k)$  be defined by (2.4) where

$$M = \det \left( \frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_k)}{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})} \right)$$

and put

$$\tilde{M} = \det \left( \frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_{k-1}, \Phi)}{\partial(x_{i_1}, x_{i_2}, \dots, x_{i_k})} \right) = M \cdot \tilde{\Phi}.$$

Then

$$\begin{aligned} \mu(\Phi_1, \Phi_2, \dots, \Phi_{k-1}, \Phi) &\geq \deg(\tilde{M}^0) + \sum_{j=1}^k g_{i_j} - \sum_{i=1}^{k-1} \deg(\Phi_i^0) - \deg(\Phi^0) \\ &= \deg(M^0) + \sum_{j=1}^k g_{i_j} - \sum_{i=1}^k \deg(\Phi_i^0) + \deg(\tilde{\Phi}^0) + \deg(\Phi_k^0) - \deg(\Phi^0) \\ &= \mu(\Phi_1, \Phi_2, \dots, \Phi_k) + \deg(\tilde{\Phi}^0) + \deg(\Phi_k^0) - \deg(\Phi^0) > \mu(\Phi_1, \Phi_2, \dots, \Phi_k). \quad \square \end{aligned}$$

*Proof of Lemma 2.1.* Suppose that (2.1<sup>0</sup>) is reduction of the system (2.1) with respect to the similarity transformation (2.2). If  $\Phi$  is a rational first integral of (2.1) then  $\Phi^0$  is a rational first integral of the reduced system (2.1<sup>0</sup>). Indeed, if  $\frac{d}{dt}$

and  $\frac{d^0}{dt}$  are the time derivatives along the flows of (2.1) and (2.1<sup>0</sup>) respectively,

and  $\frac{d^0}{dt} \Phi^0 \neq 0$ , then  $\frac{d^0}{dt} \Phi^0 = \left( \frac{d}{dt} \Phi \right)^0 \equiv 0$ . Using Lemma 2.2 we conclude that if the system (2.1) possesses  $k$  rational functionally independent first integrals, then the reduced system (2.1<sup>0</sup>) also possesses  $k$  rational functionally independent first integrals. □

### 3. Existence of logarithmic singularities and non-existence of rational first integrals

Suppose that the system (1.1) possesses an additional rational first integral. According to the corollary of Section 2 there are three possibilities for the parameters  $A, B, C, x_0, y_0,$  and  $z_0$ . As in the Euler case an additional first integral exists for any choice of  $\lambda_1, \lambda_2,$  and  $\lambda_3$ , then we shall consider only Lagrange and Kovalevskaya cases. Suppose that either  $A = B, x_0 = y_0 = 0,$  or  $A = B = 2C, y_0 = z_0 = 0$ . Suppose also that  $\lambda_1, \lambda_2,$  and  $\lambda_3$  are fixed real constants such that  $\lambda_1^2 + \lambda_2^2 \neq 0$ . For an arbitrary  $v \in \mathbb{C}$  consider the system

$$\begin{cases} J \frac{d}{dt} \omega = (J\omega + v \cdot \lambda) \times \omega + \varepsilon e \times r \\ \frac{d}{dt} \omega = e \times \omega \end{cases} \tag{3.1'}$$

Obviously the system (3.1<sup>0</sup>) which is obtained from (3.1<sup>v</sup>) after substituting  $v = 0$  coincides with the system (1.1<sup>0</sup>).

Under the above assumptions the following three lemmas hold

LEMMA 3.1. *The system (3.1<sup>v</sup>) possesses a family of complex solutions of the form*

$$\begin{aligned} \omega_i(t) &= t^{-1} \left( \sum_{j=0}^{\infty} \omega_i^j t^j \right), \quad i = 1, 2, 3, \\ e_i(t) &= t^{-2} \left( \sum_{j=0}^{\infty} e_i^j t^j \right), \quad i = 1, 2, 3, \end{aligned} \tag{3.2}$$

depending upon five free parameters  $\alpha_0, \alpha_1, \dots, \alpha_4$ , and convergent for all  $t \in \mathbb{C}$ , such that for arbitrary fixed parameters  $|t|$  is sufficiently small and  $|\arg t|$  is bounded.  $\omega_i^j$  and  $e_i^j$  are polynomials with respect to  $v$ ,  $v \cdot \beta \cdot \log(t) + \alpha_1, \alpha_2, \alpha_3, \alpha_4$ , whose coefficients depend algebraically upon  $\alpha_0$ , and  $\beta = \beta(\alpha_0) \neq 0$  is an algebraic function. For any fixed  $v \in \mathbb{C}$  the trajectories of the complex solutions (3.2) form a set of non-empty interior in the phase  $\mathbb{C}^6$ , and this set is parameterized by  $\alpha_0, \alpha_1, \dots, \alpha_4, t \in \mathbb{C}$ .

If  $(\omega(t), e(t))$  is a solution of the system (1.1) then  $(v^{-1}\omega(vt), v^{-2}e(vt))$  is a solution of the system (3.1<sup>v</sup>). It is concluded that if  $H(\omega, e)$  is a rational first integral of the system (1.1) then the function

$$H^v(\omega, e) = v^{\deg(H^0)} \cdot H(v^{-1}\omega, v^{-2}e)$$

is a rational first integral of the system (3.1<sup>v</sup>), where  $H^0$  is a weight-homogeneous part of  $H$  as it is defined in Section 2,  $\deg(\omega_i) = 1, \deg(e_i) = 2$ . Denote  $x = (\omega, e)$  and  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_4)$ . It is clear that  $H^1(x) \equiv H(x)$  and  $H^v(x)|_{v=0} = H^0(x)$ .

LEMMA 3.2. *Let  $H^v(x)$  be a rational first integral of (3.1<sup>v</sup>) corresponding to the rational first integral  $H(x)$  of the system (1.1) and let  $x(t, \log(t), \alpha, v)$  denotes the solution (3.2) of (3.1<sup>v</sup>). Then*

$$h(\alpha, v) = H^v(x(t, \log(t), \alpha, v))$$

is an algebraic function in  $\alpha, v$ , which does not depend upon  $\alpha_1$ .

LEMMA 3.3. *Let  $H_1^0, H_2^0, H_3^0, H_4^0$  be rational functionally independent first integrals of the system (1.1<sup>0</sup>). Then for all generic constants  $c_1, c_2, c_3, c_4$  the one-dimensional algebraic set*

$$\Gamma = \{ \alpha \in \mathbb{C}^5 : H_i^0(x(t, \log(t), \alpha, 0)) = c_i, i = 1, 2, 3, 4 \}$$

does not contain any of the lines  $\{ \alpha \in \mathbb{C}^5 : \alpha_i = \text{const}, i = 0, 2, 3, 4 \}$ .

*Proof of Theorem 1.1 assuming the above lemmas.* If  $H_1, H_2, H_3, H_4$  are rational functionally independent first integrals of the system (1.1) then  $H_1^y, H_2^y, H_3^y, H_4^y$  are rational functionally independent first integrals of the system (3.1'). According to Lemma 3.2 the algebraic functions

$$h_i(\alpha, \nu) = H_i^y(x(t, \log(t), \alpha, \nu)), \quad (i = 1, 2, 3, 4)$$

do not depend upon  $\alpha_1$  and hence the functions  $h_i(\alpha, 0)$  do not depend upon  $\alpha_1$  too. Thus the set  $\Gamma$  is a finite union of lines  $\{\alpha \in \mathbb{C}^5: \alpha_i = \text{const}, i = 0, 2, 3, 4\}$  which contradicts to Lemma 3.3. It follows that if  $A = B, x_0 = y_0 = 0$ , or  $A = B = 2C, y_0 = z_0 = 0$ , and the equations of heavy gyrostat (1.1) possess an additional rational first integral then  $\lambda_1^2 + \lambda_2^2 = 0$ , i.e.  $\lambda_1 = \lambda_2 = 0$ . As in that case an additional rational first integral does exist (see Section 1) then Theorem 1.1 is proved.  $\square$

To this end we shall prove Lemma 3.1, Lemma 3.2, and Lemma 3.3. Let the system (2.1<sup>0</sup>) be reduction of the system (2.1) with respect to the similarity transformation (2.2), and suppose that the system (2.1<sup>0</sup>) possesses a particular solution

$$x_i(t) = x_i^0 \cdot t^{-g_i}, \quad (i = 1, 2, \dots, n).$$

**DEFINITION.** The matrix

$$K = \left[ \frac{\partial F_i^0}{\partial x_j} (x_1, x_2, \dots, x_n) + g_i \cdot \delta_{ij} \right]_{i,j=1,1}^{n,n} \quad (\delta_{ij} \text{ is the Kronecker delta})$$

is called the Kovalevskaya's matrix of the system (2.1) and its eigenvalues are called the Kovalevskaya's exponents.

Suppose that the Kovalevskaya's exponents are integers and denote by  $\rho_1, \rho_2, \dots, \rho_s$  ( $\rho_i \leq \rho_{i-1}$ ) the positive ones. For the sake of simplicity we shall also suppose that the matrix  $K$  is diagonalizable (i.e. the linear space  $\mathbb{C}^n$  splits into one-dimensional proper subspaces of the linear operator  $K$ ). Let us denote the eigenvectors corresponding to  $\rho_1, \rho_2, \dots, \rho_s$  by  $\vartheta_1, \vartheta_2, \dots, \vartheta_s$  respectively. The following theorem may be easily derived from the classical memoir of Horn [10].

**THEOREM.** *The system (2.1) possesses a  $s$ -parameter family of solutions of the form*

$$x_i(t) = t^{-g_i} \left( \sum_{j=0}^{\infty} x_i^j \cdot t^j \right), \quad (i = 1, 2, \dots, n). \quad (3.4)$$

where the coefficients  $x^j = (x_1^j, x_2^j, \dots, x_n^j)$ ,  $j = 1, 2, \dots, n$ , are polynomials with respect to the  $s$  parameters  $\alpha_1, \alpha_2, \dots, \alpha_s$ , and possibly to  $\log(t)$ . Moreover the

expansion (3.4) is convergent for all values of  $t \in \mathbb{C}$ , such that for arbitrary fixed values of the parameters  $|t|$  is sufficiently small and  $\arg(t)$  is bounded. Each parameter  $\alpha_k$  appears for the first time in the coefficient  $x^{\rho_k}$ , and

$$x^{\rho_k} = \alpha_k \vartheta_k + \left( \begin{array}{l} \text{vector whose entries are polynomials in} \\ \alpha_1, \alpha_2, \dots, \alpha_{k-1} \text{ and possibly in } \log(t) \end{array} \right).$$

If at least one coefficient  $x^1, x^2, \dots$  depends upon  $\log(t)$  then  $\log(t)$  appears for a first time in  $x^{\rho_j}$  for some  $j = 1, 2, \dots, s$ , and

$$x^{\rho_j} = (\beta \cdot \log(t) + \alpha_j) \vartheta_j + \left( \begin{array}{l} \text{vector whose entries are} \\ \text{polynomials in } \alpha_1, \alpha_2, \dots, \alpha_{j-1} \end{array} \right)$$

where  $\beta$  is a constant. At last any formal solution of the system (2.1)

$$\tilde{x}_i(t) = t^{-g_i} \left( \sum_{j=0}^{\infty} \tilde{x}_i^j \cdot t^j \right), \quad (i = 1, 2, \dots, n),$$

where the coefficients  $\tilde{x}_i^j$  are polynomials in  $\log(t)$  coincides with (3.4) for a suitable choice of the parameters  $\alpha_1, \alpha_2, \dots, \alpha_s$ .

*Sketch of the proof of Horn's theorem.* In his original paper [10] Horn has studied systems of the form

$$t \cdot \frac{d}{dt} z_i = G_i(z, t), \quad G_i(0, 0) = 0, \quad (i = 1, 2, \dots, n), \tag{3.5}$$

where  $z = (z_1, z_2, \dots, z_n)$  and  $G_1, G_2, \dots, G_n$  are analytical functions in  $z$  and  $t$  in a neighbourhood of the origin in  $\mathbb{C}^{n+1}$ . He has proved that the system (3.5) possesses a solution (convergent for sufficiently small  $|t|$  and bounded  $\arg(t)$ ) of the form

$$z_i(t) = \sum_{j=1}^{\infty} \sum_{k=0}^{2j-1} x_i^{j,k} \cdot t^j \cdot (\log(t))^k, \quad (i = 1, 2, \dots, n), \tag{3.6}$$

where  $x_i^{j,k}$  are constants. Indeed, after substituting (3.6) into (3.5) we obtain the following recurrent system for the coefficients  $x_i^{j,k}$

$$\begin{aligned} (K - j \cdot I) \cdot x^{j,k} - (k + 1) \cdot x^{j,k+1} &= \tilde{G}^{j,k}, \quad (k \leq 2j - 2) \\ (K - j \cdot I) \cdot x^{j,2j-1} &= 0, \quad K = \left( \frac{\partial G_i}{\partial z_j} (0, 0) \right)_{i,j=1,1}^{n,n} \\ I &= \text{diag}(1, 1, \dots, 1), \quad x^{j,k} = (x_1^{j,k}, x_2^{j,k}, \dots, x_n^{j,k})^t \end{aligned} \tag{3.7}$$

where  $\tilde{G}^{j,k}$  denotes a vector whose entries are polynomials (determined uniquely

by (3.5)) in  $x_i^{l,m}$ ,  $i = 1, 2, \dots, n$ ,  $m = 0, 1, \dots, 2l - 1$ ,  $l = 1, 2, \dots, j - 1$ . If  $j$  is an eigenvalue of the matrix  $K$  then the linear systems (3.7) are overdetermined and hence possibly they are unsolvable. The reader can, however, easily check that if  $x_i^{l,m}$ ,  $i = 1, 2, \dots, n$ ,  $m = 0, 1, \dots, 2l - 1$ ,  $l = 1, 2, \dots, r - 1$ , satisfy the linear systems (3.7) for  $j = 1, 2, \dots, r - 1$ , then there exists constants  $x_i^{r,m}$ ,  $i = 1, 2, \dots, n$ ,  $m = 0, 1, \dots, 2r - 1$ , such that the linear systems (3.7) are also satisfied for  $j = r$ . The convergence proof of the expansion (3.6) is an adaptation of the majorant method.

**DEFINITION.** The matrix  $K$  defined by (3.7) is called the Kovalevskaya's matrix of the system (3.5).

Let  $\rho_1, \rho_2, \dots, \rho_s$  be the positive integer eigenvalues of  $K$  and  $\vartheta_1, \vartheta_2, \dots, \vartheta_s$  be the corresponding eigenvectors. The linear systems (3.7) imply that if (3.6) is a solution of (3.5) then

$$z(t) = \sum_{j=1}^{\infty} \sum_{k=0}^{2j-1} \tilde{x}^{j,k} \cdot t^j \cdot (\log(t))^k, \quad (i = 1, 2, \dots, n), \tag{3.8}$$

is again a solution of (3.5) where

$$\tilde{x}^{j,k} = \begin{cases} x^{j,k}, & \text{if } (j, k) \neq (\rho_\nu, 0) \text{ for } \nu \in \{1, 2, \dots, s\} \\ x^{j,k} + \alpha_\nu \vartheta_\nu, & \text{if } (j, k) = (\rho_\nu, 0) \text{ for some } \nu \in \{1, 2, \dots, s\} \end{cases}$$

and  $\alpha_1, \alpha_2, \dots, \alpha_s$  are free parameters. Moreover, as the matrix  $K$  is diagonalizable, we obtain in this way all possible solutions (3.6) of the system (3.5). To complete the proof of Horn's theorem we note that the system (2.1) is equivalent to the system (3.5). Indeed after a change of the variables  $(x_1, x_2, \dots, x_n) \rightarrow (z_1, z_2, \dots, z_n)$  defined by

$$x_i(t) = t^{-\vartheta_i} \cdot (x_i^0 + z_i(t)), \quad (i = 1, 2, \dots, n)$$

(recall that (3.3) is a particular solution of the reduction (2.1<sup>0</sup>) the system (2.1) turns into the system (3.5) where

$$G_i(z, t) = G_{i0}(t) + G_{i1}(z, t)G_{i2}(z, t),$$

$$G_{i0}(t) = g_i x_i^0 + t^{\vartheta_i+1} \cdot F_i(x_1^0 t^{-\vartheta_1}, x_2^0 t^{-\vartheta_2}, \dots, x_n^0 t^{-\vartheta_n}),$$

$$G_{i1}(z, t) = \sum_{j=1}^n K_{ij}(t) \cdot z_j,$$

$$K_{ij}(t) = t^{\vartheta_i+1} \cdot \frac{\partial F_i}{\partial x_j}(x_1^0 t^{-\vartheta_1}, x_2^0 t^{-\vartheta_2}, \dots, x_n^0 t^{-\vartheta_n}),$$

$$G_{i2}(z, t) = t^{\vartheta_i+1} \cdot \sum_{|m| \geq 2} \frac{\partial^{|m|}}{\partial^m x} F_i(x_1^0 t^{-\vartheta_1}, x_2^0 t^{-\vartheta_2}, \dots, x_n^0 t^{-\vartheta_n}) \cdot t^{-\langle m, \vartheta \rangle} \cdot z^m / m!.$$

Here the following notations are used

$$m = (m_1, m_2, \dots, m_n), \quad g = (g_1, g_2, \dots, g_n), \quad |m| = \sum_{i=1}^n m_i, \quad m! = \prod_{i=1}^n (m_i!),$$

$$\langle m, g \rangle = \sum_{i=1}^n m_i g_i, \quad z^m = \prod_{i=1}^n z_i^{m_i}, \quad \frac{\partial^{|m|}}{\partial^m x} = \frac{\partial^{|m|}}{\partial^{m_1} x_1 \partial^{m_2} x_2 \dots \partial^{m_n} x_n}.$$

As the system (2.1<sup>0</sup>) is reduction of the system (2.1) with respect to the similarity transformation (2.2), and (3.3) is a particular solution of the system (2.1<sup>0</sup>) then  $G_{i0}(0) = 0$ , and  $G_{i0}(t)$ ,  $K_{ij}(t)$  are analytic functions in a neighbourhood of the origin in  $\mathbb{C}$ . It follows that the Kovalevskaya's matrix of the system (3.5) coincides with the Kovalevskaya's matrix of the system (2.1). In a similar way one proves that  $G_{i2}(z, t)$ ,  $i = 1, 2, \dots, n$ , are analytic functions in a suitable neighbourhood of the origin in  $\mathbb{C}^{n+1}$ , and  $G_{i2}(0, 0) = 0, i = 1, 2, \dots, n$ . This completes the proof of Horn's theorem.  $\square$

*Proof of Lemma 3.1.* Let

$$\omega_i = \omega_i^0/t, \quad e_i = e_i^0/t^2, \quad (i = 1, 2, 3) \tag{3.9}$$

be a particular solution of the Euler-Poisson equations (1.1<sup>0</sup>). The Kovalevskaya's matrix of the system (3.1<sup>v</sup>) takes the form

$$K = \begin{pmatrix} 1 & \frac{B-C}{A} \omega_3^0 & \frac{B-C}{A} \omega_2^0 & 0 & \frac{\varepsilon z_0}{A} & -\frac{\varepsilon y_0}{A} \\ \frac{C-A}{B} \omega_3^0 & 1 & \frac{C-A}{B} \omega_1^0 & -\frac{\varepsilon z_0}{B} & 0 & \frac{\varepsilon x_0}{B} \\ \frac{A-B}{C} \omega_2^0 & \frac{A-B}{C} \omega_1^0 & 1 & \frac{\varepsilon y_0}{C} & -\frac{\varepsilon x_0}{C} & 0 \\ 0 & -e_3^0 & e_2^0 & 2 & \omega_3^0 & -\omega_2^0 \\ e_3^0 & 0 & -e_1^0 & -\omega_3^0 & 2 & \omega_1^0 \\ -e_2^0 & e_1^0 & 0 & \omega_2^0 & -\omega_1^0 & 2 \end{pmatrix}$$

Suppose that  $A = B, x_0 = y_0 = 0$ . Without loss of generality we may also suppose that  $A = B = 1, z_0 = 1$ . The system (1.1<sup>0</sup>) possesses a particular solution (3.9) where

$$\begin{aligned} \omega_1^0 &= -2\sqrt{-1} \alpha_0, & \omega_2^0 &= 2\sqrt{\alpha_0^2 - 1}, & \omega_3^0 &= 0, & e_1^0 &= \omega_2^0, \\ e_2^0 &= -\omega_1^0, & e_3^0 &= 2 \end{aligned} \tag{3.10}$$

( $\alpha_0$  is a free parameter) and the eigenvalues of the Kovalevskaya's matrix  $K$  are  $-1, 0, 1, 2, 3$ , and  $4$ . In the case  $A = B = 2C, y_0 = z_0 = 0$ , we may suppose without loss of generality that  $A = B = 2C = 1$  and  $\varepsilon x_0 = 1$ . Now a particular solution of the system (1.1<sup>0</sup>) is (3.9) where

$$\omega_1^0 = \alpha_0, \quad \omega_2^0 = \sqrt{-1} \alpha_0, \quad \omega_3^0 = 2\sqrt{-1}, \quad e_1^0 = 1, \quad e_2^0 = \sqrt{-1}, \quad e_3^0 = 0 \tag{3.11}$$

and the eigenvalues of the Kovalevskaya's matrix  $K$  are again  $-1, 0, 1, 2, 3$ , and  $4$ . Denote the eigenvectors of  $K$  corresponding to the Kovalevskaya's exponents  $1, 2, 3, 4$  by  $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$  respectively. Horn's theorem implies that both in Lagrange and Kovalevskaya cases the system (3.1<sup>v</sup>) possesses a five parameter family of solutions (3.2), as the particular solutions (3.10) and (3.11) depend upon the free parameter  $\alpha_0$ . Moreover these solutions possess logarithmic branch points provided that  $\lambda_1^2 + \lambda_2^2 \neq 0$ . Indeed, if the coefficient  $x^1 = (\omega_1^1, \omega_2^1, \omega_3^1, e_1^1, e_2^1, e_3^1)$  does not depend upon  $\log(t)$  then the linear system (3.7),  $j = 1$ , for determining the coefficient  $x^1 = z^{1,0}$  takes the form

$$(K - I)x^1 = v \cdot \left( \frac{\lambda_3 \omega_2^0 - \lambda_2 \omega_3^0}{A}, \frac{\lambda_1 \omega_3^0 - \lambda_3 \omega_1^0}{B}, \frac{\lambda_2 \omega_1^0 - \lambda_1 \omega_2^0}{C}, 0, 0, 0 \right)^t \tag{3.12}$$

Trivial (but tedious) computations show that if  $\lambda_1^2 + \lambda_2^2 \neq 0$  then for all generic values of  $\alpha_0$  the system (3.12) is not compatible. We conclude that

$$x^1 = \bar{x}^1 + (\alpha_1 + \beta \cdot v \cdot \log(t))\vartheta_1 \tag{3.13}$$

where  $\bar{x}^1$  does not depend upon  $t$  and  $\alpha_1$ , and  $\beta = \beta(\alpha_0)$  and the entries of  $\bar{x}^1$  are algebraic functions in  $\alpha_0$ .

Denote the above five parameter family of solutions by  $x(t, \log(t), \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, v)$ . According to Horn's theorem for all fixed generic values of  $\alpha_0$  the coefficients  $x^j, j = 1, 2, \dots, \infty$  (see (3.4)) are vectors whose entries are polynomials in  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ . As the Kovalevskaya's matrix  $K$  depends upon  $x^0$ , and it does not depend upon  $v$ , then the coefficients  $x_i^j, i = 1, 2, \dots, 6, j = 1, 2, \dots, \infty$ , are also polynomials in  $v$ , rational functions in  $x_i^0, i = 1, 2, \dots, 6$ , and hence they are algebraic functions in  $\alpha_0$ . Suppose that the entries of the vectors  $x^j, j = 1, 2, \dots, \infty$ , can not be presented as polynomials in  $v, (\alpha_i + \beta \cdot v \cdot \log(t)), \alpha_2, \alpha_3, \alpha_4$ , and consider the following new solution  $\tilde{x}$  of the system (3.1<sup>v</sup>)

$$\tilde{x} = x(t, (\beta \cdot v \cdot \log(t) + \alpha_1)/(\beta \cdot v), \alpha_0, 0, \alpha_2, \alpha_3, \alpha_4, v)$$

where the algebraic function  $\beta = \beta(\alpha_0)$  is defined by (3.13). We shall prove that

each coefficient  $\tilde{x}_i^j$  of the solution  $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_6)$

$$\tilde{x}_i = t^{-\theta_i} \cdot \left( \sum_{j=0}^{\infty} \tilde{x}_i^j \cdot t^j \right), \quad (j = 1, 2, \dots, 6)$$

is a polynomial in  $v$ . Indeed  $x^0 \equiv \tilde{x}^0$  and according to (3.13)  $x^1 \equiv \tilde{x}^1$ . Consider the coefficients  $x^2$  and  $\tilde{x}^2$ . We have

$$\begin{aligned} x^2 &= (\log(t))^2 \cdot a(\alpha_0, \alpha_1, v) + \log(t) \cdot b(\alpha_0, \alpha_1, v) + f(\alpha_0, \alpha_1, v) + \alpha_2 \cdot \vartheta_2, \\ \tilde{x}^2 &= (\log(t))^2 \cdot \tilde{a}(\alpha_0, \alpha_1, v) + \log(t) \cdot \tilde{b}(\alpha_0, \alpha_1, v) + \tilde{f}(\alpha_0, \alpha_1, v) + \alpha_2 \cdot \vartheta_2 \\ &= (\log(t) + \alpha_1/(\beta \cdot v))^2 \cdot a(\alpha_0, 0, v) + (\log(t) + \alpha_1/(\beta \cdot v)) \cdot b(\alpha_0, 0, v) \\ &\quad + f(\alpha_0, 0, v) + \alpha_2 \cdot \vartheta_2, \end{aligned}$$

where  $a, b, f, \tilde{a}, \tilde{b}, \tilde{f}$  are suitable vectors (in our case  $x^{2,3} = \tilde{x}^{2,3} = 0$ ). The linear systems (3.7),  $j = 2$ , imply that  $x^2 - \tilde{x}^2 = k \cdot \vartheta_2$  where  $k = k(\alpha_0, \alpha_1, v)$ . It follows that  $a \equiv \tilde{a}$ ,  $b \equiv \tilde{b}$ , and as  $a$  and  $b$  are polynomials in  $v$  then  $\tilde{a}$  and  $\tilde{b}$  are also polynomials in  $v$ . We have

$$\tilde{f}(\alpha_0, \alpha_1, v) = a(\alpha_0, 0, v) \cdot \frac{a_1^2}{\beta^2 \cdot v^2} + b(\alpha_0, 0, v) \cdot \frac{\alpha_1}{\beta \cdot v} + f(\alpha_0, 0, v).$$

As the generic solutions of Lagrange and Kovalevskaya top can be explicitly expressed in terms of theta functions [7] then each generic solution of these tops can have no worse than pole singularities. Thus the expansion (3.2) of the solution  $x(t, \log(t), \alpha, 0)$  is a Laurent power series. It follows that  $a(\alpha_0, \alpha_1, 0) = b(\alpha_0, \alpha_1, 0) = 0$  and hence  $b(\alpha_0, \alpha_1, v)/v$  is a polynomial in  $v$ . If  $a(\alpha_0, 0, v)/v^2$  is not a polynomial in  $v$  then the polynomial  $\tilde{b}(\alpha_0, \alpha_1, v) = 2a(\alpha_0, 0, v) \cdot \alpha_1/\beta \cdot v + b(\alpha_0, 0, v)$  will not vanish at  $v = 0$ . We conclude that  $a(\alpha_0, 0, v)/v^2$ , and hence  $\tilde{f}(\alpha_0, \alpha_1, v)$  is a polynomial in  $v$ .

Let us change the parameter  $\alpha_2$  in  $x(t, \log(t), \alpha, v)$  as

$$\alpha_2 \rightarrow \alpha_2 + k(\alpha_0, \alpha_1, v)$$

where the function  $k(\alpha_0, \alpha_1, v)$  is defined by the identity  $x^2 - \tilde{x}^2 = f - \tilde{f} = k \cdot \vartheta_2$ . Thus we obtain another solution of the system (3.1 $^v$ ), which we will also denote by  $x(t, \log(t), \alpha, v)$ . It has the property that  $x_i^j = \tilde{x}_i^j$  for  $j = 0, 1, 2, i = 1, 2, \dots, 6$ , and its coefficients  $x_i^j, i = 1, 2, \dots, 6, j = 0, 1, \dots, \infty$  are polynomials in  $v, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ , and depend algebraically upon  $\alpha_0$ .

In a quite similar way one proves that the coefficients  $\tilde{x}_i^j, j = 3, 4$ ,

$i = 1, 2, \dots, 6$ , are polynomials in  $v$ , and (after a suitable change of the parameters  $\alpha_3, \alpha_4$ )  $x_i^j = \tilde{x}_i^j$  for  $j = 3, 4, i = 1, 2, \dots, 6$ . At last we note that if  $x_i^j = \tilde{x}_i^j, i = 1, 2, \dots, 6, j = 0, 1, \dots, r, r \geq 4$ , then Horn's theorem implies that  $x^{r+1} = \tilde{x}^{r+1}$ . Thus we proved that the coefficients  $x_i^j$  of the expansion (3.4) of the solution  $x(t, \log(t), \alpha, v)$  are polynomials with respect to  $v, v \cdot \beta \cdot \log(t) + \alpha_1, \alpha_2, \alpha_3, \alpha_4$ , whose coefficients depend algebraically upon  $\alpha_0$ , and  $\beta = \beta(\alpha_0) \neq 0$  is an algebraic function.

To complete the proof of Lemma 3.1 we have to show that the five parameter family of solutions (3.2) form a set of non-empty interior in the complex phase space  $\mathbb{C}^6$ . Indeed, if  $\alpha$  and  $v$  are fixed and  $|t|$  is sufficiently small, then

$$\begin{aligned} \frac{d}{dt} x &= (-g_1 \cdot x_1^0 \cdot t^{-g_1-1} \cdot (1 + O(|t|)), \dots, -g_6 \cdot x_6^0 \cdot t^{-g_6-1} \cdot (1 + O(|t|)))^t, \\ \frac{\partial}{\partial \alpha_0} x &= \left( t^{-g_1} \frac{\partial}{\partial \alpha_0} x_1^0 \cdot (1 + O(|t|)), \dots, t^{-g_6} \frac{\partial}{\partial \alpha_0} x_6^0 \cdot (1 + O(|t|)) \right)^t, \\ \frac{\partial}{\partial \alpha_i} x &= (t^{-g_1} \vartheta_{i1} \cdot (1 + O(|t|)), \dots, t^{-g_6} \cdot \vartheta_{i6} \cdot (1 + O(|t|)))^t, \end{aligned}$$

where  $g_1 = g_2 = g_3 = 1, g_4 = g_5 = g_6 = 2, x_i^0 = \omega_i^0, x_{i+3}^0 = e_i^0, i = 1, 2, 3; \vartheta_i = (\vartheta_{i1}, \vartheta_{i2}, \dots, \vartheta_{i6})^t, i = 1, 2, 3, 4$ , are the eigenvectors of the Kovalevskaya's matrix  $K$  corresponding to the Kovalevskaya's exponents 1, 2, 3, 4, and

$$\begin{aligned} \vartheta_0 &= \left( \frac{\partial}{\partial \alpha_0} x_1^0, \frac{\partial}{\partial \alpha_0} x_2^0, \dots, \frac{\partial}{\partial \alpha_0} x_6^0 \right), \\ \vartheta_{-1} &= (-g_1 \cdot x_1^0, -g_2 \cdot x_2^0, \dots, -g_6 \cdot x_6^0) \end{aligned}$$

are the eigenvectors of  $K$  corresponding to the Kovalevskaya's exponents 0 and  $-1$ . At last

$$\det \left( \frac{dx}{dt}, \frac{\partial}{\partial \alpha_0} x, \frac{\partial}{\partial \alpha_1} x, \dots, \frac{\partial}{\partial \alpha_4} x \right) = \det(\vartheta_{-1}, \vartheta_0, \dots, \vartheta_1) + O(|t|) \neq 0$$

for all  $t \in \mathbb{C}$  such that  $|t|$  is sufficiently small. □

*Proof of Lemma 3.2.* Consider the function

$$h(\alpha, v) = H^v(x(t, \log(t), \alpha, v))$$

where  $H^v$  is the rational first integral of the system (3.1<sup>v</sup>) corresponding to the rational first integral  $H$  of the system (3.1<sup>0</sup>). When the time  $t$  makes  $m$  turns

around the origin in  $\mathbb{C}$  along an appropriate closed path the solution (3.4) changes as

$$x(t, \log(t), \alpha, v) \rightarrow x(t, \log(t) + 2\pi m \cdot \sqrt{-1}, \alpha, v).$$

By making use of Lemma 3.1 we obtain

$$\begin{aligned} h(\alpha, v) &= H^v(x(t, \log(t), \alpha, v)) = H^v(x(t, \log(t) + 2\pi m \cdot \sqrt{-1}, \alpha, v)) \\ &= H^v(x(t, \log(t), \alpha_0, \alpha_1 + 2\pi m \cdot \beta v \cdot \sqrt{-1}, \alpha_2, \alpha_3, \alpha_4, v)) \\ &= h(\alpha_0, \alpha_1 + 2\pi m \cdot \beta v \cdot \sqrt{-1}, \alpha_2, \alpha_3, \alpha_4, v). \end{aligned}$$

However, as the coefficients  $x_i^j, i = 1, 2, \dots, 6, j = 0, 1, \dots, \infty$ , of the solution  $x(t, \log(t), \alpha, v)$  are algebraic functions in  $\alpha_0, \alpha_1, \dots, \alpha_4, v$ , and  $H^v$  is a rational function then  $h(\alpha, v)$  is an algebraic function. If  $\beta \neq 0, v \neq 0$ , then the above identities imply that  $h(\alpha, v)$  is a periodic function in  $\alpha_1$ , and hence it does not depend upon  $\alpha_1$ . □

*Proof of Lemma 3.3.* Let  $H_1^0, H_2^0, H_3^0, H_4^0$ , be rational functionally independent first integrals of the system (1.1<sup>0</sup>). It is well known that Lagrange and Kovalevskaya tops are non-degenerated Hamiltonian systems, i.e. the trajectory's closure of each generic (complex) solution is a smooth two-dimensional (complex) manifold given by

$$\mathbb{A} = \{H_1^0 = h_1, H_2^0 = h_2, H_3^0 = h_3, H_4^0 = h_4\}.$$

where  $h_1, h_2, h_3, h_4$  are appropriate constants. Each generic invariant complex manifold  $\mathbb{A}$  of Kovalevskaya top can be extended to a complex algebraic torus (Abelian variety) after adjoining some divisor, and the phase variables  $x_i = x_i(t)$  are meromorphic functions on  $\mathbb{A}$  [2, 14]. This divisor coincides with the one-dimensional complex algebraic set  $\Gamma$  defined in Lemma 3.3. Thus each irreducible component of  $\Gamma$  is an algebraic curve lying on an Abelian variety. However, on an Abelian variety a genus zero curve cannot live [9]. This implies that the generic algebraic set  $\Gamma$  does not contain lines.

Consider now the Lagrange top. Unfortunately the generic invariant manifold  $\mathbb{A}$  cannot be completed to a complex algebraic torus and hence the above arguments do not apply. In fact one may easily prove that the algebraic set  $\Gamma$  in this case is actually a union of genus zero curves. Without loss of generality we may suppose that  $H_4^0 = \omega_3, A = B = 1, z_0 = 1, (x_0 = y_0 = 0)$ . The eigenvector  $\mathfrak{g}_1$  of Kovalevskaya's matrix  $K$ , corresponding to Kovalevskaya's exponent 1 reads

$$\mathfrak{g}_1 = ((2 - C) \cdot \sqrt{1 - \alpha_0^2}, \alpha_0(2 - C), 1, 2\alpha_0(1 - C), 2(C - 1) \cdot \sqrt{1 - \alpha_0^2}, 0)^t.$$

As  $x^1 = \alpha_1 \cdot \vartheta_1$ ,  $\omega_3 = x_3^1 = \alpha_1$ , then the algebraic set  $\Gamma$  does not contain any of the lines  $\{\alpha \in \mathbb{C}^5: \alpha_i = \text{const}, i = 0, 2, 3, 4\}$  which completes the proof of Lemma 3.3.  $\square$

Theorem 1.1 is announced in [6]. The author is obliged to Emil Horozov for the stimulating discussions.

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