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# Equisingular deformations below the Newton boundary

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## 1. Introduction

Let $(V, 0) \subseteq (\mathbb{A}_k^3, 0)$ be a normal 2-dimensional singularity ($k$ = fixed algebraically closed field with $\operatorname{char} k = 0$). Suppose an embedded good resolution

$$\begin{array}{ccccc} \bigcup_v D_v = D & \hookrightarrow & X & \xrightarrow{\pi} & (\mathbb{A}_k^3, 0) \\ \uparrow & & \uparrow & & \uparrow \\ \bigcup_v E_v = E & \hookrightarrow & Y & \xrightarrow{\pi} & (V, 0) \end{array}$$

($D_v$, $Y$ are smooth and transversal; $E_v := Y \cap D_v$) to be given, then the following functors on $\mathscr{C}$ (category of local Artinian algebras of finite type over $k$) are defined:

$$\begin{aligned} \mathrm{ESE}_X(A \in \mathscr{C}) := \{&\text{isomorphism classes of } A\text{-deformations } \tilde{Y}, \tilde{D}_v \hookrightarrow \tilde{X} \\ &\text{of } Y, D_v \hookrightarrow X \text{ such that } \tilde{X} \text{ blows down to } (\mathbb{A}_k^3, 0)\} \\ \mathrm{ES}_Y(A \in \mathscr{C}) := \{&\text{isomorphism classes of } A\text{-deformations } \tilde{E}_v \hookrightarrow \tilde{Y} \text{ of} \\ &E_v \hookrightarrow Y \text{ such that } \tilde{Y} \text{ blows down to } (V, 0)\}. \end{aligned}$$

In an obvious way we get natural transformations $\mathrm{ESE}_X \xrightarrow{\alpha} \mathrm{ES}_Y \xrightarrow{\beta} \mathrm{Def}_V$, and following Wahl [Wa2] the deformations in the image of $\beta$ (denoted by $\overline{\mathrm{ES}}$) will be called "equisingular" (this notion does not depend on the choice of the resolution $\pi$).

(1.1) Are there suitable embedded resolutions $X$ such that all equisingular deformations are induced by the corresponding functor ESE?

In case of *plane curve singularities* Wahl used the embedded resolution obtained by successive blowing ups of $(\mathbb{A}^2, 0)$. In his paper [Wa1] he showed

$$\mathrm{ES}(A) = \frac{\mathrm{ESE}(A)}{A\text{-automorphisms of } (\mathbb{A}_k^2, 0) \times_k A},$$

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in particular the natural transformation $\alpha: \mathrm{ESE} \to \mathrm{ES}$ is surjective for all $A \in \mathscr{C}$. (Note the different notations: In [Wa1] the functors ESE and ES are denoted by ES and $\overline{\mathrm{ES}}$, respectively.)

If, in *dimension* 2, the singularity $(V, 0) \subseteq (\mathbb{A}_k^3, 0)$ is given by a polynomial $f$ that is non-degenerate on the Newton boundary $\Gamma(f)$, we can use embedded resolutions given by subdividing the dual Newton polyhedron $\Sigma_0$ (cf. [Va]). Under the additional assumption that *each of the coordinate axes of* $\mathbb{R}^3$ *meets exactly one top dimensional face of* $\Gamma(f)$, the existence of a "good" subdivision $\Sigma < \Sigma_0$ making $\alpha: \mathrm{ESE}_{X_\Sigma} \to \mathrm{ES}_{Y_\Sigma}$ surjective on $\mathscr{C}$ is shown in [Al], (5.4).

In sections 2 and 3 of this paper we will show for *arbitrary non-degenerate polynomials* $f$, that any equisingular first order deformation $\xi$ of $(V, 0)$ can be split into equisingular parts $\xi = \xi_1 + \xi_2 + \xi_3$ such that each of the $\xi_i$ is induced by an equisingular deformation of an embedded resolution $\pi_i: X_i \to (\mathbb{A}_k^3, 0)$ $(i = 1, 2, 3)$ (cf. Theorem (3.4)).

This means we can give a support to the above question on the infinitesimal level in the following sense:

There are three embedded resolutions, one for each coordinate axis, such that all equisingular deformations are induced by the corresponding functors $\mathrm{ESE}_i$ $(i = 1, 2, 3)$ together.

(1.2) In section 4 we fix an arbitrary smooth subdivision $\Sigma < \Sigma_0$ and compute the image $\mathrm{Im}(\mathrm{ESE}_{X_\Sigma}(k[\varepsilon]) \to \mathrm{Def}_R(k[\varepsilon]))$ (cf. Proposition (4.6)). This together with Theorem (3.4) imply our main result – an algorithm for computing all equisingular first-order deformations in $\mathrm{Def}_R(k[\varepsilon])$ (cf. Theorem (5.1)). None of the smooth subdivisions $\Sigma < \Sigma_0$, but only the starting f.r.p.p. decomposition $\Sigma_0$ itself is used there, hence, this algorithm seems to be an easy method to determine $\overline{\mathrm{ES}}(k[\varepsilon])$ by computers. In particular, for each equation $f$ we can decide if there are equisingular deformations below $\Gamma(f)$ or not.

Finally, an example is given in (5.3).

## 2. Resolutions related to the coordinate axes of $\mathbb{R}^3$

(2.1) We will use the notations of [Al]:

Let $M := \mathbb{Z}^3$, $N := \mathbb{Z}^3$ which are regarded as being dual to each other by the canonical pairing $\langle\ ,\ \rangle$. Let $f \in (\mathbf{x})^2 k[\mathbf{x}]$ $(\mathbf{x} = (x_1, x_2, x_3))$ be an irreducible, complete polynomial (i.e. $f(0, \ldots, x_i, \ldots, 0) \not\equiv 0$ for $i = 1, 2, 3$); suppose moreover, that $f$ is non-degenerate on its Newton boundary $\Gamma(f) \subseteq M_{\mathbb{R}}$.

The dual of $\Gamma(f)$ gives a finite rational partial polyhedral (f.r.p.p.) decomposition $\Sigma_0 < \mathbb{R}^3_{\geqslant 0} \subseteq N_{\mathbb{R}}$. We consider this only by regarding the intersection

$$\Delta := \mathbb{R}^3_{\geqslant 0} \cap \left\{(a_1, a_2, a_3) \in \mathbb{R}^3_{\geqslant 0} \middle/ \sum_{i=1}^{3} a_i = 1\right\} \subseteq N_{\mathbb{R}};$$

however, rational elements of $\Delta$ will always be given with integer coordinates that are relatively prime. The vertices of $\Delta$ are in a 1 to 1-correspondence to the coordinate functions $x_1, x_2, x_3$ and will be denoted by $e^1$, $e^2$ and $e^3$, respectively.

Subdivisions of $\Sigma_0$ will always be considered not to change the boundary $\partial\Delta$. Finally, we denote by $\Omega\langle D\rangle$, $\Omega\langle D + Y\rangle$, $\Theta\langle -D\rangle$, $\Theta\langle -D - Y\rangle$ the corresponding sheaves with logarithmic poles or their duals. (In [Al] the latter two were called $S'$ and $S$, respectively.)

(2.2) DEFINITION. Let $a, b \in \Delta \cap \mathbb{Q}^3$. We denote by $P_b(a)$ the point of $\Delta \cap \mathbb{Q}^3$ characterized by the following conditions:

(1) $P_b(a) \in \overline{ab}$.
(2) If $P_b(a)$ is given by integer coordinates that are relatively prime, then $\{P_b(a), b\}$ will be a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^3 \subseteq N_{\mathbb{R}}$.
(3) The distance between $a$ and $P_b(a)$ is minimal.

REMARK. (i) Denote by

$$d := \det(a, b) := \gcd\left(\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \Big/ 1 \leqslant i < j \leqslant 3\right)$$

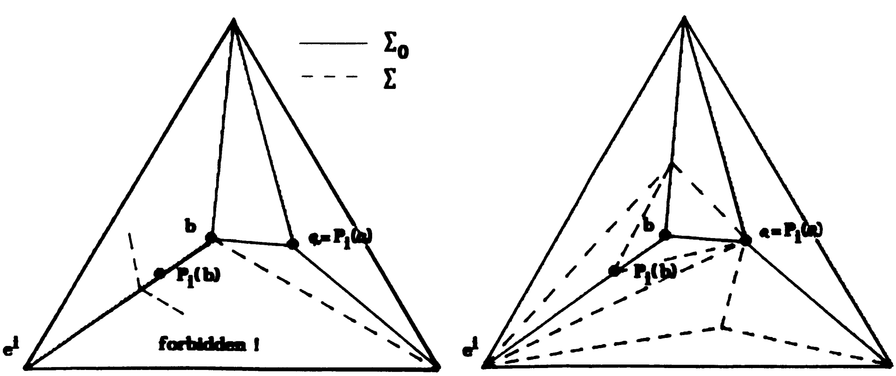
the "volume" of the corresponding cone $\langle a, b\rangle \subseteq \mathbb{R}^3$. Then, $P_b(a)$ is given by

$$P_b(a) := \frac{a + k \cdot b}{d} \in \Delta \cap \mathbb{Q}^3 \quad \text{with } 0 \leqslant k < d$$

$$d | a_i + k \cdot b_i \quad \text{for all } i.$$

(ii) As in [Al] (5.2) we abbreviate $P_{e^i}(a)$ by $P_i(a)$.

(2.3) DEFINITION. A smooth f.r.p.p. subdivision $\Sigma < \Sigma_0$ is called "good for $e^i$" if the following condition is satisfied: Whenever an element $a \in \Sigma_0^{(1)}$ (i.e. a vertex of the $\Sigma_0$-partition of $\Delta$) is contained, with $e^i$, in a common cone $\alpha \in \Sigma_0$, then there will be a cone $\tilde{\alpha} \in \Sigma$ containing $P_i(a)$ and $e^i$.




good for $e^i$

REMARK. In section 5 of [Al] the property of $\Sigma$ to be good for all $e^i$ ($i = 1, 2, 3$) is called "good" and is shown to be sufficient for making the natural transformation $\mathrm{ESE}_{X_\Sigma} \to \mathrm{ES}_{Y_\Sigma}$ smooth. Nevertheless, in most of the cases drawn above (i.e. with non-simplicial "boundary-cones" of $\Sigma_0$) such subdivisions $\Sigma$ do not exist. (There are exceptions: In Example (5.3) the fact $P_2(a) \in \overline{be^3}$ yields good subdivisions of $\Sigma_0$.)

(2.4) Let $i \in \{1, 2, 3\}$ be fixed. Now, we give a construction of a $\Sigma_i < \Sigma_0$ which will be good for $e^i$:

STEP 1. Connecting $e^i$ with all $a \in \Sigma_0^{(1)}$ contained, with $e^i$, in a common $\Sigma_0$-cone. We get a new $\tilde{\Sigma}_0$ with the 2-skeleton $\tilde{\Sigma}_0^{(2)}$.

STEP 2. Dividing all line segments $\overline{ab} \in \tilde{\Sigma}_0^{(2)}$ into the "canonical primitive sequence" (cf. [O], (3.5)(i)): $b^0 := b$;

$$b^{\nu+1} := P_{b^\nu}(a) \quad (\nu \geq 0)$$

stops with $b^N = a$. (If $\overline{ab}$ was already smooth, then $b^1 = P_b(a) = a$. Hence, in difference to [O], nothing is to do with that line segment.) We obtain a new $\tilde{\tilde{\Sigma}}_0^{(2)}$.

STEP 3. By a non-canonical division of the 3-dimensional cones of $\tilde{\Sigma}_0$ we obtain a smooth subdivision $\Sigma_i < \Sigma_0$ with $\Sigma_i^{(2)} \cap |\Sigma_0^{(2)}| = \tilde{\tilde{\Sigma}}_0^{(2)} \cap |\Sigma_0^{(2)}|$.

Let $\Sigma' := \Sigma_i^{(2)} \cap |\Sigma_0^{(2)}|$ (that means the actual subdivision of the 2-skeleton of $\Sigma_0$); this f.r.p.p.-decomposition does not depend on $i \in \{1, 2, 3\}$!

Denoting by $X_i := X_{\Sigma_i}$ and $W := X_{\Sigma'}$ the corresponding torus embeddings after a flat base change $\times_{\mathbb{A}_k^3} (\mathbb{A}_k^3, 0)$, the following geometric situation results:

$$\begin{array}{ccccc} W & \xrightarrow[\subset]{\text{open}} & X_i & \xrightarrow{\pi_i} & (\mathbb{A}_k^3, 0) \\ & & \uparrow & & \uparrow \\ & & Y_i & \xrightarrow[\text{(strict transform)}]{\pi_i} & (V, 0) \end{array}$$

are embedded resolutions of $(V, 0) \subseteq (\mathbb{A}_k^3, 0)$ for $i = 1, 2, 3$.

(2.5) PROPOSITION. *Whenever $\Sigma < \Sigma_0$ is a smooth subdivision with $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$, the strict transform $Y_\Sigma \subseteq X_\Sigma$ of $(V, 0) \subseteq (\mathbb{A}_k^3, 0)$ is contained in $W = X_{\Sigma'}$; therefore, all such resolutions $\pi\colon Y_\Sigma \to (V, 0)$ coincide and will be denoted by $\pi\colon Y^{\#} \to (V, 0)$.*

*In particular, the above three resolutions $\pi_i\colon Y_i \to (V, 0)$ are equal to $Y^{\#}$.*

*Proof.* STEP 1. There is a 1 to 1-correspondence between the top dimensional cones $\beta \in \Sigma_0$ and the vertices $r^\beta \in \Gamma(f)$:

$$\langle a, r^\beta \rangle = m(a) := \min_{r \in \Gamma_+(f)} \langle a, r \rangle \text{ iff } a \in \bar{\beta}.$$

Now, let $\Sigma < \Sigma_0$ be an arbitrary smooth subdivision, and let $\alpha \in \Sigma$ be an arbitrary cone ($\alpha \nsubseteq \partial\Delta$); we fix the following notations:

$$\begin{array}{ccccc} \alpha & < & \alpha'(\in\Sigma) & \subseteq \beta(\in\Sigma_0) & \\ \| & & \| & & \\ \langle a^1,\dots,a^k\rangle & < & \langle a^1, a^2, a^3\rangle & & (1 \leqslant k \leqslant 3). \end{array}$$

Then, $X_{\alpha'} \subseteq X_\Sigma$ is an open subset with the coordinates $y_1, y_2, y_3$, and we will look for the condition for $Y_\Sigma$ to meet the closed orbit $\overline{\text{orb}(\alpha)} \subseteq X_\Sigma$:

$(\overline{\text{orb}(\alpha)} \cap X_{\alpha'}) \cap Y_\Sigma \neq \varnothing$

iff there is a triple $(y_1, y_2, y_3) \in Y_\Sigma$ with $y_1 = \dots = y_k = 0$

iff there is a non-trivial monomial $y_{k+1}^{c_{k+1}} \cdots y_3^{c_3}$ in the equation $f_\beta(\mathbf{y}) := \Pi_{i=1}^3 y_i^{-m(a^i)} f(\mathbf{x}) = \mathbf{x}^{-r^\beta} f(\mathbf{x})$ of $Y_\Sigma$ in $X_{\alpha'}$

iff there exists an $r \in \operatorname{supp} f$ (different from $r^\beta$) with $\langle a, r\rangle = \langle a, r^\beta\rangle = m(a)$ for all $a \in \bar{\alpha}$

iff there exists a vertex $r^\gamma \in \Gamma(f)$ ($\beta \neq \gamma$) with $\langle a, r^\beta\rangle = \langle a, r^\gamma\rangle = m(a)$ for all $a \in \bar{\alpha}$.

The latter condition does not depend on the special choice of $\alpha'$.
Finally, we obtain:

$\overline{\text{orb}(\alpha)} \cap Y_\Sigma \neq \varnothing$ iff $\alpha$ is contained in two different top dimensional $\Sigma_0$-cones.

STEP 2. If $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$, we obtain

$\overline{\text{orb}(\alpha)} \cap Y_\Sigma = \varnothing$ for all $\alpha \nsubseteq |\Sigma_0^{(2)}|$, i.e. $\alpha \notin \Sigma'$.

In particular, $W = X_\Sigma \setminus \bigcup_{\alpha\in\Sigma\setminus\Sigma'} \overline{\text{orb}(\alpha)}$ contains $Y_\Sigma$. □

REMARK. As we have seen in (2.4), $Y^{\#}$ arises after a canonical subdivision of $\Sigma_0^{(2)}$. This resolution is the minimal one in the category of resolutions $Y_\Sigma$ obtained by smooth subdivisions $\Sigma < \Sigma_0$.

*Proof.* By induction only the following has to be checked:
Is $a = a^0, \dots, a^N = b$ any smooth subdivision (i.e. $\det(a^\nu, a^{\nu+1}) = 1$) of a line segment $\overline{ab}$, then $P_b(a)$ is among the elements $a^\nu$ ($\nu = 0, \dots, N$).
We can assume $b = e^1$; let $d := \gcd(a_2, a_3)$.

Now, any point $c$ of the line $ae^1$ (in $\Delta \cap \mathbb{Q}^3$) can be written as

$$c\left(\frac{p}{q}\right) = c(p, q) = \left(p, q\cdot\frac{a_2}{d}, q\cdot\frac{a_3}{d}\right) \quad \text{with } p, q \in \mathbb{N}; (p, q) = 1,$$

and this notation admits the following properties:

(1) $c\colon \mathbb{Q}_{\geqslant 0} \cup \{\infty\} \to ae^1$ is order preserving
(2) $c(\infty) = c(1, 0) = e^1$; $c(0) = c(0, 1)$ is the intersection of $ae^1$ with $\langle e^2, e^3\rangle$
(3) $a = c(a_1/d)$; and $c(r)$ belongs to the line segment $\overline{ae^1}$ if and only if $r \geqslant a_1/d$
(4) $c(p_1/q_1)$, $c(p_2/q_2)$ yield a smooth segment if and only if

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \pm 1 \quad \left(\text{i.e. } \frac{p_1}{q_1} - \frac{p_2}{q_2} = \pm\frac{1}{q_1 q_2}\right).$$

(5) $c(r)$, $e^1$ yield a smooth segment if and only if $r \in \mathbb{Z}$; $P_1(a) = c([a_1/d] + 1)$.

Let $a^\nu = c(p_\nu/q_\nu)$, $a^{\nu+1} = c(p_{\nu+1}/q_{\nu+1})$ be adjacent elements in the above smooth subdivision of $\overline{ae^1}$. Then, by (4) the open interval $(p_\nu/q_\nu, p_{\nu+1}/q_{\nu+1})$ can not contain any integer $g$

$$\left(\frac{p_{\nu+1}}{q_{\nu+1}} - \frac{p_\nu}{q_\nu} = \left(\frac{p_{\nu+1}}{q_{\nu+1}} - g\right) + \left(g - \frac{p_\nu}{q_\nu}\right)\right.$$
$$\left. \geqslant \frac{1}{q_{\nu+1}} + \frac{1}{q_\nu} > \frac{1}{q_\nu q_{\nu+1}} \quad \text{would give a contradiction}\right);$$

in particular, we obtain $[a_1/d] + 1 \notin (p_\nu/q_\nu, p_{\nu+1}/q_{\nu+1})$. □

(2.6) Finally, we want to state here the most important property of resolutions that are good for $e^i$.

Let $\Sigma < \Sigma_0$ be a smooth subdivision, then by section 5 of [Al] we get the following diagram (with exact rows and columns)

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
\mathrm{ESE}_X(k[\varepsilon]) & \xrightarrow{\alpha} & \mathrm{ES}_Y(k[\varepsilon]) & & & & \\
\| & & \downarrow & & & & \\
H^1(X, \Theta_X\langle -D-Y\rangle) & \longrightarrow & H^1(Y, \Theta_Y\langle -E\rangle) & \xrightarrow{\psi} & H^2(X, \Theta_X\langle -D\rangle(-Y)) & \longrightarrow & 0 \\
 & & {\scriptstyle\varphi}\downarrow{\scriptstyle \oplus_j x_j} & & {\scriptstyle\Phi}\downarrow{\scriptstyle \oplus_j x_j} & & \\
 & & H^1(Y, \mathcal{O}_Y^3) & = & H^2(X, \mathcal{O}_X(-Y)^3) & &
\end{array}$$

($k[\varepsilon]$ denotes the object $k[\varepsilon] := k[\varepsilon]/\varepsilon^2$ of $\mathscr{C}$).

The sheaf $\Theta_X\langle -D\rangle$ as well as the map $\Phi$ split into

$$\Theta_X\langle -D\rangle = \bigoplus_{j=1}^{3} S_j \quad \text{with } S_j := \Theta_X\langle -D\rangle \cap \mathcal{O}_{\text{torus}} \frac{\partial}{\partial x_j}$$

(denoted by $S'_j$ in [Al])

and

$$\Phi = \bigoplus_{j=1}^{3} \Phi_j \quad \text{with } \Phi_j : H^2(X, S_j(-Y)) \xrightarrow{x_j} H^2(X, \mathcal{O}_X(-Y)).$$

Now, the proof of Proposition (5.4) of [Al] yields:

*If $\Sigma < \Sigma_0$ is good for $e^i$, then $\Phi_i$ will be injective.*

### 3. Dividing $\mathbf{ES}_{Y^\#}(k[\varepsilon])$ into a sum

(3.1) Let $\tilde{\Sigma} < \Sigma < \Sigma_0$ be two smooth subdivisions with a canonical $|\Sigma_0^{(2)}|$-part $\Sigma'$; let $\sigma : \tilde{X} \to X$ be the corresponding map of the torus embeddings. We get an injection $\Theta_{\tilde{X}} \hookrightarrow \sigma^*\Theta_X$, and the canonical map

$$\bigoplus_{\mu} \mathcal{N}_{\tilde{D}_\mu|\tilde{X}} \to \bigoplus_{\nu} \mathcal{N}_{\sigma^*D_\nu|\tilde{X}} = \bigoplus_{\nu} \sigma^*\mathcal{N}_{D_\nu|X}$$

induces the injection

$$i : \Theta_{\tilde{X}}\langle -\tilde{D}\rangle \hookrightarrow \sigma^*\Theta_X\langle -D\rangle.$$

(Locally we can illustrate the situation as follows:
Let $\sigma^* : B \to A$ be the ring homomorphism corresponding to $\sigma$;
$D_\nu \subseteq X$ correspond to equations $g_\nu \in B$,
$\tilde{D}_\mu \subseteq \tilde{X}$ correspond to equations $f_\mu \in A$, and the pull backs $\sigma^*D_\nu$ are given by

$$\sigma^*(g_\nu) = \prod_{\mu} f_{\nu,\mu}^{c_{\nu,\mu}}.$$

Then, the map $\oplus_\mu \mathcal{N}_{\tilde{D}_\mu|\tilde{X}} \to \oplus_\nu \mathcal{N}_{\sigma^*D_\nu|\tilde{X}}$ is given by

$$\bigoplus_{\mu} \operatorname{Hom}\left(\frac{f_\mu}{f_\mu^2}, \frac{A}{f_\mu}\right) \to \bigoplus_{\nu} \operatorname{Hom}\left(\frac{\sigma^*(g_\nu)}{\sigma^*(g_\nu)^2}, \frac{A}{\sigma^*(g_\nu)}\right)$$

$$[f_\mu \mapsto a_\mu \in A]_\mu \mapsto \left[\sigma^*(g_\nu) \mapsto \sum_{\mu} c_{\nu,\mu} \frac{\sigma^*(g_\nu)}{f_{\nu,\mu}} a_{\nu,\mu}\right]_\nu.$$

Is $D: A \to A$ a derivation (i.e. $D \in \Theta_{\tilde{X}}$), we can compute the image of $D$ in the above sheaves as

$$[f_\mu \mapsto D(f_\mu)]_\mu \mapsto \left[\sigma^*(g_\nu) \mapsto \sum_\mu c_{\nu,\mu} \frac{\sigma^*(g_\nu)}{f_{\nu,\mu}} D(f_{\nu,\mu})\right]_\nu$$

$$\| $$

$$D\left(\prod_\mu f_{\nu,\mu}^{c_{\nu,\mu}}\right) = D(\sigma^* g_\nu).$$

Finally, $D(\sigma^* g_\nu) = D(g_\nu)$ if $D$ is considered as an element of $\sigma^*\Theta_X$ (i.e. as a derivation $B \to A$).)

The two sheaves are both contained in $\bigoplus_{j=1}^{3} \mathcal{O}_{\text{torus}} \dfrac{\partial}{\partial x_j}$;

therefore,

$$S_j^{(\tilde{X})} = \sigma^* S_j^{(X)} \cap \Theta_{\tilde{X}}\langle -\tilde{D}\rangle,$$

and the injection $i$ will split into $i_j: S_j^{(\tilde{X})} \hookrightarrow \sigma^* S_j^{(X)}$ ($j = 1, 2, 3$). Now, the exact sequence

$$0 \to \Theta_X\langle -D\rangle(-Y^\#) \hookrightarrow \Theta_X\langle -D - Y^\#\rangle \to \Theta_{Y^\#}\langle -E\rangle \to 0$$

together with the analogous one for $\tilde{X}$ yields the surjections

$$\begin{array}{ccc} & & H^2\left(\tilde{X}, \bigoplus_{j=1}^{3} S_j^{(\tilde{X})}(-Y^\#)\right) \\ & \nearrow & \downarrow H^2(i) \\ H^1(Y^\#, \Theta_{Y^\#}\langle -E\rangle) & \searrow & \\ & & H^2\left(X, \bigoplus_{j=1}^{3} S_j^{(X)}(-Y^\#)\right) \end{array}$$

((a) Existence of $H^2(i)$:

$$H^2(\tilde{X}, (\sigma^* S_j^{(X)})(-Y^\#)) = H^2(\tilde{X}, \sigma^*[S_j^{(X)}(-Y^\#)]) \quad (\sigma \text{ is an isomorphism on } W, \text{ hence, } \sigma^*(Y^\#) = Y^\#)$$

$$= \mathbb{H}^2(X, (\mathbb{R}^+\sigma_*)\sigma^*[S_j^{(X)}(-Y^\#)])$$

$$= H^2(X, S_j^{(X)}(-Y^\#))$$

($S_j^{(X)}(-Y^{\#})$ is invertible, and $\mathcal{O}_X \to (\mathbb{R}^+\sigma_*)\sigma^*\mathcal{O}_X$ is a quasiisomorphism).

(b) The horizontal maps are surjective by

$$\left. H^2(\tilde{X}, \Theta_{\tilde{X}}\langle -\tilde{D} - Y^{\#}\rangle) = H^2(X, \Theta_X\langle -D - Y^{\#}\rangle) = 0 \quad \text{(cf. [Al], (4.2)).}\right)$$

(3.2) LEMMA. *There exists a universal surjection of k-vectorspaces*

$$\psi: H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle) \twoheadrightarrow \bigoplus_{j=1}^{3} P_j$$

*such that for arbitrary smooth subdivisions* $\Sigma < \Sigma_0$ (*with* $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$) *the surjection* $H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle) \twoheadrightarrow H^2(X, \oplus_{j=1}^3 S_j^{(X)}(-Y^{\#}))$ *will factorize (uniquely) through* $P_j \twoheadrightarrow H^2(X, S_j^{(X)}(-Y^{\#}))$ $(j = 1, 2, 3)$.

*Proof.* Two smooth subdivisions of $\Sigma_0$ coinciding in $|\Sigma_0^{(2)}|$ admit a common finer one without changing the part in $|\Sigma_0^{(2)}|$.

Therefore, we can define $P_j$ as the limit of the successive surjections

$$H^2(\tilde{X}, S_j^{(\tilde{X})}(-Y^{\#})) \twoheadrightarrow H^2(X, S_j^{(X)}(-Y^{\#}))$$

described in (3.1); by $\dim_k H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle) < \infty$, this process stops after finitely many steps. □

(3.3) Now, we can define some subspaces of $\mathrm{ES}_{Y^{\#}}(k[\varepsilon])$ and $H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle)$: For $j = 1, 2, 3$ let

$$F'_j := \psi^{-1}(P_j) \subseteq H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle) \quad \text{and} \quad F_j := F'_j \cap \mathrm{ES}_{Y^{\#}}(k[\varepsilon]).$$

PROPOSITION.

(1) $\Sigma_{j=1}^3 F'_j = H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle)$.

(2) *For arbitrary smooth subdivisions* $\Sigma < \Sigma_0$ *with* $\Sigma \cap |\Sigma_0^{(2)}| = \Sigma'$ *the surjection*

$$H^1(Y^{\#}, \Theta_{Y^{\#}}\langle -E\rangle) \twoheadrightarrow H^2\left(X, \bigoplus_{j=1}^{3} S_j^{(X)}(-Y^{\#})\right)$$

*maps* $F'_j$ *onto* $H^2(X, S_j^{(X)}(-Y^{\#}))$.

(3) $\Sigma_{j=1}^3 F_j = \mathrm{ES}_{Y^{\#}}(k[\varepsilon])$.

(4) *If the image of* $F_j$ *under the map* $\mathrm{ES}_{Y^{\#}}(k[\varepsilon]) \to \mathrm{Def}_R(k[\varepsilon])$ *is denoted by* $\bar{F}_j \subseteq \overline{ES}(k[\varepsilon])$, *then* $\bigcap_{j=1}^3 \bar{F}_j$ *contains all "above* $-\Gamma(f)-$ *deformations" of* $f$ (*i.e. deformations* $\tilde{f}$ *with* $\Gamma(\tilde{f}) = \Gamma(f)$).

*Proof.* The parts (1) and (2) are clear by definition or by the previous Lemma. For (3) and (4) take a smooth subdivision $\Sigma < \Sigma_0$ $(\Sigma \cap |\Sigma_0^{(2)}| = \Sigma')$ that realizes

$\oplus_{j=1}^3 P_j$, i.e. $H^2(X, S_j(-Y^\#)) = P_j$ $(j = 1, 2, 3)$. Now, we regard the diagram of (2.6):

*Part* (3). Let $\xi \in \mathrm{ES}_{Y^\#}(k[\varepsilon])$ with $\xi = \xi_1 + \xi_2 + \xi_3$ $(\xi_j \in F'_j)$. Then, $\varphi = \Phi \circ \psi$ maps each of the $\xi_j$ into the corresponding factor of $H^1(Y^\#, \mathcal{O}_{Y^\#})^3$, and we get

$$0 = \varphi(\xi) = \varphi(\xi_1) + \varphi(\xi_2) + \varphi(\xi_3) \in H^1(Y^\#, \mathcal{O}_{Y^\#})^3,$$

hence $\varphi(\xi_1) = \varphi(\xi_2) = \varphi(\xi_3) = 0$.

But this means $\xi_j \in \mathrm{ES}_{Y^\#}(k[\varepsilon])$, i.e. $\xi_j \in F_j$ $(j = 1, 2, 3)$.

*Part* 4. By construction of the subspaces $F_j \subseteq \mathrm{ES}_{Y^\#}(k[\varepsilon])$ we have

$$\begin{aligned}
&\mathrm{Im}(\mathrm{ESE}_X(k[\varepsilon] \xrightarrow{\alpha} \mathrm{ES}_{Y^\#}(k[\varepsilon]))\\
&\quad = \mathrm{Im}(H^1(X, \Theta_X\langle -D - Y^\#\rangle) \to H^1(Y^\#, \Theta_{Y^\#}\langle -E\rangle))\\
&\quad = \mathrm{Ker}\, \psi\\
&\quad = \bigcap_{j=1}^{3} F_j, \text{ hence}
\end{aligned}$$

$$\mathrm{Im}(\mathrm{ESE}_X(k[\varepsilon]) \to \overline{\mathrm{ES}}(k[\varepsilon])) \subseteq \bigcap_{j=1}^{3} \bar{F}_j.$$

On the other hand, all "above-$\Gamma(f)$-deformations" can be lifted to elements of $\mathrm{ESE}_X(k[\varepsilon])$ (cf. [Al], (2.1)). □

(3.4) Finally, let us return to the situation of section 2: We had three embedded resolutions

$$\begin{array}{ccc}
\pi_i: X_i & \longrightarrow & (\mathbb{A}_k^3, 0)\\
\uparrow & & \uparrow f\\
\pi: Y^\# & \longrightarrow & (V, 0),
\end{array}$$

and each $X_i$ was good for $e^i$ $(i = 1, 2, 3)$. Regarding the corresponding diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
\mathrm{ESE}_{X_i}(k[\varepsilon]) & \longrightarrow & \mathrm{ES}_{Y^\#}(k[\varepsilon]) & & \\
\| & & \downarrow & & \\
H^1(X_i, \Theta_{X_i}\langle -D^{(i)} - Y^\#\rangle) & \longrightarrow & H^1(Y^\#, \Theta_{Y^\#}\langle -E\rangle) & \xrightarrow{\psi_i} & H^2(X_i, \oplus_{j=1}^3 S_j^{(X_i)}(-Y^\#))\\
 & & \downarrow \varphi & & \downarrow \oplus_{j=1}^3 \Phi_j^{(i)}\\
 & & H^1(Y^\#, \mathcal{O}_{Y^\#}^3) & = & H^2(X_i, \mathcal{O}_{X_i}(-Y^\#)^3)
\end{array}$$

of (2.6), we obtain

THEOREM. *For* $i = 1, 2, 3$, *the subspace* $F_i \subseteq \mathrm{ES}_{Y^\#}(k[\varepsilon])$ *is contained in* $\mathrm{Im}(\alpha_i)$. *In particular, the map between the tangent spaces*

$$\bigoplus_{i=1}^{3} \mathrm{ESE}_{X_i}(k[\varepsilon]) \xrightarrow{\oplus \alpha_i} \mathrm{ES}_{Y^\#}(k[\varepsilon])$$

*is surjective.*

*Proof.* Let $\xi \in F_i \subseteq \mathrm{ES}_{Y^\#}(k[\varepsilon])$, then, by Proposition (3.3), the image $\psi_i(\xi)$ is contained in the factor $H^2(X_i, S_i^{(X_i)}(-Y^\#))$.

Now, $\Phi_i^{(i)}$ is injective (cf. (2.6)), and from $\varphi(\xi) = 0$ we obtain the vanishing of $\psi_i(\xi)$, i.e. $\xi$ is contained in the image of $\mathrm{ESE}_{X_i}(k[\varepsilon])$. □

(3.5) REMARK. (1) Equisingular deformations coming from some ESE are equimultiple (cf. [Ka], (4.6)). So the above theorem illustrates the fact that over $k[\varepsilon]$ equisingularity alone is sufficient for equimultiplicity (cf. [Ka], (2.8)).

(2) In [Al] the smoothness of ESE and the surjectivity of some $\mathrm{ESE}(k[\varepsilon]) \to \mathrm{ES}(k[\varepsilon])$ are used to show the smoothness of ES. It is not possible to apply this schema of proof to our situation – the surjectivity of $\oplus_{i=1}^{3} \mathrm{ESE}_{X_i}(k[\varepsilon]) \to \mathrm{ES}_{Y^\#}(k[\varepsilon])$ cannot be lifted to any Artinian rings $A$ of higher order because there is no possibility of defining the sum of two $A$-deformations!

## 4. Computation of $\mathrm{Im}(\mathrm{ESE}_X(k[\varepsilon]) \xrightarrow{\gamma} \mathrm{Def}_R(k[\varepsilon]))$ (for a fixed embedded resolution $X$)

For this section we fix an arbitrary smooth f.r.p.p. subdivision $\Sigma < \Sigma_0$ with the corresponding good resolution $\pi: X \to \mathbb{A}_k^3$.

(4.1) The connecting morphism of the cohomology sequences of

$$0 \to \Theta_X\langle -D - Y\rangle \to \Theta_X\langle -D\rangle \to \mathcal{N}_{Y|X} \to 0$$

yields the following diagram:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^0(X, \mathcal{O}_X(Y)) & \twoheadrightarrow & H^0(X, \mathcal{N}_{Y|X}) & \longrightarrow & H^1(X, \Theta_X\langle -D - Y\rangle) \\
\cap\downarrow & & \cap\downarrow & & \downarrow\gamma \\
H^0(X\backslash D, \mathcal{O}_X(Y)) & \twoheadrightarrow & H^0(X\backslash D, \mathcal{N}_{Y|X}) & \twoheadrightarrow & H^1(X\backslash D, \Theta_X\langle -D - Y\rangle) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_D(X, \mathcal{O}_X(Y)) & \xrightarrow{\sim} & H^1_D(X, \mathcal{N}_{Y|X}) & \xrightarrow{\psi} & H^2_D(X, \Theta_X\langle -D - Y\rangle) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

This diagram can be identified with

$$
\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\langle \text{monomials} \geqslant \Gamma(f)\rangle & \twoheadrightarrow & \langle \text{monomials} \geqslant \Gamma(f)\rangle/(f) & \longrightarrow & \mathrm{ESE}_X(k[\varepsilon]) \\
\cap\downarrow & & \cap\downarrow & & \downarrow\gamma \\
k[\mathbf{x}] & \twoheadrightarrow & R & \twoheadrightarrow & \mathrm{Def}_R(k[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
k[\mathbf{x}]/\langle \text{monomials} \geqslant \Gamma(f)\rangle & \xrightarrow{\sim} & R/\langle \text{monomials} \geqslant \Gamma(f)\rangle & \overset{\psi}{\twoheadrightarrow} & \mathrm{Coker}\,\gamma \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

(The first columns are identified according to [Al] (2.2) – we take

$$k[\mathbf{x}] = H^0\left(X\backslash D, \mathcal{O}_X\left(-\sum_{a\in\Sigma^{(1)}} m(a)D_a\right)\right) \xrightarrow[\sim]{\cdot 1/f} H^0(X\backslash D, \mathcal{O}_X(Y))$$

and

$$H^1(X, \mathcal{O}_X(Y)) = H^1(X, \mathcal{N}_{Y|X}) = 0;$$

for the right hand side we use (2.5)($\gamma$) and (4.2) of [Al] – in the latter one the vanishing of $H^2(X, \Theta_X\langle -D - Y\rangle)$ has been proved.)

DEFINITION. For $\xi = \Sigma_{r\geqslant 0}\xi_r\cdot\mathbf{x}^r \in k[\mathbf{x}] \cong H^0(X\backslash D, \mathcal{O}_X(Y))$ we denote by $\xi_{<\Gamma(f)}$ the image of $\xi$ in

$$\frac{k[\mathbf{x}]}{\langle \text{monomials} \geqslant \Gamma(f)\rangle} = H^1_D(X, \mathcal{O}_X(-\Sigma\, m(a)D_a)) \cong H^1_D(X, \mathcal{O}_X(Y)).$$

Taking the canonical section of $k[\mathbf{x}] \twoheadrightarrow k[\mathbf{x}]/\langle \text{monomials} \geqslant \Gamma(f)\rangle$, we get

$$\xi_{<\Gamma(f)} = \sum_{\substack{r\geqslant 0\\ r<\Gamma(f)}} \xi_r\cdot\mathbf{x}^r.$$

(4.2) PROPOSITION. (1) *For $i = 1, 2, 3$ the vertices $e^i \in \Delta$ correspond to the – non exceptional – divisors $\overline{\mathrm{orb}(e^i)} \subseteq X$. Denote these divisors also by $e^i$, and let*

$$\varphi_i: H^1_D(X, \mathcal{O}_X(e^i)) \to H^1_D(X, \mathcal{O}_X(-\Sigma\, m(a)D_a))$$

*be the multiplication by $x_i(\partial f/\partial x_l)$. Under the isomorphism*

$$H^1_D(X, \mathcal{O}_X(-\Sigma m(a)D_a)) \xrightarrow[\sim]{\cdot 1/f} H^1_D(X, \mathcal{O}_X(Y)) \xrightarrow{\sim} H^1_D(X, \mathcal{N}_{Y|X}),$$

*then*

$$\operatorname{Coker}\gamma = \operatorname{Coker}\left(\bigoplus_{i=1}^{3} \varphi_i\right).$$

(2) *Let $\xi = \Sigma_{r \geqslant 0} \xi_r \cdot \mathbf{x}^r \in k[\mathbf{x}]$ define an element of* $\mathrm{Def}_R(k[\varepsilon])$ (*the infinitesimal deformation $\tilde{f}(\mathbf{x}, \varepsilon) = f(\mathbf{x}) - \varepsilon\xi(\mathbf{x})$*). *Then, this deformation is induced by* $\mathrm{ESE}_X(k[\varepsilon])$ *if and only if*

$$\xi_{<\Gamma(f)} \in \operatorname{Im}\left(\bigoplus_{i=1}^{3} \varphi_i\right).$$

*Proof.* (1) By the second diagram of (4.1) it holds

$$\operatorname{Coker}\gamma = \frac{H^1_D(X, \mathcal{N}_{Y|X})}{\operatorname{Ker}\psi} = \operatorname{Coker}(H^1_D(X, \Theta_X\langle -D\rangle) \to H^1_D(X, \mathcal{N}_{Y|X})).$$

On the other hand, we can lift the surjection $\Theta_X\langle -D\rangle \twoheadrightarrow \mathcal{N}_{Y|X}$ to the homomorphism $\Theta_X\langle -D\rangle \to \mathcal{O}_X(Y)$ given by $\eta \mapsto [\eta(f)/f]$:

(i) In local coordinates (take the same notations as in the proof of (2.5): $f = \mathbf{x}^{r^\alpha} \cdot f_\alpha$) we obtain

$$\frac{\eta(f)}{f} = \frac{\eta(f_\alpha)}{f} + \frac{\eta(\mathbf{x}^{r^\alpha})}{\mathbf{x}^{r^\alpha}}.$$

Since $\eta \in \Theta_X\langle -D\rangle$, the section $[\eta(x^{r\alpha})/x^{r\alpha}]$ is regular on $X$, and $\eta(f)/f$ is indeed an element of the sheaf $\mathcal{O}_X(Y)$.

(ii) The projections $\Theta_X\langle -D\rangle \twoheadrightarrow \mathcal{N}_{Y|X}$ and $\mathcal{O}_X(Y) \twoheadrightarrow \mathcal{N}_{Y|X}$ are locally given by

$$\eta \mapsto \left[f_\alpha \in \frac{(f_\alpha)}{(f_\alpha^2)} \mapsto \eta(f_\alpha) \in \frac{\mathcal{O}_X}{(f_\alpha)}\right]$$

and

$$a \mapsto \left[f_\alpha \in \frac{(f_\alpha)}{(f_\alpha^2)} \mapsto af_\alpha \in \frac{\mathcal{O}_X}{(f_\alpha)}\right],$$

respectively. Then, the congruence

$$\frac{\eta(f)}{f}\cdot f_\alpha = \eta(f_\alpha) + \frac{\eta(\mathbf{x}^{r^\alpha})}{\mathbf{x}^{r^\alpha}}\cdot f_\alpha \equiv \eta(f_\alpha) \quad (\text{modulo } f_\alpha)$$

shows that the diagram

$$\begin{array}{ccc} & & \mathcal{O}_X(Y) \\ & \nearrow & \downarrow \\ \Theta_X\langle -D\rangle & \twoheadrightarrow & \mathcal{N}_{Y|X} \end{array} \quad \text{commutes.}$$

Since $H^1_D(X, \mathcal{O}_X(Y)) \xrightarrow{\sim} H^1_D(X, \mathcal{N}_{Y|X})$ is an isomorphism, we obtain

$$\operatorname{Coker}\gamma = \operatorname{Coker}(H^1_D(X, \Theta_X\langle -D\rangle) \to H^1_D(X, \mathcal{O}_X(Y))).$$

Finally, the first claim follows by the equation

$$\eta(f) = \sum_{i=1}^{3}\left(x_i\frac{\partial f}{\partial x_i}\right)\cdot\frac{\eta(x_i)}{x_i}$$

and taking the isomorphism

$$\Theta_X\langle -D\rangle \xrightarrow{\sim} \bigoplus_{i=1}^{3} \mathcal{O}_X(e^i)$$

$$\eta \mapsto \left(\frac{\eta(x_1)}{x_1}, \frac{\eta(x_2)}{x_2}, \frac{\eta(x_3)}{x_3}\right).$$

(2) $\xi \in k[\mathbf{x}] = H^0(X\backslash D, \;\mathcal{O}_X(-\Sigma\, m(a)D_a)) \cong H^0(X\backslash D, \;\mathcal{O}_X(Y))$ maps onto $0 \in \operatorname{Coker}\gamma$ if and only if

$$\xi_{<\Gamma(f)} \in H^1_D(X, \mathcal{O}_X(-\Sigma\, m(a)D_a)) \cong H^1_D(X, \mathcal{O}_X(Y))$$

vanishes in $\operatorname{Coker}(\bigoplus_{i=1}^{3} \varphi_i)$. □

(4.3) Our next task will be to describe the maps $\varphi_i$ by the methods of torus embeddings. For this purpose it is useful to regard the dual version of these maps:

$$\varphi_i^*: H^2(X, \omega_X(\Sigma\, m(a)D_a)) \to H^2(X, \omega_X(-e^i)),$$

and the homomorphisms are still given by multiplication by $x_i(\partial f/\partial x_i)$. *Now, for* $r, t \in M$ we define the following sets:

$$A_r := \{a \in \Delta / \langle a, -r \rangle \leqslant -m(a)\} = \{a \in \Delta / \langle a, r \rangle \geqslant m(a)\},$$

$$B^{\Sigma}_{i,t} := \{a \in \Delta / \langle a, -t \rangle \leqslant \psi_i(a)\} \quad \text{with } \psi_i(a) := \begin{cases} 0 & \text{for } a \in \Sigma^{(1)}, a \neq e^i \\ 1 & \text{for } a = e^i \end{cases}$$

(linear on the $\Sigma$-cones),

$$\mathbb{H}_t := \{a \in \Delta / \langle a, t \rangle < 0\}.$$

Then, the convex sets $(\Delta \backslash \mathbb{H}_t)$ are contained in $B^{\Sigma}_{i,t}$, and the maps $\varphi_i^*$ are equal to some homomorphisms

$$\begin{array}{ccl} \varphi_i^*: \bigoplus_{r \in M} H^1(A_r, k) & \longrightarrow \bigoplus_{t \in M} H^1(B^{\Sigma}_{i,t}, k) & (i = 1, 2, 3). \\ \Big\| \text{(cf. (4.5))} & & \\ \bigoplus_{\substack{r \geqslant 0 \\ r < \Gamma(f)}} k \cdot x^{-r} & & \end{array}$$

(As we are really interested in the dual of, for instance, $H^2(X, \omega_X(\Sigma\, m(a) D_a))$, the notations are chosen such that $A_r$ describes the cohomology of the $-r$(th) factor of this sheaf. The relations "$\leqslant$" or "$\geqslant$" – instead of the strict ones – in the definitions of $A_r$ and $B^{\Sigma}_{i,t}$ are induced by taking $\omega_X$(divisor) instead of $\mathcal{O}_X$(divisor).)

But, what does $\varphi_i^*$ look like? We have to make some general remarks concerning the computation of cohomology on torus embeddings:

(4.4) Denote by $j: T \hookrightarrow X_\Sigma$ a torus embedding in the sense of [Ke].

(1) Let $\mathbb{L} \subseteq j_* \mathcal{O}_T = j_* k[M]^{\sim}$ be an $M$-graded invertible sheaf with order function $\Phi: |\Sigma| \to \mathbb{R}$; for $r \in M$ let $A_r := \{a \in \Delta / \langle a, r \rangle < \Phi(a)\}$.

Then, if $\alpha \in \Sigma$ is an arbitrary cone, we obtain

$$\mathbb{L}(r)_{|X_\alpha} = \begin{cases} \underline{k} & (\forall a \in \alpha: \langle a, r \rangle \geqslant \Phi(a)) \\ 0 & (\exists a \in \alpha: \langle a, r \rangle < \Phi(a)) \end{cases},$$

hence $\mathbb{L}(r)_{|X_\alpha} = H^0(\alpha, \alpha \cap A_r) \otimes \underline{k}$. In particular, the sheaf $\mathbb{L}(r)$ and the pair $(\Delta, A_r)$ yield exactly the same Cech complexes.

(2) Let $\mathbb{L}^1, \mathbb{L}^2 \subseteq j_* \mathcal{O}_T$ be $M$-graded invertible sheaves with $\Phi^1, \Phi^2$ and $A^1_r, A^2_r$ as before. Assume that there is an $s \in M$ with $\mathbf{x}^s \cdot \mathbb{L}^1 \subseteq \mathbb{L}^2$ (equivalent: $\Phi^1 + s \geqslant \Phi^2$ as functions on $\Delta$).

Then, for each $r \in M$ there is an inclusion $A^2_{r+s} \subseteq A^1_r$, which provides the commutative diagram

$$\begin{array}{ccc} \Gamma(X_\alpha, \mathbb{L}^1(r)) & \xrightarrow{\cdot \mathbf{x}^s} & \Gamma(X_\alpha, \mathbb{L}^2(r+s)) \\ \| & & \| \\ H^0(\alpha, \alpha \cap A^1_r) & \hookrightarrow & H^0(\alpha, \alpha \cap A^2_{r+s}). \end{array}$$

Again by taking Cech cohomology we obtain a description of the multiplication by $\mathbf{x}^s$ on the cohomological level:

$$\begin{array}{ccc} H^n(X, \mathbb{L}^1) & \xrightarrow{\cdot \mathbf{x}^s} & H^n(X, \mathbb{L}^2) \\ \| & & \| \\ \bigoplus_{r \in M} H^n(\Delta, A^1_r) & \xrightarrow{\varphi} & \bigoplus_{r \in M} H^n(\Delta, A^2_r) \end{array}$$

($\varphi$ is induced by the inclusion $A^2_{r+s} \subseteq A^1_r$; in particular, $\varphi$ is homogeneous of degree $s$.)

(3) Let $\mathbb{L}^i, \Phi^i, A^i_r$ ($i = 1, 2$) as before, assume that there is a Laurent polynomial $g(\mathbf{x}) \in k[M]$ with $g(\mathbf{x}) \cdot \mathbb{L}^1 \subseteq \mathbb{L}^2$.

Then, by $M$-graduation of both sheaves $\mathbb{L}^1$ and $\mathbb{L}^2$, this fact is equivalent to

$$\mathbf{x}^s \cdot \mathbb{L}^1 \subseteq \mathbb{L}^2 \text{ for all } s \in \operatorname{supp} g.$$

Hence, the method of (2) can be applied to describe the maps

$$H^n(X, \mathbb{L}^1) \xrightarrow{\cdot g(\mathbf{x})} H^n(X, \mathbb{L}^2).$$

(4.5) The third part of the previous general remark applies exactly to the special maps $\varphi_i^*$ regarded in (4.3). Denoting by $\Delta_i^\Sigma \subseteq \Delta$ the union of all closed $\Sigma$-cones not containing $e^i$, we obtain the following

LEMMA. (1) $H^1(A_r, k) = \begin{cases} k \cdot \mathbf{x}^{-r} & (\textit{for } r \geq 0 \textit{ and } r < \Gamma(f)) \\ 0 & (\textit{otherwise}) \end{cases}$,

*and the perfect pairing with*

$$H^1_D(X, \mathcal{O}_X(-\Sigma\, m(a) D_a)) = \bigoplus_{\substack{r \geq 0 \\ r < \Gamma(f)}} k \cdot \mathbf{x}^r$$

*is built in the obvious way.*

(2) *For $i = 1, 2, 3$ and $t \in M$ the cohomology group $H^1(B^{\Sigma}_{i,t}, k)$ is equal to*

(i) $H_0(\Delta^{\Sigma}_i \cap \mathbb{H}_t) \cdot \mathbf{x}^{-t}$ (*for $t_i = -1$ and $t_v \geqslant 0$ for all $v \neq i$*),

(ii) $\dfrac{H_0(\Delta^{\Sigma}_i \cap \mathbb{H}_t)}{H_0(\{e^j\})} \cdot \mathbf{x}^{-t}$ (*for $t_i = -1, t_j \leqslant -1$ ($j \neq i$), and the remaining component is $\geqslant 0$*);

(iii) $0$ (*for $t_i \neq -1$ or $t \leqslant -(1, 1, 1)$*).

(3) *Let $f(\mathbf{x}) = \Sigma_{s \in \operatorname{supp} f} \lambda_s \cdot \mathbf{x}^s$ be the explicit description of our starting equation. Let $r$, $i$ and $t$ be such that $H^1(A_r, k)$, $H^1(B^{\Sigma}_{i,t}, k) \neq 0$ (i.e. $r \geqslant 0$, $r < \Gamma(f)$ and $t_i = -1$, $t \nleqslant -(1, 1, 1)$, respectively).*

*Then, the $\mathbf{x}^{-t}$ part of $\varphi^*_i(\mathbf{x}^{-r})$ is given by*

$$s_i \cdot \lambda_s \cdot [H_0(\{a^*\}) \in H_0(\Delta^{\Sigma}_i \cap \mathbb{H}_t)]$$

*with $s := -t + r$ (because of $(-t) = s + (-r)$) and $a^* \in \Sigma^{(1)}_0$ such that $\langle a^*, r \rangle < m(a^*)$.*

*In particular, this part of $\varphi^*_i(\mathbf{x}^{-r})$ vanishes, unless $s \geqslant \Gamma(f)$.*

*Proof.* (1) $A_r = \Delta \backslash \{a \in \Delta / \langle a, r \rangle < m(a)\} = \Delta \backslash$(convex set), and the above conditions for $r$ arise by $r \geqslant 0$ iff $\partial\Delta \subseteq A_r$ and

$$r < \Gamma(f) \quad \text{iff } A_r \neq \Delta.$$

(2) $\Delta \backslash \mathbb{H}_t \subseteq B^{\Sigma}_{i,t}$, and the only vertex of $\Sigma^{(1)}$ in which both sets can differ is $e^i$. Hence, the non-vanishing of $H^1(B^{\Sigma}_{i,t}, k)$ implies $e^i \notin \Delta \backslash \mathbb{H}_t$, $e^i \in B^{\Sigma}_{i,t}$, and we obtain $t_i = \langle e^i, t \rangle = -1$.

Assuming this from now on, we see that $B^{\Sigma}_{i,t}$ contains exactly the same elements of $\Sigma^{(1)}$ as $\Delta \backslash [\Delta^{\Sigma}_i \cap \mathbb{H}_t]$. In particular, both subsets of $\Delta$ (consisting of open or closed halfspaces in every cone of $\Sigma$) are homotopy equivalent and yield the same cohomology. Without loss of generality we take $i = 1$ and consider the above three cases:

(i) $t_2, t_3 \geqslant 0$: Then, $\partial\Delta \subseteq \Delta \backslash [\Delta^{\Sigma}_1 \cap \mathbb{H}_t]$, and

$$H^1(\Delta \backslash [\Delta^{\Sigma}_1 \cap \mathbb{H}_t], k) = H_0(\Delta^{\Sigma}_1 \cap \mathbb{H}_t)$$

follows by the Alexander duality.

(ii) $t_2 \leqslant -1$, $t_3 \geqslant 0$: This means $e^1, e^3 \in (\Delta \backslash [\Delta^{\Sigma}_1 \cap \mathbb{H}_t])$, $e^2 \notin (\Delta \backslash [\Delta^{\Sigma}_1 \cap \mathbb{H}_t])$ and therefore, the connected component $C$ of $e^2$ in $\Delta_1^\Sigma \cap \mathbb{H}_t$ has no influence on the cohomology:

$$H^1(\Delta\backslash[\Delta_1^\Sigma \cap \mathbb{H}_t], k) = H^1(\Delta\backslash[(\Delta_1^\Sigma \cap \mathbb{H}_t)\backslash C], k)$$
$$= H_0([\Delta_1^\Sigma \cap \mathbb{H}_t]\backslash C) = \frac{H_0(\Delta_1^\Sigma \cap \mathbb{H}_t)}{H_0(\{e^2\})}.$$

(The middle equality again follows by the Alexander duality.)

(iii) $t_2, t_3 \leqslant -1$: By $\mathbb{H}_t = \Delta$ we obtain

$$\Delta\backslash[\Delta_1^\Sigma \cap \mathbb{H}_t] = \Delta\backslash\Delta_1^\Sigma,$$

and this set can be contracted to the point $e^1$.

(3) The linear map $H^1(A_r, k) \to H^1(B_{i,t}^\Sigma, k)$ is constructed by the inclusion $B_{i,t}^\Sigma \subseteq A_r$ (cf. (4.4)); in dual terms this means that $H_0(\Delta\backslash A_r) \to H_0(\Delta_i^\Sigma \cap \mathbb{H}_t)/\cdots$ is induced by

$$(\Delta\backslash A_r) \subseteq (\Delta\backslash B_{i,t}^\Sigma) \sim (\Delta_i^\Sigma \cap \mathbb{H}_t):$$

Take an element $a^* \in \Sigma_0^{(1)}$ with $\langle a^*, r\rangle < m(a^*)$ (i.e. $a^* \in \Delta\backslash A_r$); assuming $s \geqslant \Gamma(f)$, we obtain

$$\langle a^*, t\rangle = \langle a^*, r\rangle - \langle a^*, s\rangle > 0 \quad (\text{i.e. } a^* \in \mathbb{H}_t),$$

and $\mathbf{x}^{-r}$ maps onto the corresponding connected component in $\Delta_i^\Sigma \cap \mathbb{H}_t$ (multiplied by the coefficient of $\mathbf{x}^s$ in $x_i(\partial f/\partial x_i)$). □

REMARK. As $\Delta\backslash A_r$ is convex, we obtain for (3):

$$H_0(\{a^*\}) \in H_0(\Delta\backslash B_{i,t}^\Sigma) = H_0(\Delta_i^\Sigma \cap \mathbb{H}_t)$$

does not depend on the choice of $a^* \in \Sigma_0^{(1)}$ with $\langle a^*, r\rangle < m(a^*)$.

(4.6) Now, we are in the position to determine the deformations of

$$\mathrm{Im}(\mathrm{ESE}_X(k[\varepsilon]) \xrightarrow{\gamma} \mathrm{Def}_R(k[\varepsilon]))$$

exactly:

DEFINITION. (1) We choose (and fix) a map

$$a: \{r \geqslant 0/r < \Gamma(f)\} \to \Sigma_0^{(1)}\backslash\{e^1, e^2, e^3\}$$
$$r \mapsto a(r) \text{ with } \langle a(r), r\rangle < m(a(r)).$$

($r < \Gamma(f)$ means that $r$ sits below some faces of $\Gamma(f)$. Now, by the map $a$, one of these is selected. $a(r)$ plays exactly the role of $a^*$ in the poposition above, and we have seen that all constructions are independent of the special choice the map $a$.)

(2) For $i = 1, 2, 3$ let $M_i := \{r \in M / r \geqslant 0, \Gamma(f) - e_i \leqslant r < \Gamma(f)\}$ ($\{e_1, e_2, e_3\}$ denotes the canonical $\mathbb{Z}$-basis of $M$).

Recall the definitions

$$\mathbb{H}_t := \{a \in \Delta / \langle a, t \rangle < 0\} \text{ (for } t \in M)$$

and

$$\Delta_i^\Sigma := \bigcup \{\bar{\alpha}/\bar{\alpha} \in \Sigma, e^i \notin \bar{\alpha}\} \subseteq \Delta.$$

PROPOSITION. (I) *Given the following data*

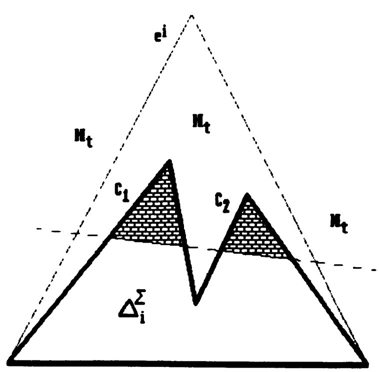
(1) $i \in \{1, 2, 3\}$,
(2) $t \in M$ *with*: (a) $t_i = -1$
 (b) (i) $t_\nu \geqslant 0$ (*i.e.* $e^\nu \notin \mathbb{H}_t$) *for all* $\nu \neq i$, *or*
 (ii) $t_j \leqslant -1$ $(i \neq j)$ *and the remaining component is* $\geqslant 0$,
 (c) *there exists an* $r \in M_i$ *with* $r - t \geqslant \Gamma(f)$ *and* $\langle a(r), t + e_i \rangle \geqslant 0$,
(3) *a connected component* $C$ *of* $\Delta_i^\Sigma \cap \mathbb{H}_t$ *not containing any of the vertices* $e^1$, $e^2$, $e^3$,

*then, the deformation defined by*

$$\sum_{\substack{r \in M_i \\ a(r) \in C}} (r_i + 1)\lambda_{r-t} \cdot x^r = \left( x^{t+e_i} \cdot \frac{\partial f}{\partial x_i} \right)\Bigg|_{M_i \cap a^{-1}(C)}$$

*comes from* $\mathrm{ESE}_X(k[\varepsilon])$.

(II) $\mathrm{Im}(\gamma) \subseteq \mathrm{Def}_R(k[\varepsilon])$ *as a k-vectorspace is spanned by the above-*$\Gamma(f)$*-deformations and all deformations constructed in the above way.*



*Proof.* By Proposition (4.2), Im($\gamma$) is spanned by the above-$\Gamma(f)$-deformations together with the images of the maps $\varphi_i$ ($i = 1, 2, 3$). However, in Lemma (4.5)(2) it is shown that the data $\{i, t, C\}$ meeting (1), (2a), (2b) and (3) of the claim form a $k$-basis of

$$\bigoplus_{i=1}^{3} H_D^2(X, \mathcal{O}_X(e^i)) = \bigoplus_{i=1}^{3} \bigoplus_{t\in M} H^1(B_{i,t}^\Sigma, k) \quad \text{(or its } k\text{-dual);}$$

finally, part (3) of the same Lemma gives

$$\varphi_i(\{i, t, C\})_{|k\cdot \mathbf{x}^r} = \begin{cases} (r_i + 1)\lambda_{r-t} & (\text{for } a(r)\in C) \\ 0 & (\text{otherwise}) \end{cases}.$$

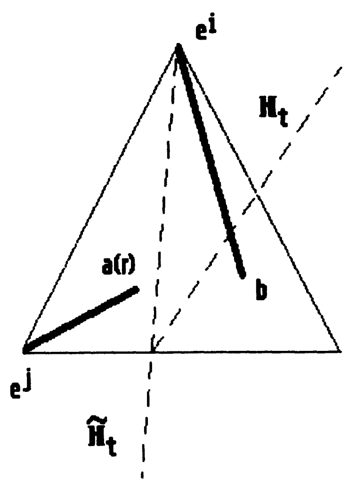
It remains to prove that we are able to restrict ourselves to $r\in M_i$ (instead of $r \geqslant 0$, $r < \Gamma(f)$) and that the additional assumption (2c) for $t$ can be made:

Let $\{i, t, C\}$ be as before and take an $r \geqslant 0$, $r < \Gamma(f)$ such that $\varphi_i(\{i, t, C\})_{|k\cdot \mathbf{x}^r} \neq 0$.

CLAIM. $\langle a(r), t\rangle \geqslant -a(r)_i$.

$\langle a(r), t\rangle < -a(r)_i$ would imply that there is an $j \neq i$ with $t_j \leqslant -1$ (cf. case (ii)), and we would obtain the following situation:

$\tilde{\mathbb{H}}_t := \{a\in\Delta / \langle a, t\rangle < -a_i\} \subseteq \mathbb{H}_t$ contains $a(r)$ and $e^j$, but not the vertex $e^i$. Hence, there is no cone $\overline{be^i}\in\Sigma$, $b\notin\tilde{\mathbb{H}}_t$ ($b\notin\mathbb{H}_t$) meeting $\overline{a(r)e^j}$, and $a(r)$ and $e^j$ must be contained in the same connected component of $\Delta_i^\Sigma\cap\mathbb{H}_t$.




Therefore, the $\mathbf{x}^{-t}$-part of $\varphi_i^*(\mathbf{x}^{-r})$ would be killed by dividing out $H_0(\{e^j\}) \subseteq H_0(\Delta_i^\Sigma \cap \mathbb{H}_t)$ to get $H^1(B_{i,t}^\Sigma, k)$.

Now, $(r - t) \in \operatorname{supp} f$ implies $r - t \geqslant \Gamma(f)$; in particular, we obtain

$$\langle a(r), r - t \rangle \geqslant m(a(r))$$

and therefore

$$\langle a(r), r \rangle \geqslant m(a(r)) - a(r)_i.$$

Finally, each face $a$ of $\Gamma(f)$ with $\langle a, r \rangle < m(a)$ could be taken instead of $a(r)$, and we obtain $r \in M_i$. □

REMARK. Condition (2c) guarantees that there are only a few (in particular a finite number of) $t \in M$ fulfilling (2).

(4.7) In (2.6)–(2.8) of [Al] we already tried to describe the image of $\gamma$.

For elements $\xi \in H^1(X, \Theta_X\langle -D - Y \rangle)$ (given explicitly by a 1-cocycle $\{\xi_{\alpha\beta}\}$) the induced deformation $\gamma(\xi)_{<\Gamma(f)}$ was computed directly. Now, we want to give a short dictionary to understand this formulae in the cohomological language used here. This language is more suitable to see what really happens.

(i) For $i = 1, 2, 3$ we obtain elements $\xi(x_i) \in H^1(X, \mathcal{O}_X(-\Sigma_{a>0} a_i D_a))$ (given by $\xi_{\alpha\beta}(x_i)$ in [Al]).

(ii) The exact sequence

$$0 \to \mathcal{O}_X\left(-\sum_{a>0} a_i D_a\right) \to \mathcal{O}_X \to \mathcal{O}_{\Sigma a_i D_a} \to 0,$$

together with $H^1(X, \mathcal{O}_X) = 0$, shows that $\xi(x_i)$ can be lifted to an element $b_i \in H^0(X, \mathcal{O}_{\Sigma a_i D_a})$.

(In [Al] these sections are given locally by $b_i^\alpha \in \mathcal{O}_X$:

$$\xi_{\alpha\beta}(x_i) = b_i^\beta - b_i^\alpha \text{ for every two cones } \alpha, \beta \in \Sigma.)$$

(iii) Multiplying by $\partial f / \partial x_i$ provides a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-\Sigma_{a>0} a_i D_a) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\Sigma a_i D_a} & \longrightarrow & 0 \\ & & \big\downarrow \cdot(\partial f/\partial x_i) & & \big\downarrow \cdot(\partial f/\partial x_i) & & \big\downarrow \cdot(\partial f/\partial x_i) & & \\ 0 & \longrightarrow & \mathcal{O}_X(-\Sigma\, m(a) D_a) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\Sigma m(a) D_a} & \longrightarrow & 0. \end{array}$$

Therefore, we obtain $\Sigma_{i=1}^3 (\partial f / \partial x_i) b_i \in H^0(X, \mathcal{O}_{\Sigma m(a) D_a})$ – still written as a local $\mathcal{O}_X$-section in [Al].

(iv) Finally, we recall the isomorphism

$$H^0_{(D)}(X, \mathcal{O}_{\Sigma m(a)D_a}) \tilde{\to} H^1_D(X, \mathcal{O}_X(-\Sigma m(a)D_a)) = \frac{k[\mathbf{x}]}{\langle \text{monomials} \geqslant \Gamma(f)\rangle}.$$

## 5. An algorithm to determine the equisingular deformations below $\Gamma(f)$

(5.1) Analogously to Proposition (4.6) it is possible to compute all deformations of $\overline{\mathrm{ES}}(k[\varepsilon]) \subseteq \mathrm{Def}_R(k[\varepsilon])$. The corresponding algorithm does not use any of the smooth subdivisions of $\Sigma_0$ regarded before, but only the starting f.r.p.p. decomposition $\Sigma_0$ itself.

Let $\Delta_i := \cup\{\bar{\alpha}/\bar{\alpha} \in \Sigma_0,\ e^i \notin \bar{\alpha}\} \subseteq \Delta$ $(i = 1, 2, 3)$ and take the definition of $M_i \subseteq M$, $a: M_i \to \Sigma_0^{(1)}$ and $\mathbb{H}_t$ of (4.6).

THEOREM. (I) *Given the following data*

(1) $i \in \{1, 2, 3\}$,
(2) $t \in M$ *with*: (a) $t_i = -1$
(b) (i) $t_\nu \geqslant 0$ (*i.e.* $e^\nu \notin \mathbb{H}_t$) *for all* $\nu \neq i$, *or*
(ii) $t_j \leqslant -1$ $(i \neq j)$ *and the remaining component is* $\geqslant 0$,
(c) *there exists an* $r \in M_i$ *with* $r - t \geqslant \Gamma(f)$ *and* $\langle a(r), t + e_i\rangle \geqslant 0$,
(3) *a connected component* $C$ *of* $\Delta_i \cap \mathbb{H}_t$, *not containing any of the vertices* $e^1, e^2, e^3$,

*then, the deformation defined by*

$$\sum_{\substack{r \in M_i \\ a(r) \in C}} (r_i + 1)\lambda_{r-t} \cdot \mathbf{x}^r = \left(\mathbf{x}^{t+e_i} \cdot \frac{\partial f}{\partial x_i}\right)\Bigg|_{M_i \cap a^{-1}(C)}$$

*is contained in* $\overline{\mathrm{ES}}(k[\varepsilon])$.

(II) $\mathrm{ES}(k[\varepsilon]) \subseteq \mathrm{Def}_R(k[\varepsilon])$ *as a k-vectorspace is spanned by the above-*$\Gamma(f)$*-deformations and all deformations constructed in the above way.*

*Proof.* Take the three resolutions $\Sigma_\nu$ $(\nu = 1, 2, 3)$ of (2.4). Then, by Theorem (3.4) and Proposition (4.6) the above claim were valid if the $\Delta_i$ would be replaced by $\Delta_i^{\Sigma_\nu}$ and the resulting elements of $\overline{\mathrm{ES}}(k[\varepsilon])$ were put together for $\nu = 1, 2, 3$.

STEP 1. *Each deformation that is induced by a* $\Delta_i^{\Sigma_\nu} \cap \mathbb{H}_t$ *can also be obtained by using* $\Delta_i \cap \mathbb{H}_t$.

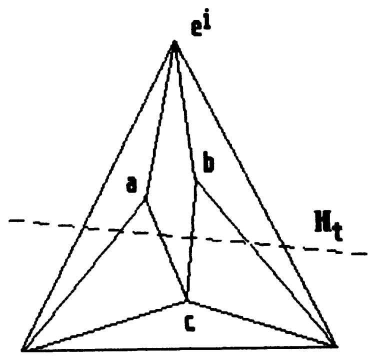
Let $i, \nu \in \{1, 2, 3\}$, $t \in M$ be fixed. By construction it is clear that $\Delta_i \subseteq \Delta_i^{\Sigma_\nu}$, hence $\Delta_i \cap \mathbb{H}_t \subseteq \Delta_i^{\Sigma_\nu} \cap \mathbb{H}_t$.

Now, both sets contain the same elements of $\Sigma_0^{(1)}$, and the connected components of $\Delta_i^{\Sigma_\nu} \cap \mathbb{H}_t$ (restricted to $\Delta_i \cap \mathbb{H}_t$) are built by taking the union of several complete components of $\Delta_i \cap \mathbb{H}_t$.

For the deformations induced by $\Delta_i^{\Sigma_v} \cap \mathbb{H}_t$ this means that they split into sums of deformations induced by $\Delta_i \cap \mathbb{H}_t$.

STEP 2. *The connected components of $\Delta_i \cap \mathbb{H}_t$ and $\Delta_i^{\Sigma_i} \cap \mathbb{H}_t$ correspond to each other and contain the same elements of $\Sigma_0^{(1)}$*:

Let $a, b \in \Sigma_0^{(1)} \cap [\Delta_i \cap \mathbb{H}_t]$ be contained in different components of $\Delta_i \cap \mathbb{H}_t$, then they can be separated by a line segment $\overline{ce^i}$ (contained in a cone of $\Sigma_0$) with $c \notin \mathbb{H}_t$.




By the construction of $\Sigma_i$ (cf. (2.4)), this f.r.p.p. decomposition contains $\overline{P_i(c)e^i}$ as one cone of the canonical partition of $\overline{ce^i}$. Because of $t_i = -1$, $\langle c, t\rangle \geqslant 0$ implies $\langle P_i(c)\, t\rangle \geqslant 0$, and $\overline{P_i(c)e^i}$ will separate $a$ and $b$ as elements of $\Delta_i^{\Sigma_i} \cap \mathbb{H}_t$. (The opposite direction was already done in step 1.) □

REMARK. (1) $\overline{\mathrm{ES}}(k[\varepsilon])/\langle \text{monomials} \geqslant \Gamma(f)\rangle$ is generated by the columns of the following matrix $A$:

The *rows* correspond to elements $r \in \bigcup_{i=1}^{3} M_i$,

the *columns* correspond to triples $(i, t, C)$ with (I), (1)–(3) of the above Theorem, and

$$a_{r,(i,t,C)} := \begin{cases} (r_i + 1)\cdot \lambda_{r-t} & \text{for } a(r) \in C \\ 0 & \text{otherwise} \end{cases}.$$

(Of course, this matrix does not depend on the special choice of the function $a: M_i \to \Sigma_0^{(1)}$.)

To make the computation of all possible $t$ easier, it is useful to give coarser restrictions than those of (I)(2):

Let $i = 1$, hence $t_1 = -1$.

$r - t \geqslant \Gamma(f)$ implies $t \leqslant r$; together with $\langle a(r), t + e_1 \rangle \geqslant 0$ this gives the following conditions for $t$:

There exists an $r \in M_1$: $\quad -\dfrac{a_3(r)}{a_2(r)} \leqslant t_2 \leqslant r_2$

$$-\frac{a_2(r)}{a_3(r)} \leqslant t_3 \leqslant r_3$$

with $t_2$ and $t_3$ not both negative.

(2) All type-($i$)-deformations in $\overline{\mathrm{ES}}(k[\varepsilon])$ (cf. (I.2.b) of the Theorem) consist of pieces of trivial deformations (i.e. trivial deformations in which some terms are dropped). If, moreover, $\Delta_i \cap \mathbb{H}_t$ is connected, then the corresponding deformation will be really trivial.

(3) Compare with Theorem (5.8) of [Al]: If the sets $\Delta_i$ are convex, there will be no type-(ii)-deformations, and all deformations of type (i) will be trivial. Hence, all equisingular deformations are above $\Gamma$.

(5.2) COROLLARY. *The k-vectorspace* $\overline{\mathrm{ES}}(k[\varepsilon])/\langle\text{monomials} \geqslant \Gamma(f)\rangle$ *and, in particular, the fact whether* $\overline{\mathrm{ES}}$ *is exactly the functor of above-*$\Gamma(f)$*-deformations or not, are independent of the coefficients* $\lambda_s$ *of* $f$ *with*

$$\langle a, s \rangle \geqslant m(a) + \max\{a_1, a_2, a_3\} \quad \text{for all } a \in \Sigma_0^{(1)} \backslash \{e^1, e^2, e^3\}.$$

*Proof.* Let $\lambda_s$ be a coefficient of $f$ that appears in the matrix $A$ (defined in the previous remark). If

$$a_{r,(i,t,C)} = s_i \cdot \lambda_s \quad (s = r - t),$$

then we take the element $a := a(r) \in \Sigma_0^{(1)} \backslash \{e^1, e^2, e^3\}$, and now we obtain

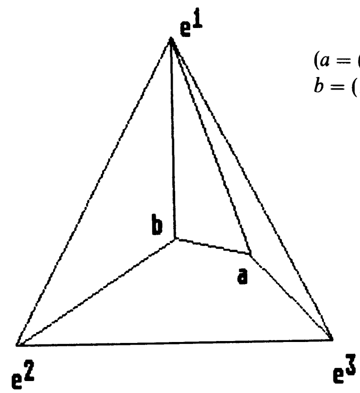
$$\langle a, r \rangle < m(a) \quad \text{(by definition of } a(r)),$$
$$\langle a, t \rangle \geqslant -a_i \quad \text{(by (I)(2c) of the Theorem)},$$

hence,

$$\langle a, s \rangle < m(a) + a_i. \qquad \square$$

(5.3) EXAMPLE. Let $f(x, y, z) := x^5 + y^6 + z^5 + y^3z^2$ (cf. [Al], §3); we get

$\Sigma_0 =$




($a = (12, 10, 15)$; $m(a) = 60$ and $b = (1, 1, 1)$; $m(b) = 5$).

(1) $\Delta_1$ is convex, hence, the case $i = 1$ yields only trivial deformations (cf. remark (5.1)(3)).

(2) Let $i = 2$.

*Computation of* $M_2$:
$r \in M_2$ iff $r \geqslant 0$,

$$12r_1 + 10r_2 + 15r_3 \geqslant 60 - 10 = 50,$$
$$r_1 + r_2 + r_3 \geqslant 5 - 1 = 4 \quad \text{and}$$
$$[12r_1 + 10r_2 + 15r_3 < 60 \quad \text{or} \quad r_1 + r_2 + r_3 < 5].$$

We obtain

$$M_2 = \{(0, 0, 4); (0, 1, 3); (0, 2, 2); (1, 0, 3); (1, 1, 2); (2, 0, 2); (3, 0, 1); (0, 4, 1); (0, 5, 0); (1, 3, 1); (1, 4, 0); (2, 2, 1); (2, 3, 0); (3, 2, 0); (4, 1, 0)\}.$$

*Conditions for* $t$:
For $a(r)$ ($r \in M_2$) there are only two candidates: (12, 10, 15) and (1, 1, 1). We obtain the conditions

$$-\tfrac{15}{12} \leqslant t_1 \leqslant 4; \quad t_2 = -1;$$
$$-1 \leqslant t_3 \leqslant 4 \quad (t_1, t_3 \text{ not both negative}).$$

*Connected components of* $\Delta_2 \cap \mathbb{H}_t$:
The only possibility for $\Delta_2 \cap \mathbb{H}_t$ to have at least two components is

$$e^3, (1, 1, 1) \in \mathbb{H}_t$$
$$e^1, (12, 10, 15) \notin \mathbb{H}_t.$$

Hence,

$$0 \leqslant t_1 \leqslant 4, \qquad t_2 = t_3 = -1,$$
$$t_1 + (-1) + (-1) < 0, \quad 12t_1 + (-10) + (-15) \geqslant 0,$$

and we get a contradiction.

(3) Let $i = 3$.
By the same methods as in the previous case we obtain

$$M_3 = M_2 \cup \{(0, 3, 1); (1, 2, 1); (2, 1, 1); (3, 1, 0); (4, 0, 0)\}$$

and the conditions for $t$:

$$-1 \leqslant t_1 \leqslant 4, \quad -\tfrac{12}{10} \leqslant t_2 \leqslant 5 \quad \text{and} \quad t_3 = -1 \qquad (t_1, t_2 \text{ not both negative}).$$

*Connected components of* $\Delta_3 \cap \mathbb{H}_t$:
The only possibility for $\Delta_3 \cap \mathbb{H}_t$ to have at least two components is

$$e^2, (12, 10, 15) \in \mathbb{H}_t$$
$$e^1, (1, 1, 1) \notin \mathbb{H}_t.$$

Hence,

$$0 \leqslant t_1 \leqslant 4, \qquad t_2 = t_3 = -1,$$
$$12t_1 + (-10) + (-15) < 0, \qquad t_1 + (-1) + (-1) \geqslant 0.$$

We obtain the only solution $t^0 = (2, -1, -1)$, and our matrix $A$ consists of exactly one column $(i = 3, t^0, C)$ (with $C \subseteq \Delta_3 \cap \mathbb{H}_{t^0}$ is the connected component containing the vertex (12, 10, 15)).

*Computation of the entries of* $A$:

$$a_{r,(3,t^0,C)} := \begin{cases} (r_3 + 1)\lambda_{r-t^0} & \text{for } a(r) \in C \\ 0 & \text{otherwise} \end{cases}.$$

Which elements $r \in \bigcup_{i=1}^{3} M_i \subseteq \{r \geqslant 0 / r < \Gamma(f)\}$ have the property $r - t^0 \geqslant \Gamma(f)$?

$$\langle (12, 10, 15), t^0 \rangle = -1, \qquad \langle (1, 1, 1), t^0 \rangle = 0;$$

therefore, such an $r$ has to meet the following conditions:

$$12r_1 + 10r_2 + 15r_3 = 60 - 1 = 59$$
$$r_1 + r_2 + r_3 \geqslant 5,$$

and the only solution is $r^0 = (2, 2, 1)$.

That means the only non-vanishing element in our matrix $A$ is

$$(r_3^0 + 1) \cdot \lambda_{r^0 - t^0} = 2 \cdot \lambda_{(0,3,2)} \text{ (in the row corresponding to } r^0 = (2, 2, 1)).$$

Since $(0, 3, 2) \in M$ represents a vertex of $\Gamma(f)$, the coefficient $\lambda_{(0,3,2)}$ can never vanish ($\lambda_{(0,3,2)} = 1$ in our special example). Therefore, we have proved

$$\frac{\overline{\mathrm{ES}}(k[\varepsilon])}{\langle \text{monomials} \geqslant \Gamma(f) \rangle} = k \cdot x^2 y^2 z,$$

not only for the special equation $f$, but for all equations having this special Newton boundary.

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