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## Perfect powers in arithmetical progression (II)

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### Section 1

It is an old question how long arithmetical progressions of equal powers of positive integers can be. There exist infinitely many triples of coprime squares in arithmetical progression, but Fermat proved that there are no four squares in arithmetical progression, see [8, p. 21]. Dénes [2] conjectured that for  $l \geq 3$  there are no three  $l$ -th powers in arithmetical progression, and proved this assertion for  $3 \leq l \leq 30$  and for sixty other prime values of  $l$ . Faltings' celebrated result [5] implies that for any  $l \geq 5$  there are only finitely many triples of coprime  $l$ -th perfect powers in arithmetical progression. Without loss of generality we can restrict our attention to the following situation:

$l$  is a prime number,  $l \geq 3$ , and  $m$ ,  $d$  and  $k$  are positive integers satisfying  $\gcd(m, d) = 1$  and  $k \geq 3$  such that each of the numbers  $m, m + d, \dots, m + (k - 1)d$  is an  $l$ -th perfect power. } (1)

Let  $d_1$  be the maximal divisor of  $d$  such that all the prime factors of  $d_1$  are  $\equiv 1 \pmod{l}$ . Similarly, we define  $m_1$  as the maximal divisor of  $m$  such that all the prime factors of  $m_1$  are  $\equiv 1 \pmod{l}$ . For an integer  $v$  with  $|v| > 1$ , we write  $p(v)$ ,  $P(v)$ ,  $Q(v)$  and  $\omega(v)$ , respectively, for the least prime factor of  $v$ , the greatest prime factor of  $v$ , the greatest square free factor of  $v$  and the number of distinct prime divisors of  $v$ . Further, we set  $p(\pm 1) = P(\pm 1) = Q(\pm 1) = 1$  and  $\omega(\pm 1) = 0$ . We write  $p(l, 1)$  for the least prime  $\equiv 1 \pmod{l}$ . Recently Heath-Brown announced that he has proved

$$p(l, 1) < l^{10} \quad \text{if } l \geq C \quad (2)$$

for some effectively computable absolute constant  $C$ . Until now the best exponent was 16, see Wei [12].

It follows from the above mentioned results of Dénes and Faltings that if (1)

holds then  $k$  is bounded by some number depending only on  $l$ . We showed in [10] that there exists an upper bound for  $k$  depending only on  $\omega(d)$ . In fact, we proved in [10] that (1) implies that

$$\prod_{p < k/2} p \text{ divides } d \tag{3}$$

and that

$$\log d \geq 2k^2/13 \text{ if } k \geq C_1. \tag{4}$$

Furthermore, it follows from formula (2.7) of [11] that

$$d_1 > 1 \text{ if } k \geq C_1. \tag{5}$$

Here  $C_1$  is an effectively computable absolute constant. In this paper, we find an upper bound for  $k$  depending only on  $\omega(d_1)$ . More precisely, we prove

**THEOREM 1.** *Let  $\varepsilon > 0$ . There exists an effectively computable number  $C_2$  depending only on  $\varepsilon$  such that (1) with  $k \geq C_2$  implies that*

$$2^{\omega(d_1)} > (1 - \varepsilon)k. \tag{6}$$

Theorem 1 sharpens the assertion of Corollary 1 of [11] under a stronger assumption. We derive the following result from Theorem 1, the Brun-Titchmarsh Theorem and (3).

**COROLLARY 1.** *Let  $\varepsilon > 0$  and  $\eta(\varepsilon) = (1 - \varepsilon)/\log 2$ . There exists an effectively computable number  $C_3$  depending only on  $\varepsilon$  such that (1) with  $k \geq C_3$  implies that*

$$2P(d_1) \geq \eta(\varepsilon)l \log k \log \log k \tag{7}$$

and

$$\log Q(d_1) \geq \eta(\varepsilon)(\log k)^2. \tag{8}$$

Observe that (3) gives stronger results than (7) for small values of  $l$ . For example, we see from (3) and the Prime Number Theorem for arithmetical progressions that (1) implies that

$$P(d_1) > k/4$$

if  $l$  does not exceed a fixed power of  $\log k$  and  $k$  is sufficiently large.

It is clear from (3) that it is not possible to obtain an inequality analogous to (7) for  $p(d_1)$ . In fact,  $p(d_1)$  is bounded whenever  $l$  is bounded. More generally, we see from (3) and (2) that if (1) holds with  $k$  exceeding a sufficiently large absolute constant, then

$$p(d_1) \leq l^{10} \quad \text{if } l < C_4 k^{1/10}$$

where  $C_4 = 2^{-1/10}$ .

Next we consider analogous results for  $m$ .

**THEOREM 2.** (a) *There exists an effectively computable absolute constant  $C_5 > 0$  such that (1) implies that*

$$\log(m + 1) \geq C_5 k^2. \tag{9}$$

(b) *There exist effectively computable absolute constants  $C_6$  and  $C_7 > 0$  such that (1) with  $k \geq C_6$  implies that*

$$l^{\omega(m_1)} \geq C_7 k, \tag{10}$$

$$p(m_1) > kl \tag{11}$$

and

$$Q(m_1) \geq C_7 k^{1 + \omega(m_1)}. \tag{12}$$

**COROLLARY 2.** *If (1) holds then*

$$m_1 > 1 \quad \text{for } k \geq C_6.$$

Finally, we prove a result involving both  $m$  and  $d$ ,

**THEOREM 3.** *There exists an effectively computable absolute constant  $C_8$  such that (1) with  $k \geq C_8$  implies that*

$$P(m - d) \geq k.$$

## Section 2. Proof of Theorem 1

We may assume that (1) with  $k \geq c$  is valid where  $c$  is a sufficiently large effectively computable number depending only on  $\varepsilon$ . Further, by the result of

Dénes mentioned in Section 1, we may suppose that  $l > 30$ . The case  $l \leq 4\omega(d_1) + 2$  is treated in the following lemma.

LEMMA 1. *Suppose (1) is satisfied with  $30 < l \leq 4\omega(d_1) + 2$ . Then*

$$\omega(d_1) > \frac{3}{2} \log k. \quad (13)$$

*Proof.* First we consider the case that  $3l \leq \omega(d_1)$ . Then there exists a positive integer  $n$  satisfying

$$\frac{5\omega(d_1)}{l} < n \leq \frac{16\omega(d_1)}{3l}.$$

Consequently, there exists a divisor  $d'$  of  $d_1$  such that

$$\omega(d') = n \leq \frac{16\omega(d_1)}{3l} \quad (14)$$

and

$$d' \geq d_1^{n/\omega(d_1)} > d_1^{5/l}.$$

Therefore, by (2.1) of [11],

$$d' \geq C_5 d^{5(l-2)/l^2} \quad (15)$$

where  $C_5$  is the absolute constant appearing in (2.1) of [11]. Further, since  $l > 30$ , we observe that

$$\frac{5(l-2)}{l^2} \geq \frac{9}{2l}. \quad (16)$$

Now we proceed to apply Theorem 1(b) of [11]. For this, we observe from (15), (16) and (4) (or (2.13) of [11]) that

$$d' \geq C_8 l^{-1} d^{4/l}$$

where  $C_8$  is the constant appearing in (2.4) of [11]. Hence, by theorem 1(b) of [11],

$$l^{\omega(d')} > \frac{k}{\log k}$$

which, together with (14), implies that

$$\omega(d_1) \geq \frac{3l}{16} \frac{\log(k/\log k)}{\log l} > \frac{l \log k}{6 \log l} > \frac{3}{2} \log k, \tag{17}$$

since  $l > 30$ .

Thus we may assume that  $3l > \omega(d_1)$ . Then we may suppose that  $l > 200$ , otherwise we apply Corollary 3 of [11] to conclude that  $k < c$ .

Next we consider the general case  $30 < l \leq 4\omega(d_1) + 2$ . Then there exists a positive integer  $n$  with

$$\frac{5\omega(d_1)}{l} < n \leq \frac{10\omega(d_1)}{l}$$

and a divisor  $d'$  of  $d_1$  satisfying

$$\omega(d') \leq \frac{10\omega(d_1)}{l}, \quad d' \geq d_1^{5/l} \geq d_1^{9/2l}.$$

Now, we apply Theorem 1(b) of [11] as above, to conclude that

$$\omega(d_1) > \frac{l \log k}{12 \log l} \tag{18}$$

which, since  $l > 200$ , implies that  $\omega(d_1) > \frac{3}{2} \log k$ . □

To complete the proof of Theorem 1, we consider the case  $l > 4\omega(d_1) + 2$ . Then, by (5), there exists a divisor  $d'$  of  $d_1$  such that

$$\omega(d') = 1 \quad \text{and} \quad d' \geq d_1^{1/\omega(d_1)}.$$

Further, by (2.1) of [11], we observe that

$$d' \geq C_5 d_1^{4(1 + 1/(l-3))/l} \tag{19}$$

where  $C_5$  is the absolute constant appearing in (2.1) of [11]. Now we proceed to apply Lemma 8 of [11] with  $f(x) = \log x$ . First we observe from (19) and (2.7) of [11] that

$$d' \geq l^{-1} (\log k)^3 (dk)^{2/l}.$$

Also it is easy to see that  $|S_1| \leq \pi(k)$  where  $S_1$  is the set appearing in Lemma 8 of

[11]. For this, observe that, by (1), for every prime  $p \leq k$  there is at most one  $\mu$  with  $0 \leq \mu < k$  such that  $m + \mu d$  is divisible by  $p$ . (In fact, (3) is an immediate consequence of this assertion.) Hence we apply Lemma 8 of [11] to conclude that

$$l \geq (1 - \varepsilon_1)k, \quad \varepsilon_1 = \varepsilon/2. \quad (20)$$

We write

$$m + \mu d = x_\mu^l \quad \text{for } 0 \leq \mu < k. \quad (21)$$

Therefore, for  $\mu > \nu \geq 0$ , we have

$$(\mu - \nu)d = (x_\mu - x_\nu) \left( \frac{x_\mu^l - x_\nu^l}{x_\mu - x_\nu} \right). \quad (22)$$

We put

$$x_{\mu,\nu} = \frac{x_\mu^l - x_\nu^l}{x_\mu - x_\nu} \quad \text{for } \mu > \nu \geq 0.$$

Except possibly  $l$ , every prime factor of  $x_{\mu,\nu}$  is  $\equiv 1 \pmod{l}$  and  $l^2 \nmid x_{\mu,\nu}$ . In particular, we observe from (20) that

$$p(x_{\mu,\nu}) \geq l \geq (1 - \varepsilon_1)k$$

which implies that

$$\gcd(x_{\mu,\nu}, \mu - \nu) = 1 \quad \text{for } \mu > \nu \geq [2\varepsilon_1 k] - 2. \quad (23)$$

Now we fix  $\nu = [2\varepsilon_1 k] - 2$ . We conclude from (22) that for  $\mu > \nu$  there exist positive integers  $d_{1,\mu} = d_{1,\mu,\nu}$  and  $d_{2,\mu} = d_{2,\mu,\nu}$  such that

$$d = d_{1,\mu} d_{2,\mu}, \quad \gcd(d_{1,\mu}, d_{2,\mu}) = 1 \text{ or } l$$

and

$$d_{1,\mu} | x_{\mu,\nu}, \quad d_{2,\mu} | x_\mu - x_\nu.$$

Furthermore, we derive from (23) that  $x_{\mu,\nu} = d_{1,\mu}$  and  $d_{1,\mu} l^{-\delta}$  is a divisor of  $d_1$

where

$$\delta = \begin{cases} 1 & \text{if } l \mid d, \\ 0 & \text{if } l \nmid d. \end{cases}$$

For  $\mu_1 > \mu_2 > \nu$ , we observe that  $x_{\mu_1, \nu} > x_{\mu_2, \nu}$ . Now we apply the Box Principle to conclude that

$$2^{\omega(d_1)} \geq k - \nu - 1 > (1 - \varepsilon)k. \quad \square$$

**PROOF OF COROLLARY 1.** Let  $\varepsilon > 0$  and  $\varepsilon_1 = \varepsilon/2$ . Suppose that (1) with  $k \geq C_3$  is valid. Further, we may assume that  $C_3$  is sufficiently large. Now we see from (6) that

$$\omega(d_1) \geq (1 - \varepsilon_1) \frac{\log k}{\log 2}. \quad (24)$$

We recall that every prime divisor of  $d_1$  is  $\equiv 1 \pmod{l}$  and we apply the Brun-Titchmarsh Theorem to derive (7) from (24).

Next we proceed to prove (8). First we assume that  $l \geq 4\omega(d_1) + 2$ . Now we apply Lemma 8 of [11] as in the proof of Theorem 1 to conclude (20). Further we observe that

$$\log Q(d_1) \geq \omega(d_1) \log l. \quad (25)$$

We combine (25), (24) and (20) to derive (8) if  $l > 4\omega(d_1) + 2$ .

Thus we may assume  $l \leq 4\omega(d_1) + 2$ . Then we apply Theorem 1(b) of [11] as in the proof of Lemma 1 to conclude (18). We combine (25) and (18) to derive

$$\log Q(d_1) \geq (l \log k)/12.$$

Thus we may suppose that  $l \leq 12\eta(\varepsilon) \log k$ . Then we see from (3) and the Prime Number Theorem for arithmetical progressions that

$$\log Q(d_1) \geq k/(\log k)^3$$

which implies (8). □



**Section 3**

We denote by  $c_1, \dots, c_9$  effectively computable absolute constants. Suppose that (1) is valid with  $k \geq c_1$  where  $c_1$  is sufficiently large. We write (21). As already remarked, for every prime  $q \leq k$  there is at most one  $\mu$  with  $0 \leq \mu < k$  such that  $m + \mu d$  is divisible by  $q$ . Therefore there exists a set  $T$  with  $|T| \geq k - \pi(k)$  consisting of integers  $\mu$  satisfying  $0 \leq \mu < k$  and  $p(x_\mu) > k$ . Further, for  $\mu \in T$  and  $\nu \in T$  with  $\mu > \nu$ , we observe from (21) that

$$\mu x_\nu^l - \nu x_\mu^l = (\mu - \nu)m.$$

Then we have

$$\frac{\mu}{\gcd(\mu, \nu)} x_\nu^l - \frac{\nu}{\gcd(\mu, \nu)} x_\mu^l = \frac{\mu - \nu}{\gcd(\mu, \nu)} m.$$

We apply the sieve-theoretic Lemma 1 of Erdős [3] to conclude that there exist positive integers  $P, Q$  and  $R$  such that

$$\max(P, Q, R) \leq c_2, \quad \gcd(P, Q) = 1$$

and that

$$Px_\nu^l - Qx_\mu^l = Rm \tag{26}$$

is satisfied by at least  $c_3 k$  pairs  $x_\mu, x_\nu$  with  $p(x_\mu) > k$ ,  $p(x_\nu) > k$  and  $\gcd(x_\mu, x_\nu) = 1$ .

**PROOF OF THEOREM 2(a).** We may assume that  $k \geq c_1$ , otherwise (9) follows immediately. Now we apply a theorem of Baker [1] on linear forms in logarithms to conclude from (26) that

$$m \geq x_\mu^{l - c_4 \log l}$$

which implies that

$$m \geq d^{1 - (c_5 \log l)/l}. \tag{27}$$

This inequality is trivial if  $c_5 \log l \geq l$  which implies that  $l \leq c_6$ . If  $l \leq c_6$ , we apply an estimate of Feldman [6] on the magnitude of integral solutions of Thue's equation (cf. [9], Chapter 5) to (26) for deriving

$$m \geq d^{c_7} \quad \text{if } l \leq c_6. \tag{28}$$

Now we combine (27) and (28) to obtain  $m \geq d^{c_8}$  which, together with (4), implies (9).

**PROOF OF THEOREM 2(b).** Assume (1). We may suppose that  $k \geq c_1$  and, by part (a), that  $m \geq c_2^5$ . Then there are at least  $c_3k$  pairs  $x_\mu, x_\nu$  with  $p(x_\mu) > k$ ,  $p(x_\nu) > k$  and  $\gcd(x_\mu, x_\nu) = 1$  satisfying (26). Observe that

$$\gcd(P, m) = \gcd(Q, m) = \gcd(x_\mu, x_\nu) = 1.$$

Now we apply Theorem 2 of Evertse [4] on an upper bound for the number of solutions of (26) to conclude that

$$c_3k \leq R(l, m) + 2 \leq c_9 l^{\omega(m_1)}$$

which implies (10).

We put  $p = p(m_1)$ . Then, by Corollary 2, we observe that  $p \geq 2$ . Since  $\text{ord}_p(m) \geq l$  and  $\gcd(m, d) = 1$ , we see that  $\text{ord}_p(m + pd) = 1$ . If  $p < k$ , then this is not possible. Thus we derive that

$$p \geq k. \tag{29}$$

By (21), we have

$$\mu d \equiv x_\mu^l \pmod{p} \quad \text{for } 0 \leq \mu < k. \tag{30}$$

Further, by (29) and  $\gcd(m, d) = 1$ , we observe that

$$\mu d \not\equiv \nu d \pmod{p} \quad \text{for } 0 \leq \nu < \mu < k. \tag{31}$$

On the other hand, since  $p \equiv 1 \pmod{l}$ , we see that  $x_0^l, x_1^l, \dots, x_{k-1}^l$  lie in at most  $(p - 1)/l$  distinct residue classes mod  $p$ . Hence we conclude from (30) and (31) that  $(p - 1)/l \geq k$  which implies (11). Finally we observe that  $Q(m_1) \geq p^{\omega(m_1)}$  which, together with (11) and (10), implies (12).  $\square$

#### Section 4. Proof of Theorem 3.

We denote by  $c_{10}$ ,  $c_{11}$  and  $c_{12}$  effectively computable positive absolute constants. Assume (1) with  $k \geq c_{10}$  where  $c_{10}$  is sufficiently large. Put  $|m - d| = M$ . Observe that  $M \neq 0$ , since  $\gcd(m, d) = 1$ . By (21), we have

$$M = |2x_0^l - x_1^l|.$$

As in the proof of Theorem 2(a), we derive from the theorems of Baker and Feldman that

$$\log M \geq c_{11} \log d$$

which, together with (4), implies

$$\log M \geq c_{12} k^2. \tag{32}$$

For a prime  $q \leq k$  dividing  $m - d$ , we see from (21) that  $\text{ord}_q(m - d + qd) \geq l$ . Therefore, since  $\text{gcd}(m, d) = 1$ , we derive that  $\text{ord}_q(m - d) \leq 1$  for every prime  $q \leq k$ . Assume that  $P(M) \leq k$ . Then

$$\log M \leq \sum_{q \leq k} \log q \leq 2k$$

which contradicts (32). □

**REMARK.** Estimate (9) can be sharpened. Suppose that (1) is satisfied and let  $\varepsilon > 0$ . We refer to (27) and (4) to derive that

$$\log(m + 1) \geq \left( \frac{2}{13} - \varepsilon \right) k^2 \quad \text{if } \min(k, l) \geq C_9,$$

where  $C_9$  is an effectively computable number depending only on  $\varepsilon$ . For all  $l$ , we apply Theorem 1 of [7] to (26), together with (4) and  $l > 30$ , to obtain

$$\log(m + 1) \geq \left( \frac{29}{403} - \varepsilon \right) k^2 \quad \text{if } k \geq C_{10}$$

where  $C_{10}$  is an effectively computable number depending only on  $\varepsilon$ .

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