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Compositio Mathematica, tome 82, n° 2 (1992), p. 119-136
<http://www.numdam.org/item?id=CM_1992__82_2_119_0>

Compositio Mathematica **82**: 119–136, 1992.

Perfect powers in products of terms in an arithmetical progression (II)

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Received 11 December 1989; accepted 22 August 1991

Section 1

For an integer x with $|x| > 1$, we write $p(x)$, $P(x)$, $Q(x)$ and $\omega(x)$, respectively, for the least prime factor of x, the greatest prime factor of x, the greatest square free factor of x and the number of distinct prime factors of x. Further, we put $p(\pm 1) = P(\pm 1) = Q(\pm 1) = 1$ and $\omega(\pm 1) = 0$. We consider the equation

$$m(m + d) \cdots (m + (k - 1)d) = by^l \tag{1}$$

in positive integers b, d, k, l, m, y subject to $P(b) \leqslant k$, $\gcd(m, d) = 1$, $k > 2$, $l > 1$ and $P(y) > k$. There is no loss of generality in assuming that l is a prime number. Let d_1 be the maximal divisor of d such that all the prime factors of d_1 are $\equiv 1 (\text{mod } l)$. Similarly, we define m_1 as the maximal divisor of m such that all the prime divisors of m_1 are $\equiv 1 (\text{mod } l)$. For $\varepsilon > 0$, we put $\eta(\varepsilon) = (1 - \varepsilon)/\log 2$,

$$\eta'(\varepsilon) = \eta'(l, \varepsilon) = \begin{cases} (1-\varepsilon)/\log 2 & \text{if } l > 30 \\ (1-\varepsilon)/\log l & \text{if } l \leqslant 30 \end{cases}$$

and

$$\tau(\varepsilon) = \tau(l, \varepsilon) = \begin{cases} 2 & \text{if } l > 30 \\ l+\varepsilon & \text{if } l \leqslant 30. \end{cases}$$

Notice that

$$\eta'(l, \varepsilon) \geqslant (1 - \varepsilon)/\log 30 \text{ and } \tau(l, \varepsilon) \leqslant 30 + \varepsilon$$

for every l. We shall follow the above notation without reference.

Erdös and Selfridge [4] confirmed an old conjecture by proving that the product of two or more consecutive positive integers is never a power. Marszalek [6] showed that (1) with $b = 1$ implies that k is bounded by an effectively computable number C depending only on d. Shorey [7] showed that we can take C depending only on $P(d)$ whenever $l \geqslant 3$. According to Shorey and Tijdeman [9], Corollary 3, the number C can be taken to depend only on $l, \omega(d_1)$ if $l \geqslant 7$ and only on $\omega(d)$ if $l \leqslant 5$. Further, Shorey and Tijdeman [9], Corollary 4(a), proved that (1) implies that

$$\log d_1 \geqslant C_1 \frac{\log k \operatorname{loglog} k}{\operatorname{logloglog} k}, \quad \log d_1 \geqslant C_1' \log k \operatorname{loglog} k \tag{2}$$

where $C_1 > 0$ is an effectively computable absolute constant and $C_1' > 0$ is an effectively computable number depending only on l. In this paper, we sharpen (2) as follows:

THEOREM 1. *Let* $\varepsilon > 0$. *There exists an effectively computable number* C_2 *depending only on* ε *such that* (1) *with* $k \geqslant C_2$ *and* $l \geqslant 7$ *implies that*

$$\log d_1 \geqslant \eta'(\varepsilon) \log k \operatorname{loglog} k. \tag{3}$$

Next, we extend the result of Shorey mentioned above by proving that (1) implies that k is bounded by an effectively computable number depending only on $p(d_1)/l$ and $\omega(d_1)$. More precisely, we prove

THEOREM 2. *Let* $\varepsilon > 0$. *There exist effectively computable numbers* C_3 *and* $C_4 > 0$ *depending only on* ε *such that* (1) *with* $k \geqslant C_3$ *implies that*

$$p(d_1) \geqslant C_4 kl(\tau(\varepsilon))^{-\omega(d_1)} \quad \text{if } l \geqslant 7 \tag{4}$$

and

$$p(d_1) \geqslant C_4 kl(\tau(\varepsilon))^{-\omega(d)} \quad \text{if } l \in \{2, 3, 5\}. \tag{5}$$

We apply Theorem 2 to obtain the following quantitative version of the result of Shorey.

COROLLARY 1. *Let* $\varepsilon > 0$. *There exists an effectively computable number* C_5 *depending only on* ε *such that* (1) *with* $k \geqslant C_5$ *implies that*

$$2P(d_1) \geqslant \eta'(\varepsilon) l \log k \operatorname{loglog} k \quad \textit{if } l \geqslant 7, \tag{6}$$

$$P(d) \geqslant \eta'(\varepsilon) \log k \operatorname{loglog} k \quad \textit{if } l \in \{2, 3, 5\} \tag{7}$$

and

$$Q(d_1) \geqslant \max(lk^{1-\varepsilon}, k^{2-\varepsilon}) \quad \textit{if } l \geqslant 7, \tag{8}$$

$$Q(d) \geqslant \max(lk^{1-\varepsilon}, k^{2-\varepsilon}) \quad \textit{if } l \in \{2, 3, 5\}. \tag{9}$$

If $m, m+d, \ldots, m+(k-1)d$ are lth perfect powers, we may, by a result of Dénes [2], assume that $l > 30$ and therefore, (6) includes (6) of [10]. Further, under this restrictive assumption, it is shown in [10] that (8) can be replaced by

$$\log Q(d_1) \geqslant \eta(\varepsilon)(\log k)^2 \quad \text{if } k \geqslant C_5.$$

Further, we observe from Theorem 2 that for $k \geqslant C_3$ and $l \geqslant 7$, either

$$(\tau(\varepsilon))^{\omega(d_1)} \geqslant k^{1-\varepsilon} \tag{10}$$

or

$$p(d_1) \geqslant c_4 lk^{\varepsilon}. \tag{11}$$

As observed in [10], the assertion that (1) implies (11) is false. Also, notice that (1) with $l \leqslant 4\omega(d_1)+2$ implies (10). See Lemma 6. In the opposite case we prove

THEOREM 3. *Suppose that* (1) *with* $l > 4\omega(d_1)+2$ *is satisfied. Then*

(a) *There exists an effectively computable absolute constant* $C_6 > 0$ *such that*

$$p(d_1) \geqslant C_6 k. \tag{12}$$

(b) *There exists an effectively computable absolute constant* $C_7 > 0$ *such that*

$$p(d_1) \geqslant C_7 l \log k \tag{13}$$

or

$$p(m_1) \geqslant lk/(33C_7 \log k). \tag{14}$$

In particular, we derive from Theorem 3 that (1) with $l > 4\omega(d_1)+2$ implies that k is bounded by an effectively computable number depending only on $p(d_1)/l$ and $p(m_1)/l$. Next, we give lower bounds for $P(m)$. Shorey [8] showed that there exist effectively computable absolute constants C_8 and C_9 such that (1) with $k \geqslant C_8$ implies that

$$m \geqslant d^{1-C_9\Delta_l} \quad \text{where } \Delta_l = l^{-1}(\log l)^2(\log\log(l+1)). \tag{15}$$

In particular, we derive from (15) that (1) implies that

$$m \geqslant d^{1/2} \quad \text{if } \min(k, l) \geqslant C_{10} \tag{16}$$

where C_{10} is an effectively computable absolute constant. We combine this result with (2) and Corollary 3 of [9] to derive that (1) implies that k is bounded by an effectively computable number C_{11} depending only on m and $\omega(d)$. We extend this result by showing that we can take C_{11} depending only on $P(m)$ and $\omega(d)$. This is a consequence of the following result (Theorem 4(a)) and Corollary 3 of [9]. Compare Theorem 4(b) with Theorem 3 of [10].

THEOREM 4(a). *Let* $\varepsilon > 0$. *There exists an effectively computable number* C_{12} *depending only on* ε *such that* (1) *with* $k \geqslant C_{12}$ *and* $l > 4\omega(d_1)+2$ *implies that*

$$\min_{0 \leqslant \mu < k} P(m + \mu d) \geqslant k^{1/3-\varepsilon}. \tag{17}$$

(b) *There exists an effectively computable absolute constant* C_{13} *such that* (1) *with* $k \geqslant C_{13}$, $l > 4\omega(d_1)+2$ *and* $b=1$ *implies that*

$$P(m - d) > k. \tag{18}$$

Section 2

For $0 \leqslant i < k$, we see from (1) that

$$m + id = A_i X_i^l \tag{19}$$

where

$$P(A_i) \leqslant k \quad \text{and} \quad \gcd\left(X_i, \prod_{p \leqslant k} p\right) = 1.$$

Notice that

$$\gcd(X_i, X_j) = 1 \quad \text{for } i \neq j.$$

We put

$$S_1 = \{A_0, A_1, \ldots, A_{k-1}\}.$$

For $\alpha > 0$, we denote by $S_2(\alpha)$ the set of all $A_\mu \in S_1$ satisfying $A_\mu \leqslant \alpha k$. Further, we write

$$S_2 = S_2(1) \quad \text{and} \quad s_2 = |S_2|.$$

Let T be the set of all μ with $0 \leqslant \mu < k$ and $A_\mu \in S_2$. We shall always write

$$p = p(d_1), \qquad q = p(m_1). \tag{20}$$

We assume that k exceeds some effectively computable large absolute constant. Then, by (2), we have $p \geqslant 2$. Finally, as stated in Section 1, we assume in (1) that $P(y) > k$. Therefore, by (1),

$$m + (k-1)d \geqslant k^l$$

which implies that

$$m + d \geqslant k^{l-1}. \tag{21}$$

In addition to the notation of Section 1, we shall follow the above notation without reference.

In this section, we prove lemmas for the proofs of our theorems. We start with an estimate on the number of elements of a subset of S_1.

LEMMA 1. *Let S_3 be a subset of S_1 and z_3 be the maximum of the elements of S_3. Then*

$$|S_3| \leqslant (z_3 + p - 1)/l \tag{22}$$

where p is given by (20).

Proof. First, since $p \equiv 1 (\bmod l)$, we observe that $X_0^l, X_1^l, \ldots, X_{k-1}^l$ are contained in at most $(p-1)/l$ distinct residue classes mod p. Then, by reading (19) mod p, we derive that the elements of S_3 are contained in at most $(p-1)/l$ distinct residue classes mod p. Consequently,

$$|S_3| \leqslant ([z_3/p] + 1)\left(\frac{p-1}{l}\right)$$

which implies (22). □

For μ_0 with $0 \leqslant \mu_0 < k$, we denote by $\nu(A_{\mu_0})$ the number of distinct μ with $0 \leqslant \mu < k$ such that $A_\mu = A_{\mu_0}$. We observe that there exists $A_{\mu_0} \in S_3$ such that

$$\nu(A_{\mu_0}) \geqslant |T_3|/|S_3|$$

where T_3 denotes the set of all μ with $0 \leqslant \mu < k$ and $A_\mu \in S_3$. Therefore, Lemma 1 gives a lower bound for $v(A_{\mu_0})$. On the other hand, we prove

LEMMA 2. *Suppose that* (1) *with* $l \geqslant 3$ *is satisfied. Then*

$$v(A_{\mu_0}) \leqslant 2^{\omega(d_1)}\left(\left[\frac{k}{A_{\mu_0}l}\right]+1\right) \tag{23}$$

for every μ_0 *with* $0 \leqslant \mu_0 < k$.

Proof. For every μ_1 with $0 \leqslant \mu_1 < k$ and $A_{\mu_1} = A_{\mu_0}$, it suffices to show that the number of μ with $0 \leqslant \mu < k$, $A_\mu = A_{\mu_1}$ and $0 \leqslant \mu - \mu_1 < A_{\mu_1} l$ is at most $2^{\omega(d_1)}$. Let μ satisfy $0 < \mu - \mu_1 < A_{\mu_1} l$ and $A_\mu = A_{\mu_1}$. Then, we derive from (19) and $\gcd(m, d) = 1$ that

$$\left(\frac{\mu-\mu_1}{A_{\mu_1}}\right) d = (X_\mu - X_{\mu_1})\left(\frac{X_\mu^l - X_{\mu_1}^l}{X_\mu - X_{\mu_1}}\right)$$

where $(\mu - \mu_1)/A_{\mu_1}$ is a positive integer $< l$. For $\mu \neq v$, we put

$$X_{\mu,v} = \frac{X_\mu^l - X_v^l}{X_\mu - X_v}, \qquad X'_{\mu,v} = X_{\mu,v} l^{-\delta}$$

where $\delta = 0$ if $l \nmid d$ and $\delta = 1$ if $l \mid d$. We observe that every prime factor of $X'_{\mu,v}$ is $\equiv 1 (\mathrm{mod}\ l)$. Consequently, $p(X_{\mu,\mu_1}) \geqslant l$ and

$$\gcd\left(\frac{\mu-\mu_1}{A_{\mu_1}}, X_{\mu,\mu_1}\right) = 1$$

which implies that $X_{\mu,\mu_1} \mid d$. Since $X'_{\mu,\mu_1} > 1$ is monotonic increasing in μ and has $< 2^{\omega(d_1)}$ divisors, we conclude that the number of μ with $0 < \mu - \mu_1 < A_{\mu_1} l$ and $A_\mu = A_{\mu_1}$ is less than $2^{\omega(d_1)}$. □

As an immediate consequence of Lemma 2, we have

COROLLARY 2. *Let* $\alpha_1 > 0$. *Suppose that* (1) *with*

$$l \geqslant \alpha_1 k \tag{24}$$

is satisfied. Then, for every $A_{\mu_0} \in S_1$, *we have*

$$v(A_{\mu_0}) \leqslant c 2^{\omega(d_1)} \tag{25}$$

for an effectively computable number c depending only on α_1.

For applying Corollary 2, we obtain (24) under certain assumptions in the next three lemmas.

LEMMA 3. *Let $\varepsilon > 0$. There exists an effectively computable number c_1 depending only on ε such that* (1) *with $k \geqslant c_1$ and $l > 4\omega(d_1)+2$ implies that*

$$l \geqslant (1-\varepsilon)k\,\frac{\text{loglog } k}{\log k}. \tag{26}$$

This is (2.11) of [9]. The estimate (26) is slightly weaker than (24). In the next two lemmas, we sharpen (26) to (24) under additional assumptions.

LEMMA 4. *Let $\varepsilon > 0$ and $\varepsilon_1 > 0$. Suppose that* (1) *with $l > 4\omega(d_1)+2$ is satisfied. There exists an effectively computable number c_2 depending only on ε and ε_1 such that*

$$l \geqslant c_2 k \tag{27}$$

or

$$\varepsilon_1^{-1}\pi(k) < |T| < \varepsilon k. \tag{28}$$

Proof. We may assume that $0 < \varepsilon < 1/2$, $0 < \varepsilon_1 < 1/2$ and that k exceeds a sufficiently large number depending only on ε and ε_1. We write $\theta = 4e^{4\varepsilon_1^{-1}}$. Further, we observe from $l > 4\omega(d_1)+2$ that $l \geqslant 7$ and there exists a divisor d' of d_1 satisfying

$$\omega(d') = 1 \tag{29}$$

and

$$d' \geqslant d_1^{1/\omega(d_1)} \geqslant d_1^{4/(l-3)}$$

which, together with (2.1) of [9], implies that

$$d' \geqslant C_5 d^{4(1-1/(l-3))/l} \tag{30}$$

where C_5 is the absolute constant occurring in (2.1) of [9].

First, we assume that

$$|S_2(\theta)| \geqslant \frac{\varepsilon k}{2}.$$

For distinct $A_\mu \in S_2(\theta)$ and $A_\nu \in S_2(\theta)$ with $\mu > \nu$, we observe from (19) that

$$A_\mu X_\mu^l - A_\nu X_\nu^l = (\mu - \nu)d.$$

Now, we apply the Sieve-theoretic Lemma 1 of Erdös [3] to derive that there exist positive integers P, Q and R such that

$$\max(P, Q, R) \leqslant c_3, \ \gcd(P, Q) = 1$$

and

$$PX_\mu^l - QX_\nu^l = Rd =: Nd'$$

is satisfied by at least $c_4 k$ pairs X_μ, X_ν where c_3, c_4 and the subsequent letter c_5 are effectively computable numbers depending only on ε and ε_1. Further, by (30), (2.7) of [9] and $l \geqslant 7$, we observe that

$$N \leqslant (d')^{2l/5-1}.$$

Now, we apply Corollary 1(b) of Evertse [5] and (29) to conclude that

$$c_4 k \leqslant c_5 l^{\omega(d')} = c_5 l.$$

Thus, we may assume that

$$|S_2(\theta)| < \frac{\varepsilon k}{2}. \tag{31}$$

First, we consider the case that $|T| \geqslant \varepsilon k$. Then, we derive from (31) that

$$|S_1| = |S_2| + |S_1 - S_2| < \frac{\varepsilon k}{2} + (1-\varepsilon)k = \left(1 - \frac{\varepsilon}{2}\right)k.$$

Now, we apply Lemma 8 of [9] with $f(k) = (\varepsilon \log k)/4$ and a divisor d' of d satisfying (29) and (30). In view of (30) and (2.7) of [9], we see that assumption (4.28) of [9] is satisfied. Hence, we conclude that

$$l \geqslant (1-\varepsilon)\frac{\varepsilon k}{4}.$$

Consequently, it remains to consider the case $|T| \leqslant \varepsilon_1^{-1}\pi(k)$. Note that $\nu(A_\mu) = 1$ if $A_\mu \geqslant k$. We apply Lemma 5 of [9] and Lemma 6 of [9] with $g = 2\varepsilon_1^{-1}$

and $\eta = 1/2$ to derive that there exists a subset S_4 of $S_2(\theta)$ satisfying $|S_4| \geqslant k/4$. Hence, we conclude from (31) that

$$\frac{k}{4} \leqslant |S_4| \leqslant |S_2(\theta)| < \frac{\varepsilon k}{2}$$

which is not possible. Hence, (27) or (28) is valid. □

Now, we derive from Lemma 1 and Lemma 4 the following result.

LEMMA 5. *Suppose that* (1) *with* $l > 4\omega(d_1)+2$ *is satisfied. There exists an effectively computable absolute constant* $c_6 > 0$ *such that, for* $k \geqslant c_6$,

$$l \geqslant c_6^{-1}k \quad \text{and} \quad |T| \geqslant \frac{k}{32} \tag{32}$$

or

$$p(d_1) \geqslant lk/5. \tag{33}$$

Proof. Let $\varepsilon = 1/32$. We refer to Lemma 4 with $\varepsilon = \varepsilon_1 = 1/32$ to conclude that we may assume that $|T| < \varepsilon k$. Then, we apply Lemmas 5 and 6 of [9] with $g = 2\varepsilon \log k$ and $\eta = 16\varepsilon$ to derive that there exists a subset S_5 of $S_2(k^{1+4\varepsilon})$ satisfying $|S_5| \geqslant 8\varepsilon k$. Now, we apply Lemma 1 with $S_3 = S_5$ to derive that

$$8\varepsilon k \leqslant |S_5| \leqslant (k^{1+4\varepsilon} + p - 1)/l$$

which, together with (26), implies (33). □

In the above lemmas, we have considered (1) under the assumption $l > 4\omega(d_1) + 2$. On the other hand, if $l \leqslant 4\omega(d_1) + 2$, we show that $\omega(d_1)$ is so large that (3), (4) and (5) follow immediately.

LEMMA 6. *Let* $\varepsilon > 0$. *There exist effectively computable numbers* c_7 *and* $c_8 > 0$ *depending only on* ε *such that* (1) *with* $k \geqslant c_7$ *and* $l \leqslant 4\omega(d_1) + 2$ *implies that*

$$(\tau(\varepsilon))^{\omega(d_1)} > k^{1+c_8} \quad \text{if } l \geqslant 7 \tag{34}$$

and

$$(\tau(\varepsilon))^{\omega(d)} > k^{1+c_8} \quad \text{if } l \in \{2, 3, 5\}. \tag{35}$$

Proof. If $l \leqslant 30$, then (34) and (35) follow immediately from Corollary 1 of [9]. Thus we may assume $l > 30$. Now we apply Lemma 1 of [10]. It is easy to check that its proof remains valid if the condition that $m, m+d, \dots, m+(k-1)d$ are all lth perfect powers is replaced by condition (1) of the present paper. Thus inequality (13) of [10] holds which implies

$$(\tau(\varepsilon))^{\omega(d_1)} \geqslant 2^{1.5 \log k} > k^{1.03}. \qquad \square$$

In addition to the above lemmas, the proof of Theorem 3 depends on the next three lemmas. We start with a version of Lemma 1 of Erdös [3]. The proof depends on Brun's Sieve method.

LEMMA 7. *Let $\varepsilon > 0$ and $x \geqslant 3$. For a positive integer r, we write $E_r(x)$ for a set $\{a_1 < a_2 < \cdots < a_r\}$ of r positive integers not exceeding x. There exist effectively computable numbers x_0 and β depending only on ε such that for $x \geqslant x_0$ and $r \geqslant \beta x/\log x$, we can find $\beta x/4 \log x$ pairs a_i, a_j with $i > j$ satisfying*

$$\gcd(a_i, a_j) \geqslant x^{1-\varepsilon}.$$

If $0 < \beta < 1$, we can take $E_r(x)$ the set of all primes not exceeding x to observe that the assertion of Lemma 7 is no more valid.

Proof. We may assume that $0 < \varepsilon < 1$ and x_0 is sufficiently large. Let $b_1, \dots, b_s$ be the set of all integers between $x^{1-\varepsilon}$ and x such that every proper divisor of b_i is less than or equal to $x^{1-\varepsilon}$. For $b_i > x^{1-\varepsilon/2}$ and a prime p' dividing b_i,

$$x^{1-\varepsilon/2} < b_i \leqslant p' x^{1-\varepsilon}$$

which implies that $p' > x^{\varepsilon/2}$. Therefore, we apply Brun's Sieve to conclude that

$$s \leqslant x^{1-\varepsilon/2} + c_{10}\varepsilon^{-1}x/\log x \leqslant 2c_{10}\varepsilon^{-1}x/\log x$$

where c_{10} is an effectively computable absolute constant. Further, we observe that every integer between $x^{1-\varepsilon}$ and x is divisible by at least one b_i. For every b_i with $1 \leqslant i \leqslant s$, we take some $F(b_i) \in E_r(x)$, if it exists, such that $F(b_i)$ is divisible by b_i. We denote by $E_r'(x)$ the set obtained by deleting from $E_r(x)$ all $F(b_i)$ with $1 \leqslant i \leqslant s$. Further, we write $E_r''(x)$ for the set obtained by deleting from $E_r'(x)$ all the elements $\leqslant x^{1-\varepsilon}$. Observe that

$$|E_r''(x)| \geqslant r - x^{1-\varepsilon} - 2c_{10}\varepsilon^{-1}x/\log x.$$

We take $\beta = 4c_{10}\varepsilon^{-1}$. Then

$$|E_r''(x)| \geqslant \beta x/(4 \log x).$$

For $y \in E_r''(x)$, there exists an i with $1 \leqslant i \leqslant s$ such that y is divisible by b_i and hence,

$$\gcd(y, F(b_i)) \geqslant b_i > x^{1-\varepsilon}. \qquad \square$$

Now, we derive from Lemma 7 another estimate for $\nu(A_{\mu_0})$ which does not depend on $\omega(d_1)$ and which includes Corollary 2 of [10]. Compare this estimate with (23) and (25). The proof depends on (2), (16) and a theorem of Evertse [5].

LEMMA 8. *Suppose that* (1) *with* $l \geqslant 3$ *is satisfied. Further, assume that* $m_1 = 1$. *There exist effectively computable absolute constants* c_{11} *and* c_{12} *such that for* $k \geqslant c_{11}$, *we have*

$$\nu(A_{\mu_0}) \leqslant c_{12} k/\log k \tag{36}$$

for every $A_{\mu_0} \in S_1$.

Proof. Let $\varepsilon = 1/8$. We may assume that c_{11} is sufficiently large. We put

$$t = [\beta k/\log k]$$

where β is the constant appearing in Lemma 7 and we assume that there exist $0 \leqslant \mu_0 < \mu_1 < \cdots < \mu_t < k$ satisfying

$$A_{\mu_0} = A_{\mu_1} = \cdots = A_{\mu_t} \tag{37}$$

which, by (19), $\gcd(m, d) = 1$ and (21), implies that $A_{\mu_i} < k$ and $X_{\mu_i} > 1$ for $0 < i \leqslant t$. For $i > j$, we observe again from (19) and (37) that

$$(\mu_j - \mu_i)m = A_{\mu_0}(\mu_j X_{\mu_i}^l - \mu_i X_{\mu_j}^l).$$

By Lemma 7, there are at least $[t/5]$ pairs μ_i, μ_j with $i > j$ satisfying

$$\gcd(\mu_i, \mu_j) \geqslant k^{1-\varepsilon}.$$

Therefore, there exist integers $P_1 > 0$, $Q_1 > 0$ and $R_1 \neq 0$ satisfying

$$\max(P_1, Q_1, |R_1|) \leqslant k^{\varepsilon}, \qquad \gcd(P_1, Q_1) = 1$$

and

$$A_{\mu_0}(P_1 X_{\mu_i}^l - Q_1 X_{\mu_j}^l) = R_1 m \tag{38}$$

is satisfied by at least $k^{1-4\varepsilon}$ pairs X_{μ_i}, X_{μ_j} with $i > j$.

Further, we derive from (16) and (2) that there exist effectively computable absolute constants $c_{13} \geqslant 8$ and $c_{14} > 0$ such that $l \leqslant c_{13}$ or

$$\log m \geqslant (\log d)/2 \geqslant c_{14}(\log k \operatorname{loglog} k)/\operatorname{logloglog} k.$$

First, we suppose that $l > c_{13}$. Then, we apply Theorem 2 of Evertse [5] with $z = R_1$ and $d = m$ to (38). Using that $m_1 = 1$, we obtain

$$k^{1-4\varepsilon} \leqslant R(l, m) + 2 \leqslant l + 2.$$

Then, if $l \mid \gcd(A_{\mu_0}, m)$, we see from (19), (37) and $\gcd(m, d) = 1$ that $l \mid \mu_i$ for $0 \leqslant i \leqslant t$ which implies that $k > lt > k^{2-5\varepsilon}$. Thus $l \nmid \gcd(A_{\mu_0}, m)$. Let $X_{\mu_{i_1}}, X_{\mu_{j_1}}$ and $X_{\mu_{i_2}}, X_{\mu_{j_2}}$ be distinct pairs satisfying (38). Now, we see from (38) that

$$l^{\operatorname{ord}_l(m)} \mid (X^l_{\mu_{i_1}} X^l_{\mu_{j_2}} - X^l_{\mu_{i_2}} X^l_{\mu_{j_1}})$$

which implies that

$$|X_{\mu_{i_1}} X_{\mu_{j_2}} - X_{\mu_{i_2}} X_{\mu_{j_1}}| \geqslant l^{\operatorname{ord}_l(m) - 1}.$$

Further, we put

$$\Delta = (m + \mu_{i_1} d)(m + \mu_{j_2} d) - (m + \mu_{i_2} d)(m + \mu_{j_1} d).$$

Then

$$|\Delta| \geqslant l^{\operatorname{ord}_l(m)}(m+d)^{2(l-1)/l}$$

and

$$|\Delta| \leqslant 2kd(m + (k-1)d) < 2k^2 d(m + d).$$

Consequently, we derive that

$$l^{\operatorname{ord}_l(m)} < 2k^2 d^{2/l} \leqslant m^{1/2},$$

since $l \geqslant 9$. Now, we observe that

$$R'_1 = R_1 l^{\operatorname{ord}_l(m)}, \qquad m' = m/l^{\operatorname{ord}_l(m)}$$

satisfy $0 < |R'_1| < m^{3/4}$ and $m' \geqslant m^{1/2}$ which imply that

$$(m')^{(2l/5)-1} > m > |R'_1|.$$

Hence, we apply again Theorem 2 of Evertse [5] with $z = R_1'$ and $d = m'$ to (38) for concluding that $k^{1-4\varepsilon} \leqslant R(l, m') + 2 = 3$, since $m_1 = 1$ and $\gcd(m', l) = 1$. If $l \leqslant c_{13}$, we apply Theorem 1 of Evertse [5]. Again using $m_1 = 1$, we obtain

$$\begin{aligned} k^{1-4\varepsilon} &\leqslant 2R(l, |R_1|m) + 6 \leqslant 2lR(l, |R_1|) + 6 \\ &\leqslant 2l^{\omega(R_1)+1} + 6 \leqslant c_{13}^{2\varepsilon \log k/\log\log k} \end{aligned}$$

which is not possible if c_{11} is sufficiently large. □

Finally, we apply Lemma 8 and Lemma 1 to conclude the following result.

LEMMA 9. *Suppose that* (1) *is satisfied. Further, assume that* $l > 4\omega(d_1) + 2$ *and* $m_1 = 1$. *There exist effectively computable absolute constants* c_{15} *and* $c_{16} > 0$ *such that for* $k \geqslant c_{15}$, *we have*

$$p(d_1) \geqslant c_{16} l \log k. \tag{39}$$

Proof. Let $\varepsilon = 1/32$ and we may assume that c_{15} is sufficiently large. First, we consider the case that $|T| \geqslant \varepsilon k$. Then, we apply Lemma 1 with $S_3 = S_2$ and (36) to derive that

$$\varepsilon k \leqslant |T| \leqslant c_{12} \frac{k}{\log k} \left(\frac{k+p-1}{l} \right)$$

which, together with (26), implies (39). If $|T| < \varepsilon k$, we apply Lemma 5 to derive (33). □

Section 3. Proof of Theorem 1

Let $\varepsilon_1 = \varepsilon/30$ and suppose that C_2 is sufficiently large. In view of (2.12) of [9], we may suppose that $l \leqslant 4\omega(d_1) + 2$. Then, we apply Lemma 6 to derive that

$$\omega(d_1) \geqslant \eta'(\varepsilon_1) \log k.$$

Consequently, we obtain by prime number theory

$$\log d_1 \geqslant \sum_{p \leqslant (1-\varepsilon_1)\eta'(\varepsilon_1)\log k \log\log k} \log p \geqslant (1-\varepsilon_1)^2 \eta'(\varepsilon_1) \log k \log\log k$$

which implies (3). □

PROOF OF THEOREM 2. We may suppose that $C_3 > c_7$ is sufficiently large. Then, by Lemma 6 and $p > l$, we may assume that $l \geqslant 4\omega(d_1) + 3 \geqslant 7$. Now, by

Lemma 5, we may suppose (32). Then, we apply Corollary 2 to conclude that for every $A_{\mu_0} \in S_1$,

$$\nu(A_{\mu_0}) \leqslant c_{17} 2^{\omega(d_1)} \tag{40}$$

where c_{17} is an effectively computable absolute constant.

We obtain from (32) and (40) that

$$|S_2| \geqslant [k(32c_{17}2^{\omega(d_1)})^{-1}] \geqslant 2c_6, \tag{41}$$

where we may suppose the right-hand inequality of (41), otherwise (4) follows immediately from $p > l$. Finally, we apply Lemma 1 with $S_3 = S_2$ to obtain

$$|S_2| \leqslant \frac{k+p-1}{l}$$

which, together with (41) and (32), implies (4). □

PROOF OF COROLLARY 1. We may assume that C_5 is sufficiently large. For the proof of (6) and (7), we apply Theorem 2 to assume that

$$(\tau(\varepsilon))^{\omega(d_1)} \geqslant k(\log k)^{-2} \quad \text{if } l \geqslant 7 \tag{42}$$

and

$$(\tau(\varepsilon))^{\omega(d)} \geqslant k(\log k)^{-2} \quad \text{if } l \in \{2, 3, 5\}. \tag{43}$$

Now, we apply Brun–Titchmarsh Theorem to derive (6) from (42). Further, we apply Prime Number Theorem to obtain (7) from (43).

Now, we turn to the proof of (8) and (9). In view of Lemma 6, we may assume that $l > 4\omega(d_1)+2$. Now, by (26), it suffices to show that

$$Q(d_1) \geqslant lk^{1-\varepsilon} \quad \text{if } l \geqslant 7 \tag{44}$$

and

$$Q(d) \geqslant lk^{1-\varepsilon} \quad \text{if } l \in \{2, 3, 5\}. \tag{45}$$

For this, we refer to Theorem 2 to assume that

$$(\tau(\varepsilon))^{\omega(d_1)} \geqslant k^{\varepsilon/2} \quad \text{if } l \geqslant 7$$

and

$$(\tau(\varepsilon))^{\omega(d)} \geqslant k^{\varepsilon/2} \quad \text{if } l \in \{2, 3, 5\}$$

which imply (44) and (45). □

PROOF OF THEOREM 3. (a) We apply Lemma 5 to assume (32) which implies that $p > l \geqslant c_6^{-1}k$. This confirms (12) with $C_6 = c_6^{-1}$.

(b) We may assume that k exceeds a sufficiently large effectively computable absolute constant, otherwise (13) follows from $p > l$. Then, we apply Lemma 9 to assume that $m_1 > 1$. By reading (19) mod q, we have

$$\mu d \equiv A_\mu X_\mu^l (\operatorname{mod} q) \quad \text{for } 0 \leqslant \mu < k.$$

Since $q \equiv 1(\operatorname{mod} l)$ is a prime number, we see that $X_0^l, X_1^l, \ldots, X_{k-1}^l$ are contained in at most $(q-1)/l$ residue classes mod q. Let $A_{\mu_0} \in S_1$ and let $X(\operatorname{mod} q)$ be a residue class mod q. Let $0 \leqslant \mu_0 < \mu_1 \cdots < \mu_{s-1} < k$ satisfy $A_{\mu_0} = A_{\mu_1} = \cdots =$
$A_{\mu_{s-1}}$ and

$$\mu_i d \equiv A_{\mu_i} X (\operatorname{mod} q),\ 0 \leqslant i < s.$$

Then, since $\gcd(m, d) = 1$,

$$s \leqslant \left[\frac{k}{q}\right] + 1.$$

Consequently,

$$\nu(A_{\mu_0}) \leqslant \left(\left[\frac{k}{q}\right] + 1\right)\left(\frac{q-1}{l}\right) \leqslant \frac{k+q-1}{l} \tag{46}$$

for every $A_{\mu_0} \in S_1$.

Let $\varepsilon = 1/32$. If $|T| < \varepsilon k$, we apply Lemma 5 to derive (33). Thus, we may suppose that $|T| \geqslant \varepsilon k$. Then, we apply (46) and Lemma 1 with $S_3 = S_2$ to derive that

$$\varepsilon k \leqslant |T| \leqslant \left(\frac{k+q-1}{l}\right)\left(\frac{k+p-1}{l}\right)$$

which, together with (26), implies either (13) or (14). □

PROOF OF THEOREM 4. (a) We may assume $0 < \varepsilon < \frac{1}{3}$ and that C_{12} is sufficiently large. Let $0 \leqslant \mu_0 < k$ satisfy

$$P = P(m+\mu_0 d) = \min_{0 \leqslant \mu < k} P(m+\mu d) < k^{1/3-\varepsilon}.$$

Now, we apply an estimate of Yu[11] on p-adic linear forms in logarithms, as in the proof of Lemma 9 of [9], to conclude that

$$\log m \leqslant \log(m+\mu_0 d) \leqslant \sum_{p \leqslant P} \operatorname{ord}_p(m+\mu_0 d) \log p$$
$$\leqslant \left(\frac{\log l}{l}\right)(\log d)k^{1-2\varepsilon} + \sum_{p \leqslant P} 6 \log k.$$

By Corollary 6 of [9], we obtain

$$\sum_{p \leqslant P} 6 \log k \leqslant 6P \log k < k^{1/3} < \left(\frac{\log l}{l}\right)(\log d)k^{1-2\varepsilon},$$

hence

$$\log m \leqslant 2\left(\frac{\log l}{l}\right)(\log d)k^{1-2\varepsilon}.$$

By (26), we can secure that $\min(k, l) \geqslant C_{10}$. The preceding inequality combined with (16) and (26) implies that k is bounded by an effectively computable number depending only on ε.

(b) We may assume that C_{13} is sufficiently large. We suppose that $P(m-d) \leqslant k$. Let p_1 be a prime dividing $(m-d)$. If $p_1 \leqslant \log k$, we apply an estimate of Yu[11], (16) and (26) as mentioned above, to derive that

$$\operatorname{ord}_{p_1}(m-d) \leqslant c_{18}\frac{(\log k)^5}{k}(\log m) \tag{47}$$

where c_{18} and the subsequent letters c_{19}, c_{20}, c_{21} are effectively computable absolute positive constants. If $p_1 > \log k$, we see that

$$\operatorname{ord}_{p_1}(m-d) \leqslant \max_{1 \leqslant i \leqslant k} \operatorname{ord}_{p_1}(i), \tag{48}$$

otherwise, by (26),

$$0 < \operatorname{ord}_{p_1}(m(m+d)\cdots(m+(k-1)d)) \leqslant c_{19}\frac{k}{p_1} < l$$

which, since $b = 1$, is not possible. By (47),

$$\sum_{\substack{p_1 \leqslant \log k \\ p_1 \mid (m-d)}} \operatorname{ord}_{p_1}(m-d) \log p_1 \leqslant 2c_{18} \frac{(\log k)^6}{k} (\log m).$$

Further, we observe from (48) that

$$\sum_{\substack{p_1 > \log k \\ p_1 \mid (m-d)}} \operatorname{ord}_{p_1}(m-d) \log p_1 \leqslant \pi(k) \log k \leqslant 2k,$$

since $P(m-d) \leqslant k$. Consequently,

$$\log(|m-d|) \leqslant 2c_{18} \frac{(\log k)^6}{k} (\log m) + 2k. \tag{49}$$

By Lemma 5 of [9], we can find μ_1 and μ_2 with $0 \leqslant \mu_1 < k$, $0 \leqslant \mu_2 < k$ and $\mu_1 \neq \mu_2$ such that

$$A_{\mu_i} \leqslant k^2 \quad \text{for } i = 1, 2.$$

Further, we observe

$$(\mu_1 - \mu_2)(m-d) = (\mu_1 + 1)(m + \mu_2 d) - (\mu_2 + 1)(m + \mu_1 d).$$

Therefore, by (19),

$$|(\mu_1 - \mu_2)(m-d)| = |(\mu_1 + 1) A_{\mu_2} X_{\mu_2}^l - (\mu_2 + 1) A_{\mu_1} X_{\mu_1}^l|. \tag{50}$$

Now, we apply an estimate of Baker [1] on linear forms in logarithms to the right-hand side of (50) to conclude that

$$\log(k\,|m-d|) \geqslant \log m - c_{20}(\log k)^2 \left(\frac{\log l}{l}\right) \log(m + (k-1)d)$$

which, together with (2.19) of [9], (16) and (26), implies that

$$\log(k|m-d|) \geqslant \log m - \frac{(\log k)^4}{k} (\log m) \geqslant (\log m)/2. \tag{51}$$

Now, we combine (49) and (51) to derive that $\log m \leqslant 8k$. Then, we apply (16) and (2.12) of [9] to conclude that $k \leqslant c_{21}$ which is not possible if C_{13} is sufficiently large. This contradiction proves (18). □

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