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Connecting direct limit topologies with metrics on infinite-dimensional manifolds

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Abstract. Let $\mathbb{R}^\infty$ and Q^∞ be the direct limits of the towers $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$ and $Q^1 \subset Q^2 \subset Q^3 \subset \cdots$, where $Q = [-1, 1]^\omega$ is the Hilbert cube. And let σ and Σ be the spaces $\bigcup_{n\in\mathbb{N}} \mathbb{R}^n$ and $\bigcup_{n\in\mathbb{N}} Q^n$ with relative topologies inherited from the product topologies of $\mathbb{R}^\omega$ and Q^ω, respectively. Let M_i $(i = 1, 2)$ have two topologies $\tau_1^i \subset \tau_2^i$ such that (M_i, τ_1^i) is a σ-manifold and (M_i, τ_2^i) is an $\mathbb{R}^\infty$-manifold (or (M_i, τ_1^i) is a Σ-manifold and (M_i, τ_2^i) is a Q^∞-manifold). If (M_1, τ_1^1) and (M_2, τ_1^2) have the same homotopy type, their exists $h: M_1 \to M_2$ which is a homeomorphism in both topologies. This gives the answers to the questions in [We, NLC15].

0. Introduction

A manifold modeled on a given space E is called an *E-manifold*. We naturally identify $\mathbb{R}^n$ with the subset of the countable infinite product $\mathbb{R}^\omega$. The set $\bigcup_{n\in\mathbb{N}} \mathbb{R}^n$ has two natural topologies,(1) that is, the weak topology with respect to the tower $(\mathbb{R}^n)_{n\in\mathbb{N}}$ and the relative topology inherited from the product topology of $\mathbb{R}^\omega$. We denote these spaces by $\operatorname{dir\,lim} \mathbb{R}^n$ and $\mathbb{R}^\omega_f$ or simply by $\mathbb{R}^\infty$ and σ, respectively. The realization $|K|$ of a simplicial complex K has also two natural topologies, that is, the weak (or Whitehead) topology and the metric topology. These spaces are denoted by $|K|_w$ and $|K|_m$, respectively. A countable simplicial complex is called a *combinatorial ∞-manifold* if the star of each vertex is combinatorially equivalent to the countable infinite full simplicial complex Δ^∞ [Sa4]. It is shown in [Sa5] (cf. [Sa6]) that for a combinatorial ∞-manifold K, $|K|_w$ is an $\mathbb{R}^\infty$-manifold and $|K|_m$ is a σ-manifolds. In [Pa, §3], R.S. Palais pointed out a Banach manifold of fiber bundle sections admits an atlas such that the transitions are also homeomorphisms with respect to the bounded weak topology of the model (cf. [He1]). Then it is natural to ask whether a combinatorial ∞-manifold has the similar structure, that is, it admits an atlas such that the transitions are homeomorphisms in both topologies. Such a manifold may be called an *$(\mathbb{R}^\infty, \sigma)$-manifold.* Concerning $(\mathbb{R}^\infty, \sigma)$-manifolds, we can ask the same questions as [Ge, NLC9] (cf. [We, NLC15]).

Let $Q = [-1, 1]^\omega$ be the Hilbert cube. Similarly as the above, we can define $Q^\infty = \operatorname{dir\,lim} Q^n$ and $\Sigma = \bigcup_{n\in\mathbb{N}} Q^n$ as the subspace of Q^ω. Let LIP(X, Y) be the

(1)According to [Ke], such a space with two topologies is called a *bitopological space.*

space of Lipschitz maps from a compact metric space X to a metric space Y with sup-metric. For $k > 0$, let k-LIP(X, Y) be the subspace of LIP(X, Y) consisting of all k-Lipschitz maps (i.e., maps of Lipschitz constant $\leqslant k$). Then LIP(X, Y) has another topology, namely the weak topology with respect to the tower (n-LIP(X, Y))$_{n\in\mathbb{N}}$. We denote this space by LIP(X, Y)$_w$. It is proved in [Sa7] and [Sa8] that LIP(X, Y) is a Σ-manifold and LIP(X, Y)$_w$ is a Q^∞-manifold in case X is non-discrete and Y is a Euclidean polyhedron without isolated points or a Lipschitz manifold with dim $Y > 0$. In this setting, we may also ask whether LIP(X, Y) admits an atlas such that the transitions are homeomorphisms in both topologies. Such a manifold may be called a *(Q^∞, Σ)-manifold.* Concerning (Q^∞, Σ)-manifolds, we can ask similar questions (cf. [We, NLC15]).

In this paper, we give affirmative answers for these questions by proving the following theorem:

MAIN THEOREM. *Let M_i $(i=1,2)$ have two topologies $\tau_1^i \subset \tau_2^i$ such that (M_i, τ_1^i) is a σ-manifold and (M_i, τ_2^i) is an $\mathbb{R}^\infty$-manifold (or (M_i, τ_1^i) is a Σ-manifold and (M_i, τ_2^i) is a Q^∞-manifold). If (M_1, τ_1^1) and (M_2, τ_1^2) have the same homotopy type, there exists $h\colon M_1 \to M_2$ which is a homeomorphism in both topologies.*

1. Corollaries and remarks on the model spaces

Combining the main theorem with [Sa5, Corollary 3 (resp. 3′)] or the open embedding theorem for σ-manifolds (resp. Σ-manifolds) [Ch, Theorem 2.4], we have the following:

COROLLARY 1. *Any $\mathbb{R}^\infty$-manifold or any σ-manifold has the unique $(\mathbb{R}^\infty, \sigma)$-manifold structure. And any Q^∞-manifold or any Σ-manifold has the unique (Q^∞, Σ)-manifold structure.*

In the sequel corollaries, we assume the following:

(*) *M has two topologies $\tau_1 \subset \tau_2$ such that (M, τ_1) is a σ-manifold and (M, τ_2) is an $\mathbb{R}^\infty$-manifold (or (M, τ_1) is a Σ-manifold and (M, τ_2) is a Q^∞-manifold).*

Combining the main theorem with the corresponding results for σ-manifolds (or Σ-manifolds) in [Ch] or by results in [Sa6], we have the following corollaries:

COROLLARY 2 (Open embedding theorem). *There exists an open embedding $h\colon (M, \tau_1) \to \sigma$ (or $h\colon (M, \tau_1) \to \Sigma$) such that $h\colon (M, \tau_2) \to \mathbb{R}^\infty$ (or $h\colon (M, \tau_2) \to Q^\infty$) is also an open embedding. Hence M is an $(\mathbb{R}^\infty, \sigma)$-manifold (or a (Q^∞, Σ)-manifold).*

COROLLARY 3 (Triangulation theorem). *There exists a simplicial complex K and a homeomorphism $h\colon (M, \tau_1) \to |K|_m$ (or $h\colon (M, \tau_1) \to |K|_m \times Q$) such that $h\colon (M, \tau_2) \to |K|_w$ (or $h\colon (M, \tau_2) \to |K|_w \times Q$) is also a homeomorphism. (Then K must be a combinatorial ∞-manifold* [Sa6]).

COROLLARY 4 (Stability theorem). *There exists a homeomorphism* $h:(M,\tau_1)\times\sigma\to(M,\tau_1)$ *(or* $h:(M,\tau_1)\times\Sigma\to(M,\tau_1)$*) such that* $h:(M,\tau_2)\times\mathbb{R}^\infty\to(M,\tau_2)$ *(or* $h:(M,\tau_2)\times Q^\infty\to(M,\tau_2)$*) is also a homeomorphism.*

COROLLARY 5 (Negligibility theorem). *If* A *is a Z-set in* (M,τ_1) *then there exists a homeomorphism* $h: M\backslash A\to M$ *in both topologies* τ_1 *and* τ_2.

Here a closed set A in a space X is said to be a *Z-set* if the identity map of X is arbitrarily close to maps $f: X\to X\backslash A$.

The main theorem and Corollaries 2 and 4 answer affirmatively to the questions for $(\mathbb{R}^\infty,\sigma)$-manifolds (or (Q^∞,Σ)-manifold) corresponding to [Ge, NLC9] (cf. [We, NLC15]). And Corollary 2 gives affirmative answers for the questions concerning a combinatorial ∞-manifold and LIP(X, Y).

Here we observe the topologies on the model spaces. One should note that $\bigcup_{n\in\mathbb{N}}\mathbb{R}^n$ has many different metric topologies. For instance, there are the relative topologies inherited from l_p, $p\in\mathbb{N}\cup\{\infty\}$. We denote these spaces by l_p^f. For each $n\in\mathbb{N}$, let $e_n\in\mathbb{R}^\omega$ be defined by $e_n(n)=1$ and $e_n(i)=0$ if $i\neq n$. Then e_n is convergent to 0 in σ but not in l_p^f for any p. If $p>q$, $\Sigma_{i=1}^{n^q} n^{-1}\cdot e_i$ is convergent to 0 in l_p^f but not in l_q^f. However it is well-known that all l_p^f are homeomorphic to σ. The main theorem asserts that these are homeomorphic to σ by a homeomorphism of $\mathbb{R}^\infty$ onto itself. It is also known that dir lim I^n and I_f^ω are homeomorphic to $\mathbb{R}^\infty$ and σ, respectively. But now it is shown that they are homeomorphic by a single homeomorphism. One should note that Q^∞ and Σ are represented by linear topological spaces. Let $\mathbb{R}_Q^\omega$ denote the space $\bigcup_{n\in\mathbb{N}} n\cdot Q$ with the relative topology inherited from $\mathbb{R}^\omega$. Then dir lim $n\cdot Q$ and $\mathbb{R}_Q^\omega$ are linear topological spaces which are homeomorphic to Q^∞ and Σ respectively. Now it is also shown that they are homeomorphic by a single homeomorphism. Let $Q_0=\Pi_{i\in\mathbb{N}}[-2^{-i},2^{-i}]$ be another expression of the Hilbert cube. Then Q_0 is contained in all l_p as a subspace. Let $l_p^{Q_0}$ denote the space $\bigcup_{n\in\mathbb{N}} n\cdot Q_0$ with the relative topology inherited from l_p. Finally we note that dir lim $n\cdot Q_0$ and $l_p^{Q_0}$ are homeomorphic to Q^∞ and Σ by a single homeomorphism.

2. Fine homotopy equivalence

Let f, $g: X\to Y$ be maps and $\mathcal{U}$ an open cover of Y. It is said that f is *$\mathcal{U}$-homotopic* to g and denoted by $f\simeq_{\mathcal{U}} g$ if there is a homotopy $h: X\times I\to Y$ such that $h_0=f$, $h_1=g$ and each $h(\{x\}\times I)$ is contained in some $U\in\mathcal{U}$. Such a homotopy h is called a *$\mathcal{U}$-homotopy*. In case Y is a metric space and mesh $\mathcal{U}<\varepsilon$, we say that f is *ε-homotopic* to g and write $f\simeq_\varepsilon g$. In this case, we call h an *ε-homotopy*. Observe that a homotopy h is an ε-homotopy if and only if diam $h(\{x\}\times I)<\varepsilon$ for each $x\in X$. Let $X_0\subset X$. We denote $f\simeq g$ rel X_0 when there is a homotopy $h: X\times I\to Y$ such that $h_0=f$, $h_1=g$ and $h_t|X_0=f|X_0$ for all $t\in I$.

A map $f: X \to Y$ is called a *fine homotopy equivalence* if for any open cover $\mathscr{U}$ of Y there is a map $g: Y \to X$ such that

$$f \circ g \simeq_{\mathscr{U}} \mathrm{id}_Y \quad \text{and} \quad g \circ f \simeq_{f^{-1}(\mathscr{U})} \mathrm{id}_X.$$

Such a map g is called a *$\mathscr{U}$-homotopy inverse* of f.

Let $(X_n)_{n\in\mathbb{N}}$ be a tower of subsets of a metric space $X=(X,d)$. We say that $(X_n)_{n\in\mathbb{N}}$ has the *mapping absorption property for compacta* in X if for any compactum $A \subset X$, $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a map $f: A \to X_m$ for some $m \geqslant n$ such that $f|A \cap X_n = \mathrm{id}$ and $d(f, \mathrm{id}) < \varepsilon$. The proof of the following lemma is similar to [Sa3, Theorem 1].

LEMMA 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a tower of compacta in an ANR*[2] *X with the mapping absorption property for compacta in X. Then $\varphi = \mathrm{id}_X$: dir lim $X_n \to X$ is a fine homotopy equivalence.*

Sketch of the proof. For any open cover $\mathscr{U}$ of X, let $\mathscr{V}$ be an open 2-nd star-refinement of $\mathscr{U}$. Then X is $\mathscr{V}$-homotopy dominated by a simplicial complex K, i.e., there are maps $f: X \to |K|_w$ and $g: |K|_w \to X$ such that $g \circ f \simeq_{\mathscr{V}} \mathrm{id}$. Using the mapping absorption property, we have a map $g': |K|_w \to X$ which is $\mathscr{V}$-homotopic to g and maps each simplex of K into some X_n, whence $g': |K|_w \to \operatorname{dir\,lim} X_n$ is a map. Thus we have a map $\psi = g' \circ f: X \to \operatorname{dir\,lim} X_n$ such that $\varphi \circ \psi \simeq_{st\mathscr{V}} \mathrm{id}_X$. Let $h: X \times I \to X$ be a $(st\mathscr{V})$-homotopy such that $h_0 = \mathrm{id}_X$ and $h_1 = \varphi \circ \psi$. Again using the mapping absorption property, we have maps $h'_n: X_n \times I \to X$, $n \in \mathbb{N}$, such that $h'_n|X_{n-1} \times I = h'_{n-1}$, $h'_n|X_n \times \{0,1\} = h|X_n \times \{0,1\}$, $h'_n \simeq_{\mathscr{V}} h|X_n \times I$ and h'_n maps $X_n \times I$ into some X_m. These induce a $\mathscr{U}$-homotopy $h': (\operatorname{dir\,lim} X_n) \times I \to \operatorname{dir\,lim} X_n$ such that $h'_0 = \mathrm{id}$ and $h'_1 = \psi \circ \varphi$. □

As a consequence of Lemma 1, we prove the following:

LEMMA 2. *Let $(X_n)_{n\in\mathbb{N}}$ be a tower of compact ANR's in an ANR X with $X = \bigcup_{n\in\mathbb{N}} X_n$. Then* id: dir lim $X_n \to X$ *is a fine homotopy equivalence.*

Proof. By Lemma 1, it suffices to show that $(X_n)_{n\in\mathbb{N}}$ has the mapping absorption property for compacta in X. To this end, let $A \subset X$ be a compactum, $\varepsilon > 0$ and $n \in \mathbb{N}$. Put $\varepsilon_{n-1} = \varepsilon$. Since each X_i is a compact ANR, we can inductively choose $\varepsilon_i > 0$, $i \geqslant n$, so that for any two maps $f, g: Z \to X_i$ defined on an arbitrary space Z if $d(f,g) < \varepsilon_i$ and $f|Z_0 = g|Z_0$ for some $Z_0 \subset Z$ then $f \simeq_{\varepsilon_{i-1}/3} g$ rel. Z_0 [Du, Corollary 2.2, Theorem 2.5]. Let N_i be a closed neighborhood of X_i in X and $r_i: N_i \to X_i$ a retraction of N_i onto X_i such that $d(r_i, \mathrm{id}) < \varepsilon_i/3$. From compactness, $A \subset \bigcup_{i=n}^m N_n$ for some $m \geqslant n$. For each $i = n, \ldots, m$, let $A_i = A \cap N_i$ and $\tilde{A}_i = \bigcup_{j=1}^m A_j$, and let $A_0 = A \cap X_n$. Then

[2] ANR = absolute neighborhood retract for metrizable spaces.

$A_0 \subset A_i$ and $r_i | A_0 = \mathrm{id}$ for each $i = n, \ldots, m$. Observe

$$r_m | \tilde{A}_m \cap A_{m-1} \simeq_{\varepsilon_{m-2}/3} r_{m-1} | \tilde{A}_m \cap A_{m-1} \text{ rel } A_0.$$

By the homotopy extension theorem [vM, Theorem 5.1.3], $r_m | A_m \cap A_{m-1}$ extends to a map $r'_{m-1}: A_{m-1} \to X_{m-1}$ such that

$$r'_{m-1} \simeq_{\varepsilon_{m-2}/3} r_{m-1} \text{ rel } A_0.$$

We extend r_m to a map $\tilde{r}_{m-1}: \tilde{A}_{m-1} \to X_m$ by $\tilde{r}_{m-1} | A_{m-1} = r'_{m-1}$. Then $d(\tilde{r}_{m-1}, \mathrm{id}) < 2\varepsilon_{m-2}/3$ and $\tilde{r}_{m-1} | A_0 = \mathrm{id}$. Observe

$$\tilde{r}_{m-1} | \tilde{A}_{m-1} \cap A_{m-2} \simeq_{\varepsilon_{m-3}/3} r_{m-2} | \tilde{A}_{m-1} \cap A_{m-2} \text{ rel } A_0.$$

Similarly as above, $\tilde{r}_{m-1}$ extends to a map $\tilde{r}_{m-2}: \tilde{A}_{m-2} \to X_m$ such that $d(\tilde{r}_{m-2}, \mathrm{id}) < 2\varepsilon_{m-3}/3$ and $\tilde{r}_{m-2} | A_0 = \mathrm{id}$. By induction, we obtain a map $\tilde{r}_n: A = \tilde{A}_n \to X_m$ such that $d(\tilde{r}_n, \mathrm{id}) < 2\varepsilon_{n-1}/3 < \varepsilon$ and $\tilde{r}_n | A \cap X_n = \tilde{r}_n | A_0 = \mathrm{id}$. □

Under the same assumption (∗) as corollaries in the Introduction we have the following:

COROLLARY 6. *The identity* $\mathrm{id}: (M, \tau_2) \to (M, \tau_1)$ *is a fine homotopy equivalence.*

Proof. Since (M, τ_2) is an $\mathbb{R}^\infty$-manifold (or a Q^∞-manifold), $(M, \tau_2) = \operatorname{dir} \lim X_n$ for some tower $(X_n)_{n \in \mathbb{N}}$ of compact ANR's [Sa2, Proposition 1.8] (or the proof of [He2, Theorem 4]). Since (M, τ_1) is an ANR, id is a fine homotopy equivalence by Lemma 2. □

A closed set A in a space X is said to be a *strong Z-set* if the identity map of X is arbitrarily close to maps $f: X \to X$ such that $A \cap \operatorname{cl} f(X) = \phi$. It is well-known that each compact set in a σ-manifold (or a Σ-manifold) is a strong Z-set. Combining Corollary 6 with [Sa5, Proposition 1 (or 1′)], we have the following:

PROPOSITION. *Let M be an $\mathbb{R}^\infty$-manifold (or a Q^∞-manifold) and d a continuous metric on M. Then (M, d) is a σ-manifold (or a Σ-manifold) if and only if* $\mathrm{id}: M \to (M, d)$ *is a fine homotopy equivalence and each compact set in (M, d) is a strong Z-set.*

3. The proof of main theorem

We prove the main theorem by the method employed in [Sa2]. To this end, we modify Corollary 6 as follows:

LEMMA 3. *Under the assumption* $(*)$, *let* $A \subset B \subset M$ *such that* A *is* τ_2-*compact*, B *is* τ_1-*compact and finite dimensional* (*there is no dimensional assumption in the* (Σ, Q^∞)-*case*). *Then for any* τ_1-*open cover* $\mathscr{U}$ *of* M, $\mathrm{id}: (M, \tau_2) \to (M, \tau_1)$ *has a* $\mathscr{U}$-*homotopy inverse map* $h: (M, \tau_1) \to (M, \tau_2)$ *such that* $h \mid A = \mathrm{id}$ *and* $h \mid B$ *is an embedding.*

Proof. For simplicity, we write $X = (M, \tau_1)$ and $X_\infty = (M, \tau_2)$. By Corollary 6, $\varphi = \mathrm{id}: X_\infty \to X$ is a fine homotopy equivalence. Let $f: X \to X_\infty$ be a $\mathscr{V}$-homotopy inverse of φ, where $\mathscr{V}$ is an open star-refinement of $\mathscr{U}$. Then

$$f \mid A = f \circ \varphi \mid A \simeq_{\varphi^{-1}(\mathscr{V})} \mathrm{id}.$$

Since X_∞ is an absolute neighborhood extensor for metrizable spaces, we can apply the homotopy extension theorem to obtain a map $g: X \to X_\infty$ such that $g \mid A = \mathrm{id}$ and $g \simeq_{\varphi^{-1}(\mathscr{V})} f$. By [Sa2, Lemma 1.5], we have an embedding $j: B \to X_\infty$ such that $j \mid A = g \mid A = \mathrm{id}$ and $j \simeq_{\varphi^{-1}(\mathscr{V})} g \mid B$. Again using the homotopy extension theorem, we extend j to a map $h: X \to X_\infty$ such that $h \simeq_{\varphi^{-1}(\mathscr{V})} g$. Then $h \mid A = j \mid A = \mathrm{id}$, $h \mid B = j$ is an embedding and

$$\varphi \circ h \simeq_{\mathscr{V}} \varphi \circ g \simeq_{\mathscr{V}} \varphi \circ f \simeq_{\mathscr{V}} \mathrm{id}_X$$

and

$$h \circ \varphi \simeq_{\varphi^{-1}(\mathscr{V})} g \circ \varphi \simeq_{\varphi^{-1}(\mathscr{V})} f \circ \varphi \simeq_{\varphi^{-1}(\mathscr{V})} \mathrm{id}_{X_\infty}.$$

In other words, h is a $\mathscr{U}$-homotopy inverse of φ. □

Proof of main theorem. For simplicity, we write $X = (M_1, \tau_1^1)$, $X_\infty = (M_1, \tau_2^1)$, $Y = (M_2, \tau_1^2)$, $Y_\infty = (M_2, \tau_2^2)$, $\varphi = \mathrm{id}: X_\infty \to X$ and $\psi = \mathrm{id}: Y_\infty \to Y$ and we denote both admissible metrics for X and Y by the same d. We can write $X_\infty = \operatorname{dir\,lim} X_n$ and $Y_\infty = \operatorname{dir\,lim} Y_n$, where each X_n and Y_n are compact and finite dimensional (there is no dimensional restriction in the (Q^∞, Σ)-case). Note that X_n and Y_n are compact in both topologies. We may assume that X_1 is a single point. Recall that any compact set in a σ-manifold (or a Σ-manifold) is a Z-set. Since X and Y are σ-manifolds (or Σ-manifolds) having the same homotopy type, there is a homeomorphism $f: X \to Y$ [Ch, Theorem 2.2].

Put $m(1) = 1$ and $f_1 = f$. Since $X_{m(1)}$ is a single point, $f_1(X_{m(1)}) \subset Y_{n(1)}$ for some $n(1) \in \mathbb{N}$. Let $\mathscr{U}_1$ be an open cover of M such that mesh $f_1(\mathscr{U}_1) < 2^{-2}$. By Lemma 3, φ has a $\mathscr{U}_1$-homotopy inverse j_1 such that $j_1 \mid X_{m(1)} = \mathrm{id}$ and $j_1 \mid f_1^{-1}(Y_{n(1)})$ is an embedding. Then $j_1 \circ f_1^{-1}(Y_{n(1)}) \subset X_{m(2)}$ for some $m(2) > m(1)$. Since $\varphi \circ j_1 \circ f_1^{-1} \mid Y_{n(1)}$ and $f_1^{-1} \mid Y_{n(1)}$ are $\mathscr{U}_1$-homotopic Z-embeddings, we can apply the homeomorphism extension theorem [Ch, Theorem 2.25][(3)] to obtain

[(3)]The covering estimation of [Ch] was improved (cf. [Sa1]).

a homeomorphism $g_1: Y \to X$ such that $g_1 | Y_{n(1)} = \varphi \circ j_1 \circ f_1^{-1} | Y_{n(1)}$ and $g_1 \simeq_{\mathscr{U}_1} f_1^{-1}$. Then $g_1(Y_{n(1)}) \subset X_{m(2)}$ and $g_1 \circ f_1 | X_{m(1)} = \varphi \circ j_1 | X_{m(1)} = \mathrm{id}$. Since $g_1 \circ f_1$ is $\mathscr{U}_1$-near to id_X, $d(f_1, g_1^{-1}) = d(f_1 \circ g_1, \mathrm{id}_Y) < 2^{-2}$.

Let $\mathscr{V}_2$ be an open cover of Y such that $st\, \mathscr{V}_2 < \mathscr{V}_1$, mesh $\mathscr{V}_2 < 2^{-2}$ and mesh $g_1(\mathscr{V}_2) < 2^{-3}$. By Lemma 3, $\psi: Y_\infty \to Y$ has a $\mathscr{V}_2$-homotopy inverse k_2 such that $k_2 | Y_{n(1)} = \mathrm{id}$ and $k_2 | g_1^{-1}(X_{m(2)})$ is an embedding. Then $k_2 \circ g_1^{-1}(X_{m(2)}) \subset Y_{n(2)}$ for some $n(2) > n(1)$. Since $\psi \circ k_2 \circ g_1^{-1} | X_{m(2)}$ and $g_1^{-1} | X_{m(2)}$ are $\mathscr{V}_2$-homotopic Z-embeddings, we can obtain a homeomorphism $f_2: X \to Y$ such that $f_2 | X_{m(2)} = \psi \circ k_2 \circ g_1^{-1} | X_{m(2)}$ and $f_2 \simeq_{\mathscr{V}_2} g_1^{-1}$, so $d(f_2, g_1^{-1}) < 2^{-2}$. Then $f_2(X_{m(2)}) \subset Y_{n(2)}$, $f_2 \circ g_1 | Y_{n(1)} = \psi \circ k_1 | X_{m(1)} = \mathrm{id}$ and $d(g_1, f_2^{-1}) = d(g_1 \circ f_2, \mathrm{id}) < 2^{-3}$. Moreover

$$d(f_1, f_2) \leqslant d(f_1, g_1^{-1}) + d(f_2, g_1^{-1}) < 2^{-1}.$$

Let $\mathscr{U}_2$ be an open cover of X such that mesh $\mathscr{U}_2$, mesh $f_2(\mathscr{U}_2) < 2^{-3}$. By Lemma 3, $\varphi: X_\infty \to X$ has a $\mathscr{U}_2$-homotopy inverse j_2 such that $j_2 | X_{m(2)} = \mathrm{id}$ and $j_2 | f_2^{-1}(Y_{n(2)})$ is an embedding. Then $j_2 \circ f_2^{-1}(Y_{n(2)}) \subset X_{m(3)}$ for some $m(3) > m(2)$. Since $\varphi \circ j_2 \circ f_2^{-1} | Y_{n(2)}$ and $f_2^{-1} | Y_{n(2)}$ are $\mathscr{U}_2$-homotopic Z-embeddings, we can obtain a homeomorphism $g_2: Y \to X$ such that $g_2 | Y_{n(2)} = \varphi \circ j_2 \circ f_2^{-1} | Y_{n(2)}$ and $g_2 \simeq_{\mathscr{U}_2} f_2^{-1}$. Then $g_2(Y_{n(2)}) \subset X_{m(3)}$, $g_2 \circ f_2 | X_{m(2)} = \varphi \circ j_2 | X_{m(2)} = \mathrm{id}$ and $d(f_2, g_2^{-1}) = d(f_2 \circ g_2, \mathrm{id}_Y) < 2^{-3}$. Moreover

$$d(g_1, g_2) \leqslant d(g_1, f_2^{-1}) + d(g_2, f_2^{-1}) < 2^{-2}.$$

Thus by induction, we can obtain homeomorphisms $f_i: X \to Y$, $g_i: Y \to X$, $i \in \mathbb{N}$, and sequences $m(1) < m(2) <, \ldots, n(1) < n(2) < \ldots$ such that

$$\begin{aligned}
&d(f_i, f_{i+1}) < 2^{-i}, \quad g(g_i, g_{i+1}) < 2^{-i},\\
&f_{i+1} | X_{m(i)} = f_i | X_{m(i)}, \quad f_i(X_{m(i)}) \subset Y_{n(i)},\\
&g_{i+1} | Y_{n(i)} = g_i | Y_{n(i)}, \quad g_i(Y_{n(i)}) \subset X_{m(i+1)},\\
&g_i \circ f_i | X_{m(i)} = \mathrm{id} \quad \text{and} \quad f_{i+1} \circ g_i | Y_{n(i)} = \mathrm{id}.
\end{aligned}$$

Define maps $f_\infty: X_\infty \to Y_\infty$ and $g_\infty: Y_\infty \to X_\infty$ by $f_\infty | X_{m(i)} = f_i | X_{m(i)}$ and $g_\infty | Y_{n(i)} = g_i | Y_{n(i)}$. Then $f_\infty \circ g_\infty = \mathrm{id}$ and $g_\infty \circ f_\infty = \mathrm{id}$. Hence, $f_\infty: X_\infty \to Y_\infty$ is a homeomorphism. Since

$$d(f_n, f_\infty) \leqslant \sum_{i \geqslant n} d(f_i, f_{i+1}) < \sum_{i \geqslant n} 2^{-i} = 2^{-n+1},$$

f_n is uniformly convergent to f_∞. Hence, the same map $f_\infty: X \to Y$ is continuous in the metric topology. Similarly, $g_\infty: Y \to X$ is also continuous. In other words, $f_\infty: X \to Y$ is also a homeomorphism. □

REMARKS. In the proof, f_∞ can be taken arbitrarily close to f. In fact, for any open cover $\mathscr{U}$ of Y, Y has an admissible metric d such that

$$\{\{y \in Y \mid d(x, y) < 1\} \mid x \in Y\} < \mathscr{U}.$$

Using this metric, f_∞ is $\mathscr{U}$-near to f since $d_Y(f, f_\infty) < 2^0 = 1$. As a consequence, the homeomorphism of the stability theorem (Corollary 4) can be taken arbitrarily close to the projection and the one of the negligibility theorem (Corollary 5) can be taken arbitrarily close to the inclusion with respect to τ_1.

Furthermore, if $A \subset M_1$ is τ_2^1-compact and $f(A) \subset M_2$ is τ_2^2-compact, then f_∞ can be taken to satisfy $f_\infty \mid A = f \mid A$. In fact, we can assume $X_1 = A$ in the above argument, whence $f_\infty \mid A = f \mid A$. By the homeomorphism extension theorem for σ-manifolds (or Σ-manifolds), we have the following corollary under the same assumption (*) stated in the Introduction:

COROLLARY 7 (Homeomorphism extension theorem). *Let $f: A \to B$ be a homeomorphism between τ_2-compact sets in M. If f is homotopic to* id *in (M, τ_1), then f extends to $\tilde{f}: M \to M$ which is a homeomorphism in both topologies. If f is $\mathscr{U}$-homotopic to* id *for a τ_1-open cover $\mathscr{U}$ of M, $\tilde{f}$ is $\mathscr{U}$-near to* id.

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