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Congruences for numerical data of an embedded resolution

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Introduction

Let k be an algebraically closed field of characteristic zero and $f \in k[x_1, \dots, x_{n+1}]$. Let (X, h) be an embedded resolution of $f = 0$ in $\mathbb{A}^{n+1}$, constructed by means of blowing-ups according to Hironaka's Main Theorem II [Hi, p. 142]. We denote by E_i, $i \in I$, the irreducible components of $h^{-1}(f^{-1}\{0\})$, and associate to each E_i a pair of numerical data (N_i, ν_i), where N_i and $\nu_i - 1$ are the multiplicities of E_i in the divisor of respectively $f \circ h$ and $h^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1})$ on X.

In dimension one ($n = 1$) some interesting relations are known between these data. Fix one exceptional curve E with numerical data (N, ν), say E intersects k times another irreducible component and denote these components by $E_1, \dots, E_k$. Then we have

$$\sum_{i=1}^{k} N_i \equiv 0 (\operatorname{mod} N) \quad \text{and} \quad \sum_{i=1}^{k} (\nu_i - 1) + 2 \equiv 0 (\operatorname{mod} \nu). \tag{$*$}$$

Moreover we can describe the quotients $(\Sigma_{i=1}^{k} N_i)/N$ and $(\Sigma_{i=1}^{k} (\nu_i - 1) + 2)/\nu$ in the two cases as $1 + \rho$, where ρ is the number of times that a point of E occurs as center of some blowing-up during the resolution process.

When $f(x_1, x_2)$ is absolutely analytically irreducible, only $k = 1$, 2 or 3 occurs. The cases $k = 1$ and $k = 2$ were obtained by Strauss [S, Th. 1] and Meuser [M, Lemma 1], and the case $k = 3$ by Igusa [I, Lemma 2]. Loeser [L, Lemme II.2] proved the general result.

If for $i = 1, \dots, k$ we set $\alpha_i = \nu_i - \dfrac{\nu}{N} N_i$, then we can derive from $(*)$ the relation

$$\sum_{i=1}^{k} (\alpha_i - 1) + 2 = 0. \tag{$**$}$$

In [V] we have proved relations between numerical data for arbitrary polynomials in all dimensions, extending the relation (**). Now in this paper we will prove divisibility properties between numerical data of exceptional varieties, extending the congruences (*), again for arbitrary polynomials in all dimensions, and describe the quotients.

Statement of the result

We prove in Theorems 3.3, 4.3 and 4.6 essentially the following. Fix one exceptional variety E with numerical data (N, ν). There are *basic congruences* (B1 and B2) associated to the creation of E in the resolution process, generalizing (*). And there are *additional congruences* (A) associated to each blowing-up of the resolution that 'changes' E. More precisely (using the same notations as above):

The variety E in the final resolution X is in fact obtained by a finite succession of blowing-ups

$$E^0 \xleftarrow{\pi_0} E^1 \xleftarrow{\pi_1} \cdots E^i \xleftarrow{\pi_i} E^{i+1} \cdots \xleftarrow{\pi_{m-2}} E^{m-1} \xleftarrow{\pi_{m-1}} E^m = E$$

with irreducible nonsingular center $C_i \subset E^i$ and exceptional variety $\mathscr{E}_{i+1} \subset E^{i+1}$ for $i = 0, \ldots, m-1$. The variety E^0 is created at some stage of the global resolution process as the exceptional variety of a blowing-up with center D and is isomorphic to a projective space bundle $\Pi\colon E^0 \to D$ over D.

For $i = 1, \ldots, m$ and for any variety $V \subset E^j$, $0 \leqslant j \leqslant i$, let the repeated strict transform of V in E^i (by $\pi_{i-1} \circ \cdots \circ \pi_j$) be denoted by $V^{(i)}$.

There are two kinds of intersections of E with other components of $h^{-1}(f^{-1}\{0\})$. We have the repeated strict transforms $\mathscr{E}_1^{(m)}, \ldots, \mathscr{E}_m^{(m)}$ in E of the exceptional varieties $\mathscr{E}_1, \ldots, \mathscr{E}_m$; and furthermore we have the repeated strict transforms $\mathscr{E}_i^{(m)}$ in E of certain varieties $\mathscr{E}_i$, $i \in T$, (of codimension one) in E^0.

For each $i \in T \cup \{1, \ldots, m\}$ the variety $\mathscr{E}_i^{(m)}$ is (an irreducible component of) the intersection of E with exactly one other component of $h^{-1}(f^{-1}\{0\})$. Let this component have numerical data (N_i, ν_i). Then we have

$$\text{(Congruence B1)} \quad \sum_{i \in T} d_i N_i \equiv 0 (\operatorname{mod} N),$$

where d_i, $i \in T$, is the degree of the intersection cycle $\mathscr{E}_i \cdot F$ on F for a general fibre $F \cong \mathbb{P}^{n-\dim D}$ of $\Pi\colon E^0 \to D$ over a point of D; and

$$\text{(Congruence B2)} \quad \sum_{\substack{i \in T \\ d_i \neq 0}} N_i \frac{\Pi_*(\mathscr{E}_i^k)}{d_i^{k-1}} + k \sum_{\substack{i \in T \\ d_i = 0}} N_i B_i = 0$$

in $\operatorname{Pic} D / N \operatorname{Pic} D$, where $k = n + 1 - \dim D$, $\mathscr{E}_i^k$ is the kth self-intersection of $\mathscr{E}_i$ in E^0, and $\mathscr{E}_i = \Pi^* B_i$ when $d_i = 0$.

Fix one $i \in \{0, \ldots, m-1\}$. Then we have also

$$\text{(Congruence A)} \quad N_{i+1} \equiv \sum_{k \in T \cup \{1, \ldots, i\}} \mu_k N_k \pmod N$$

where μ_k, $k \in T \cup \{1, \ldots, i\}$ is the multiplicity of the generic point of C_i on $\mathscr{E}_k^{(i)}$.

We have moreover analogous congruences for the ν-data, and we can describe in each case the corresponding quotient.

At the end of the paper we give some applications of these congruences concerning the poles of the topological zeta function.

Terminology

All schemes will be quasi-projective, a variety is an irreducible and reduced scheme, and subschemes (in particular points) are assumed to be closed except when stated otherwise. The reduced scheme associated to a scheme X is denoted by X_{red}.

We make no distinction between a divisor and its divisor class; which of both is meant should be clear from the context.

Let V be a subscheme of everywhere codimension one of a nonsingular scheme X. For all $x \in X$ we define the multiplicity $\mu_x(V)$ of x on V as the maximal integer m, such that the mth power of the maximal ideal of the local ring $\mathcal{O}_{X,x}$ of x on X contains the ideal of V in $\mathcal{O}_{X,x}$.

1. Embedded resolution

Let k be an algebraically closed field of characteristic zero and $f \in k[x_1, \ldots, x_{n+1}]$ a polynomial over k. Let Y denote the zero set of f in affine $(n+1)$-space $\mathbb{A}^{n+1}$ over k and $Y_l, l \in I$, its reduced irreducible components. We exclude the trivial case $f \in k$, so Y is a subscheme of codimension one of $\mathbb{A}^{n+1}$. We fix an embedded resolution (X, h) for Y in $\mathbb{A}^{n+1}$ in the sense of Hironaka's Main Theorem II [Hi, p. 142] by means of monoidal transformations.

Recall that if $g: \tilde{Z} \to Z$ is a monoidal transformation or blowing-up of the scheme Z with center a subscheme D of Z, then the exceptional divisor $E = g^{-1}D$ is everywhere of codimension one on $\tilde{Z}$, and the restriction $g|_{\tilde{Z}\setminus E}: \tilde{Z}\setminus E \to Z\setminus D$ is an isomorphism. For any subscheme V of Z, the closure of $g^{-1}(V\setminus D)$ in $\tilde{Z}$ is called the *strict transform* of V by g. If Z and D are nonsingular varieties, then the same is true for $\tilde{Z}$ and E.

The embedded resolution (X, h) consists of the following data.

Set $X_0 = \mathbb{A}^{n+1}$, $Y^{(0)} = Y$, and $Y_l^{(0)} = Y_l$ for all $l \in I$. For $i = 0, \ldots, r-1$ we have a finite succession of monoidal transformations $g_i: X_{i+1} \to X_i$ with irreducible nonsingular center $D_i \subset X_i$ and exceptional variety $E_{i+1}^{(i+1)} \subset X_{i+1}$ subject to the following conditions.

Let $E_j^{(i+1)}$, $Y^{(i+1)}$ and $Y_l^{(i+1)}$ denote the strict transform of respectively $E_j^{(i)}$, $Y^{(i)}$ and $Y_l^{(i)}$ by g_i for $j = 1, \ldots, i$ and all $l \in I$. Then

(1) for $i = 0, \ldots, r-1$ we have $D_i \subset Y^{(i)}$, $\operatorname{codim}(D_i, X_i) \geqslant 2$, and the multiplicity on $Y^{(i)}$ of all $x \in D_i$ equals the maximal multiplicity on $Y^{(i)}$ (i.e. $\mu_x(Y^{(i)}) = \max_{y \in Y^{(i)}} \mu_y(Y^{(i)})$ for all $x \in D_i$);
(2) $\bigcup_{1 \leqslant j \leqslant i} E_j^{(i)}$ has only normal crossings and only normal crossings with D_i (in X_i) for $i = 1, \ldots, r-1$; and
(3) $(\bigcup_{1 \leqslant j \leqslant r} E_j^{(r)}) \cup (\bigcup_{l \in I} Y_l^{(r)}) = [(g_{r-1} \circ \cdots \circ g_0)^{-1}(Y)]_{\text{red}}$ has only normal crossings in X_r. In particular all $Y_l^{(r)}$, $l \in I$, are nonsingular.

A reduced subscheme E of codimension one of a nonsingular variety X is said to have only normal crossings with a subscheme D of X, if for all $x \in D$ there exists a regular system of parameters $t_1, \ldots, t_m$ in the local ring $\mathcal{O}_{X,x}$ of X at x such that the ideal in $\mathcal{O}_{X,x}$ of each irreducible component of E containing x is generated by one of the t_i, and the ideal of D in $\mathcal{O}_{X,x}$ is generated by some of the t_i. When $D = X$ we say that E has only normal crossings.

Now we set $X = X_r$ and $h = g_{r-1} \circ \cdots \circ g_0$. The numerical data of the resolution (X, h) for Y are defined as follows.

For all irreducible components E of $(h^{-1}Y)_{\text{red}}$ (i.e. for all $E_j^{(r)}$, $1 \leqslant j \leqslant r$, and all $Y_l^{(r)}$, $l \in I$), let N be the multiplicity of E in the divisor of $f \circ h$ on X, and let $\nu - 1$ be the multiplicity of E in the divisor of $h^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1})$ on X. We have N, $\nu \in \mathbb{N}_0$; and if Y is reduced, then all $Y_l^{(r)}$ have numerical data $(N, \nu) = (1, 1)$.

From now on we fix one $j \in \{1, \ldots, r\}$.

We will describe how the exceptional variety E_j and its intersections with other exceptional varieties and with the strict transform of Y change by the blowing-ups g_i, $j \leqslant i < r$. So we fix one such $g_i\colon X_{i+1} \to X_i$ and set

$$g = g_i, \quad D = D_i, \quad E_{i+1} = E_{i+1}^{(i+1)}, \quad E = E_j^{(i)}, \quad \tilde{E} = E_j^{(i+1)}.$$

When $Y_k^{(i)}$, $k \in I$, or $E_k^{(i)}$, $1 \leqslant k \leqslant i$, intersects $E_j^{(i)}$, we set also

$$Y_k = Y_k^{(i)}, \quad \tilde{Y}_k = Y_k^{(i+1)}, \quad E_k = E_k^{(i)}, \quad \tilde{E}_k = E_k^{(i+1)}.$$

Since E has normal crossings with D we have the following important fact (see e.g. [GH, p. 605]).

PROPOSITION 1.1. *The restriction $g|_{\tilde{E}}\colon \tilde{E} \to E$ of g to $\tilde{E}$ is the blowing-up of E with (nonsingular) center $D \cap E$.*

Note that $D \cap E$ can eventually be reducible. The total blow-up of E with center $D \cap E$ can then be considered as the result of consecutive blowing-ups of E with centers the irreducible components of $D \cap E$. (Because E has normal crossings with D these centers are disjoint.)

PROPOSITION 1.2. *Let E^* denote the exceptional divisor of the blowing-up $g|_{\tilde{E}}$: $\tilde{E} \to E$, and $\bar{Z}$ the strict transform in $\tilde{E}$ of any subscheme Z of E by $g|_{\tilde{E}}$. Then*

(i) $E^* = E_{i+1} \cap \tilde{E}$

and if $\operatorname{codim}(D \cap E, E) \geqslant 2$, *we have*

(ii) $\overline{E_k \cap E} = \tilde{E}_k \cap \tilde{E}$ *and* $\overline{(Y_k \cap E)_{\mathrm{red}}} = (\tilde{Y}_k \cap \tilde{E})_{\mathrm{red}}$.

(For an eventual proof see [V, Prop. 3.2].)

The remaining situation $\operatorname{codim}(D \cap E, E) = 1$ occurs if and only if $D \subset E$ and $\dim D = n - 1$. In this case $g|_{\tilde{E}}$: $\tilde{E} \to E$ is an isomorphism making E^* correspond to D. When D is not contained in respectively $(Y_k \cap E)_{\mathrm{red}}$ and $E_k \cap E$, Proposition 1.2(ii) above is still valid. In the other case we have

PROPOSITION 1.3. *Let E^* denote the exceptional divisor of the blowing-up $g|_{\tilde{E}}$: $\tilde{E} \to E$. If some irreducible component of $(Y_k \cap E)_{\mathrm{red}}$ is equal to D, then we can have in a small enough neighbourhood of the generic point of E^* either*

(i) $\tilde{Y}_k \cap \tilde{E} = \emptyset$ *or* $(\tilde{Y}_k \cap \tilde{E})_{\mathrm{red}} = E^*$.

If some irreducible component of $E_k \cap E$ is equal to D, then we have in a small enough neighbourhood of E^* always

(ii) $\tilde{E}_k \cap \tilde{E} = \emptyset$.

Now remember that we have fixed one $j \in \{1, \ldots, r\}$. In Section 2 we construct a representative of the self-intersection divisor of E_j (in the different stages of the resolution process), involving expressions in the N-data. We use this result to prove the congruences A in Section 3 and the congruences $B1$ and $B2$ in Section 4. In Section 5 we state the analogous congruences for ν-data. Then in Section 6 we give some examples where the congruences for N-data can be used in the study of the topological zeta function.

2. The self-intersection divisor on an exceptional variety

We first fix some notation concerning intersections. For any nonsingular variety V of dimension m, let $A(V)$ denote the Chow ring of V, i.e. $A(V) = \bigoplus_{i=0}^{m} A^i(V)$, where $A^i(V)$ denotes the group of cycles of codimension i on V modulo rational equivalence for $i = 0, \ldots, m$. Let $U \cdot W \in A^{k+l}(V)$ denote the intersection of $U \in A^k(V)$ and $W \in A^l(V)$. (If U and W are varieties we can consider $U \cdot W$ also in

$A^l(U)$ or $A^k(W)$.) Let also for any inclusion $\gamma\colon E \hookrightarrow X$ of codimension one of nonsingular varieties $E^2 = E \cdot E$ denote the self-intersection divisor of E in X, considered as an element of $A^1(E)(=\operatorname{Pic} E)$. Remember in this context that in $\operatorname{Pic} E$ we have $E^2 = \gamma^* E$.

Set during this section $N = N_j$, and let $\gamma\colon E_j^{(r)} \hookrightarrow X$ denote the inclusion of $E_j^{(r)}$ in X.

PROPOSITION 2.1. *Set $E = E_j^{(r)}$. Let E_i', $i \in T$, be the intersections $E_i^{(r)} \cap E$ or $Y_i^{(r)} \cap E$ of E with another exceptional variety $E_i^{(r)}$ or with a reduced irreducible component $Y_i^{(r)}$ of $Y^{(r)}$. Then in* Pic *E we have*

$$NE^2 = -\sum_{i \in T} N_i E_i'.$$

Proof. By definition of the N-data we have in $\operatorname{Pic} X$

$$\sum_{i=1}^{r} N_i E_i^{(r)} + \sum_{l \in I} N_l Y_l^{(r)} = 0$$

and thus

$$NE = -\sum_{\substack{i=1\\ i \neq j}}^{r} N_i E_i^{(r)} - \sum_{l \in I} N_l Y_l^{(r)}.$$

Because $(\bigcup_{1 \leqslant i \leqslant r} E_i^{(r)}) \cup (\bigcup_{l \in I} Y_l^{(r)})$ has only normal crossings in X, applying the pull-back homomorphism $\gamma^*\colon \operatorname{Pic} X \to \operatorname{Pic} E$ to this equality yields

$$N\gamma^* E = -\sum_{i \in T} N_i E_i'. \qquad \square$$

We shall now describe how the expression of Proposition 2.1 for the self-intersection divisor changes during the resolution process. We essentially use the following.

LEMMA 2.2. *Let Z be a nonsingular variety, D a proper nonsingular subvariety of Z, and $\Pi\colon \tilde{Z} \to Z$ the blowing-up of Z with center D and exceptional variety F. Let C be a prime divisor of Z and μ the multiplicity of the generic point of D on C (so $\mu \neq 0 \Leftrightarrow D \subset C$). Let also $\tilde{C}$ denote the strict transform of C in $\tilde{Z}$ and $\bar{\Pi}$ the restriction $\Pi|_{\tilde{C}}\colon \tilde{C} \to C$. Then in* Pic $\tilde{C}$ *we have*

$$\tilde{C}^2 = \bar{\Pi}^* C^2 - \mu \tilde{C} \cdot F.$$

Proof. Let γ and $\tilde{\gamma}$ denote the inclusion of respectively C in Z and $\tilde{C}$ in $\tilde{Z}$, and consider the commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\Pi} & Z \\ \uparrow{\scriptstyle\tilde{\gamma}} & & \uparrow{\scriptstyle\gamma} \\ \tilde{C} & \xrightarrow{\bar{\Pi}} & C. \end{array}$$

Applying the pull-back homomorphism $\tilde{\gamma}^*$ to the well-known equality $\Pi^* C = \tilde{C} + \mu F$ in $\operatorname{Pic} \tilde{Z}$ yields $\tilde{\gamma}^* \Pi^* C = \tilde{C}^2 + \mu \tilde{C} \cdot F$ in $\operatorname{Pic} \tilde{C}$. On the other hand we have $\tilde{\gamma}^* \Pi^* C = \bar{\Pi}^* \gamma^* C = \bar{\Pi}^* C^2$. Both right-hand sides then yield the stated equality. $\square$

Fix one blowing-up $g_i|_{E_j^{(i+1)}}: E_j^{(i+1)} \to E_j^{(i)}$ with $D_i \cap E_j^{(i)} \neq \emptyset$, and one irreducible component D of $D_i \cap E_j^{(i)}$. Set $E = E_j^{(i)}$ and let $g: \tilde{E} \to E$ be the blowing-up with center D, which can be considered as a composition factor of $g_i|_{E_j^{(i+1)}}$. (We suppose g to be the first blowing-up in the decomposition into such factors.) Let also E'_e denote the exceptional variety of g and $\tilde{E}'$ the strict transform by g of any prime divisor E' in E.

PROPOSITION 2.3. *First case*: $\operatorname{codim}(D, E) \geqslant 2$.
If $N\tilde{E}^2 = \Sigma_{i=1}^t a_i \tilde{E}'_i + aE'_e$ *in* $\operatorname{Pic} \tilde{E}$, *then*

(i) $NE^2 = \sum_{i=1}^{t} a_i E'_i$ *in* $\operatorname{Pic} E$,

and

(ii) $a = \sum_{i=1}^{t} \mu_i a_i - N\delta$,

where $\delta = 1$ *if* $D_i \subset E$ *and* $\delta = 0$ *if* $D_i \not\subset E$; *and* μ_i, $1 \leqslant i \leqslant t$, *is the multiplicity of the generic point of* D *on* E'_i.

Second case: $\operatorname{codim}(D, E) = 1$.
If $N\tilde{E}^2 = \Sigma_{i=1}^t a_i \tilde{E}'_i + aE'_e$ *in* $\operatorname{Pic} \tilde{E}$, *then*

(iii) $NE^2 = \sum_{i=1}^{t} a_i E'_i + (a + N)D$ *in* $\operatorname{Pic} E$.

Proof. First case

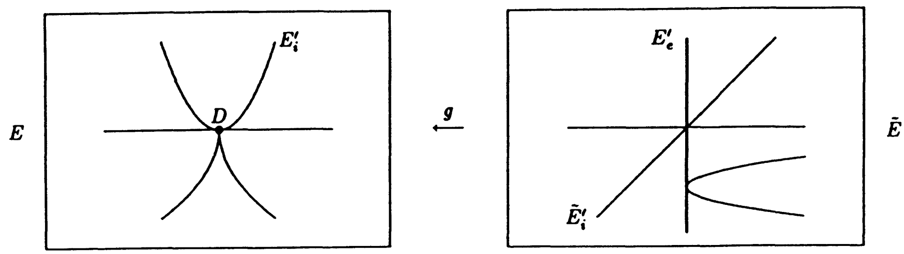

Fig. 1.

Because E has normal crossings with D_i, Lemma 2.2 implies that $\tilde{E}^2 = g^*E^2 - \delta E'_e$. So

$$\begin{aligned} g^*NE^2 - \delta NE'_e &= N\tilde{E}^2 \\ &= \sum_{i=1}^{t} a_i\tilde{E}'_i + aE'_e \\ &= \sum_{i=1}^{t} a_i(g^*E'_i - \mu_i E'_e) + aE'_e \\ &= g^*\left(\sum_{i=1}^{t} a_i E'_i\right) + \left(-\sum_{i=1}^{t} \mu_i a_i + a\right)E'_e. \end{aligned}$$

Since $\operatorname{Pic}\tilde{E} = g^* \operatorname{Pic} E \oplus \mathbb{Z}E'_e$ (where g^* is injective), we get the stated equalities.

Second case.

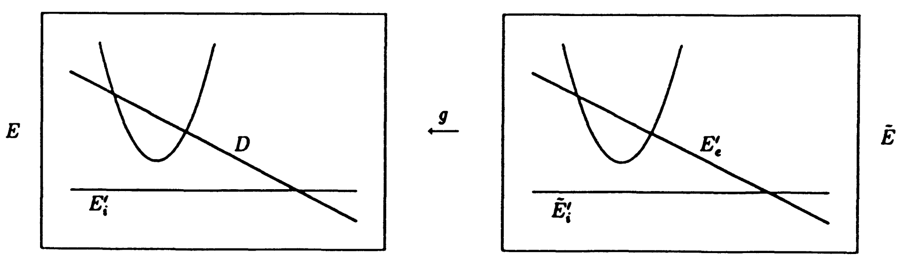

Fig. 2.

In this situation we have $D = D_i \subset E$, so Lemma 2.2 now implies that $\tilde{E}^2 = g^*E^2 - E'_e$. Since $g^*\colon \operatorname{Pic} E \to \operatorname{Pic}\tilde{E}$ is an isomorphism that makes D

correspond to E'_e, we have

$$\begin{aligned} g^*(NE^2 - ND) &= g^*NE^2 - NE'_e \\ &= N\tilde{E}^2 \\ &= \sum_{i=1}^{t} a_i\tilde{E}'_i + aE'_e \\ &= g^*\left(\sum_{i=1}^{t} a_iE'_i + aD\right), \end{aligned}$$

which implies statement (iii). □

3. Congruences associated to the blowing-ups of an exceptional variety

To simplify notations we now drop the j-indices, i.e., we set $E^{(i)} = E_j^{(i)}$ for all $i = j, \ldots, r$ and $N = N_j$.

Fix one blowing-up $g_i|_{E^{(i+1)}}: E^{(i+1)} \to E^{(i)}$ such that $D_i \cap E^{(i)} \neq \varnothing$ and $\operatorname{codim}(D_i \cap E^{(i)}, E^{(i)}) \geqslant 2$, and one irreducible component D of $D_i \cap E^{(i)}$. We will associate a congruence between numerical data to the blowing-up g of $E^{(i)}$ with center D, which can be considered as a composition factor of $g_i|_{E^{(i+1)}}$. (Here we suppose g to be the first blowing-up in the decomposition of $g_i|_{E^{(i+1)}}$ into such factors.)

The Propositions 1.1–1.3 imply the following.

PROPOSITION 3.1. (i) *Let E'_k, $k \in T$, be the reduced irreducible components of intersections of $E^{(i)}$ with other exceptional varieties $E_t^{(i)}$, $1 \leqslant t < i$, or with components $Y_l^{(i)}$, $l \in I$, of the strict transform $Y^{(i)}$ of Y. The repeated strict transform of E'_k in $E^{(r)}$ by the consecutive $g_l|_{E^{(l+1)}}: E^{(l+1)} \to E^{(l)}$, $i \leqslant l < r$, is equal to some irreducible component of the intersection of $E^{(r)}$ with another component of $(h^{-1}Y)_{\text{red}}$, say with $E_k^{(r)}$ or $Y_k^{(r)}$.*

(ii) *Let E'_e denote the exceptional variety of the blowing-up g. Also the repeated strict transform of E'_e in $E^{(r)}$ by the other factors of $g_i|_{E^{(i+1)}}$ and the consecutive $g_l|_{E^{(l+1)}}: E^{(l+1)} \to E^{(l)}$, $i + 1 \leqslant l < r$, is an irreducible component of the intersection of $E^{(r)}$ with some other exceptional variety, say with $E_e^{(r)}$.*

REMARK 3.2. In the proposition above $E_k^{(r)}$ is different from the corresponding $E_t^{(r)}$ and/or $Y_l^{(r)}$ if and only if the center of some $g_l|_{E^{(l+1)}}: E^{(l+1)} \to E^{(l)}$, $i < l < r$, contains the repeated strict transform of E'_k in $E^{(l)}$. The same remark holds for $E_e^{(r)}$ and $E_{i+1}^{(r)}$.

THEOREM 3.3. *Using the same notations as in Proposition* 3.1 *we have the* *following relation between the numerical data of $E^{(r)}$, $E_e^{(r)}$, and $E_k^{(r)}$ or $Y_k^{(r)}$, $k \in T$. Let μ_k, $k \in T$, be the multiplicity of the generic point of D on E'_k. Then*

$$\text{(Congruence A)} \quad N_e \equiv \sum_{k \in T} \mu_k N_k (\bmod N).$$

More precisely, suppose that the repeated strict transforms of E'_e and E'_k, $k \in T$, occur in the next steps of the resolution process respectively m_e and m_k times as center of blowing-ups $g_l|_{E^{(l+1)}}: E^{(l+1)} \to E^{(l)}$, $i+1 \leqslant l < r$. Then

$$\text{(Relation A)} \quad N_e = \sum_{k \in T} \mu_k N_k + \left(m_e - \sum_{k \in T} \mu_k m_k + \delta \right) N,$$

where $\delta = 1$ if $D_i \subset E^{(i)}$ and $\delta = 0$ if $D_i \not\subset E^{(i)}$.

Proof. Starting with the expression of Proposition 2.1 for N times the self-intersection of $E^{(r)}$, we consecutively apply Proposition 2.3(i) or 2.3(iii) to all composition factors of the $g_l|_{E^{(l+1)}}$, $r > l \geqslant i+1$ and to all factors of $g_i|_{E^{(i+1)}}$ except g (in the inverse order). We obtain

$$N\tilde{E}^2 = \sum_{k \in T} (-N_k + m_k N)\tilde{E}'_k + (-N_e + m_e N)E'_e,$$

where $\tilde{E}$ is the blowing-up of $E^{(i)}$ by g and $\tilde{E}'_k$, $k \in T$, is the strict transform of E'_k by g. Then by Proposition 2.3(ii) we have

$$-N_e + m_e N = \sum_{k \in T} \mu_k(-N_k + m_k N) - N\delta,$$

which is equivalent to the stated relation. □

REMARK 3.4. All N_k, corresponding to intersections on E_j arising after the creation of E_j as exceptional variety, can thus be written as linear expressions (with integer coefficients) in N and the N_k, corresponding to the intersections at its creation.

4. Congruences associated to the creation of an exceptional variety

In this section we set $E = E_j^{(j)}$, $D = D_{j-1}$, $\Pi = g_{j-1|_E}: E \to D$, and also $k = \operatorname{codim}(D, X_{j-1})$.

Let $\mathscr{I}$ be the ideal sheaf of D in X_{j-1}. Then [Ha, II, Th. 8.24] we know that E with the projection map Π is isomorphic to the projective space bundle $P(\mathscr{E})$ over D, associated to the locally free sheaf $\mathscr{E} = \dfrac{\mathscr{I}}{\mathscr{I}^2}$ of rank k on D. We denote by C the divisor corresponding to the invertible sheaf $\mathscr{O}(1)$.

In for example [F, Th. 3.3] we find

PROPOSITION 4.1. (i) *The homomorphism* $\Pi^*\colon A(D) \to A(E)$ *makes* $A(E)$ *into a free* $A(D)$*-module generated by* $1, C, C^2, \ldots, C^{k-1}$.

In particular we have

(ii) $\operatorname{Pic} E = \Pi^* \operatorname{Pic} D \oplus \mathbb{Z}C$, *where* Π^* *is injective.*

So for any prime divisor E' on E, we can write E' in $\operatorname{Pic} E$ as $E' = \Pi^* B + dC$, where $B \in \operatorname{Pic} D$ and $d \in \mathbb{N}$. The number d is the degree of the intersection cycle $E' \cdot F$ on F, where $F \cong \mathbb{P}^{k-1}$ is a general fibre of $\Pi\colon E \to D$ over a point of D. (When E is isomorphic to $\mathbb{P}^n$, d is just the degree of E'.) More precisely, F can in this determination of d be any fibre of Π that does not satisfy $F \subseteq E'$ (only occurring if $d = 0$).

The Propositions 1.1–1.3 imply

PROPOSITION 4.2. *Let* E'_i, $i \in T$, *be the reduced irreducible components of intersections of* E *with other exceptional varieties or with the strict transform of* Y. *The strict transform of* E'_i *in* $E^{(r)}$ *by the consecutive* $g_l|_{E^{(l+1)}}\colon E^{(l+1)} \to E^{(l)}, j \leqslant l < r$, *is equal to some irreducible component of the intersection of* $E^{(r)}$ *with another component of* $(h^{-1}Y)_{\text{red}}$, *say with* $E_i^{(r)}$ *or* $Y_i^{(r)}$.

THEOREM 4.3. *Using the same notations as in Proposition* 4.2, *we have the following relations between the numerical data of* $E^{(r)}$ *and* $E_i^{(r)}$ *or* $Y_i^{(r)}$, $i \in T$. *Let* $E'_i = \Pi^* B_i + d_i C$ *in* $\operatorname{Pic} E$ *for* $i \in T$. *Then*

$$\text{(Congruence B1)} \quad \sum_{i \in T} d_i N_i \equiv 0 \pmod N$$

and

$$\text{(Congruence B2')} \quad \sum_{i \in T} N_i B_i = 0 \quad \textit{in } \operatorname{Pic} D/N \operatorname{Pic} D.$$

More precisely, suppose that the repeated strict transforms of E'_i, $i \in T$, *occur in the next steps of the resolution process* m_i *times as center of blowing-ups* $g_l|_{E^{(l+1)}}\colon E^{(l+1)} \to E^{(l)}, j \leqslant l < r$. *Then we have*

$$\text{(Relation B1)} \quad \sum_{i \in T} d_i N_i = \left(1 + \sum_{i \in T} d_i m_i\right) N$$

and

$$\text{(Relation B2')} \quad \sum_{i \in T} N_i B_i = N \sum_{i \in T} m_i B_i \quad \textit{in } \operatorname{Pic} D.$$

Proof. Starting with the expression of Proposition 2.1 for N times the self-intersection of $E^{(r)}$, repeated applications of Proposition 2.3(i) or 2.3(iii) to all composition factors of the $g_l|_{E^{(l+1)}}$, $r > l \geqslant j$, yield

$$NE^2 = \sum_{i \in T} (-N_i + m_i N)E'_i$$

in $\operatorname{Pic} E$. Since $E^2 = -C$ (see [Ha, II, Th. 8.24c]) this is equivalent to

$$-NC = \Pi^* \left[\sum_{i \in T} (-N_i + m_i N) B_i \right] + \left[\sum_{i \in T} d_i(-N_i + m_i N) \right] C.$$

Now by Proposition 4.1(ii) we obtain the stated relations. □

REMARK 4.4. When $\operatorname{Pic} D$ is trivial, e.g. when D is a point, relation B2′ is of course inexistent.

EXAMPLE 4.5. When Y is a curve ($n=1$), only blowing-ups with a point as center occur in the resolution process. We have $E \cong \mathbb{P}^1$ and, since all E'_i are points on E, $d_i = 1$ for $i \in T$. So we obtain the familiar relation

$$\sum_{i \in T} N_i = \left(1 + \sum_{i \in T} m_i \right) N.$$

For those E'_i with $d_i \neq 0$, the divisor B_i has not really a "geometrical meaning". We will now rewrite relation B2′ in terms of the kth self-intersections $E_i'^k \in A^k(E)$ of the cycles E'_i in E.

THEOREM 4.6. *Using the same notations as in Proposition* 4.2, *we have the following relation between the numerical data of* $E^{(r)}$ *and* $E_i^{(r)}$ *or* $Y_i^{(r)}$, $i \in T$. *Let* $E'_i = \Pi^* B_i + d_i C$ *in* $\operatorname{Pic} E$ *for* $i \in T$. *Then*

$$\text{(Congruence B2)} \quad \sum_{\substack{i \in T \\ d_i \neq 0}} N_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} + k \sum_{\substack{i \in T \\ d_i = 0}} N_i B_i = 0 \quad \textit{in } \operatorname{Pic} D / N \operatorname{Pic} D.$$

More precisely, suppose that the repeated strict transforms of E'_i, $i \in T$, *occur in the next steps of the resolution process* m_i *times as center of blowing-ups* $g_l|_{E^{l+1}}: E^{(l+1)} \to E^{(l)}$, $j \leqslant l < r$. *Then we have*

$$\text{(Relation B2)} \quad \sum_{\substack{i \in T \\ d_i \neq 0}} N_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} + k \sum_{\substack{i \in T \\ d_i = 0}} N_i B_i = N \left[\mathfrak{e} + \sum_{\substack{i \in T \\ d_i \neq 0}} m_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} \right]$$

in $\operatorname{Pic} D$, *where* $\mathfrak{e}$ *is the first Chern class of the sheaf* $\mathscr{E}$.

Note. By the equality (*) in the proof below the expression $\frac{\Pi_*(E_i'^k)}{d_i^{k-1}}$ in the theorem can indeed be considered as an element of Pic D.

Proof. Fix $i \in T$ with $d_i \neq 0$. We can write $E_i'^k$ in $A^k(E)$ as

$$E_i'^k = d_i^k C^k + k d_i^{k-1}(\Pi^* B_i) \cdot C^{k-1} + \mathcal{O}_{k-2},$$

where $\mathcal{O}_{k-2}$ contains only terms in C^l, $0 \leqslant l \leqslant k-2$. Now by the definition of Chern classes of locally free sheaves we have

$$C^k = (\Pi^* \mathfrak{e}) \cdot C^{k-1} + \mathcal{O}'_{k-2},$$

where also $\mathcal{O}'_{k-2}$ contains only terms in C^l, $0 \leqslant l \leqslant k-2$. So

$$E_i'^k = \Pi^*(d_i^k \mathfrak{e} + k d_i^{k-1} B_i) \cdot C^{k-1} + (\mathcal{O}_{k-2} + d_i^k \mathcal{O}'_{k-2}).$$

For all $l \geqslant 2$ we have $\dim C^{k-l} > \dim D$ and thus $\Pi_*(C^{k-l}) = 0$; the projection formula (see e.g. Ha [p. 426]) then implies that

$$\Pi_*(E_i'^k) = (d_i^k \mathfrak{e} + k d_i^{k-1} B_i) \cdot \Pi_*(C^{k-1}).$$

Since $\Pi_*(C^{k-1}) = D$, we obtain

$$\Pi_*(E_i'^k) = d_i^k \mathfrak{e} + k d_i^{k-1} B_i. \tag{*}$$

We now substitute the equalities $kB_i = \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} - d_i \mathfrak{e}$, $i \in T$ and $d_i \neq 0$, in (k times) relation B2′. This yields

$$\sum_{\substack{i \in T \\ d_i \neq 0}} N_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} - \Bigg(\sum_{\substack{i \in T \\ d_i \neq 0}} d_i N_i\Bigg) \mathfrak{e} + k \sum_{\substack{i \in T \\ d_i = 0}} N_i B_i$$

$$= N \sum_{\substack{i \in T \\ d_i \neq 0}} m_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} - N\Bigg(\sum_{\substack{i \in T \\ d_i \neq 0}} d_i m_i\Bigg) \mathfrak{e} + kN \sum_{\substack{i \in T \\ d_i = 0}} m_i B_i.$$

Using relation B1 in the terms involving $\mathfrak{e}$, we get

$$\sum_{\substack{i \in T \\ d_i \neq 0}} N_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} + k \sum_{\substack{i \in T \\ d_i = 0}} N_i B_i = N\Bigg[\mathfrak{e} + \sum_{\substack{i \in T \\ d_i \neq 0}} m_i \frac{\Pi_*(E_i'^k)}{d_i^{k-1}} + k \sum_{\substack{i \in T \\ d_i = 0}} m_i B_i\Bigg].$$

One can prove by local computations that the E_i' with $d_i = 0$ never belong to the strict transform $Y^{(j)}$ of Y in $X^{(j)}$ and that consequently their associated number m_i is zero. Considering this fact we obtain the stated relation. $\square$

EXAMPLE 4.7. When Y is a surface ($n=2$), we only need blowing-ups with a point or a nonsingular curve as center in the resolution process. (When D is a point, relation B2 is of course inexistent.) If D is a nonsingular projective curve, then relation B2 becomes a numerical relation by taking degrees in $\operatorname{Pic} D$. Let $e = \deg \mathfrak{e}$ and $\kappa_i = \deg E_i'^2$, $i \in T$, the self-intersection number of E_i' in E. We obtain

$$\sum_{\substack{i\in T\\ d_i\neq 0}} \frac{\kappa_i}{d_i} N_i + 2 \sum_{\substack{i\in T\\ d_i = 0}} N_i = N\left(e + \sum_{\substack{i\in T\\ d_i\neq 0}} m_i \frac{\kappa_i}{d_i}\right).$$

(When $E_i' = \Pi^* B_i$ we must have $\deg B_i = 1$ since E_i' is irreducible.)

REMARK 4.8. In [V, Prop. 7.1 and Ex. 7.2] we showed how to compute the divisors $\Pi_*(E_i'^k)$ of Theorem 4.6 in terms of concrete intersection cycles.

5. Congruences between ν-data

Combining the relations between N-data and the relations proved in [V], we can immediately derive relations between ν-data. More precisely, a combination of respectively Theorem 3.3 and [V, Th. 4.4], Theorem 4.3 and [V, Th. 6.2], and Theorem 4.6 and [V, Th. 6.5] yields the following theorems.

THEOREM 5.1. *Using the same notations as in Proposition* 3.1 *we have the following relation between the numerical data of* $E^{(r)}$, $E_e^{(r)}$, *and* $E_k^{(r)}$ *or* $Y_k^{(r)}$, $k \in T$. *Let* μ_k, $k \in T$, *be the multiplicity of the generic point of* D *on* E_k'. *Then*

(Congruence A) $\quad \nu_e \equiv \sum_{k\in T} \mu_k(\nu_k - 1) + d \pmod{\nu},$

where $d = \operatorname{codim}(D, E^{(i)})$.

THEOREM 5.2. *Using the same notations as in Proposition* 4.2, *we have the following relations between the numerical data of* $E^{(r)}$ *and* $E_i^{(r)}$ *or* $Y_i^{(r)}$, $i \in T$. *Let* $E_i' = \Pi^* B_i + d_i C$ *in* $\operatorname{Pic} E$ *for* $i \in T$. *Then*

(Congruence B1) $\quad \sum_{i\in T} d_i(\nu_i - 1) + k \equiv 0 \pmod{\nu},$

where $k = \operatorname{codim}(D, X_{j-1})$; *and*

$$\text{(Congruence B2)} \quad \sum_{\substack{i\in T\\ d_i\neq 0}} (v_i - 1)\frac{\Pi_*(E_i'^k)}{d_i^{k-1}} + k \sum_{\substack{i\in T\\ d_i=0}} (v_i - 1)B_i - kK_D = 0$$

in $\operatorname{Pic} D/v \operatorname{Pic} D$, *where* K_D *is the canonical divisor on* D.

Moreover we can fully write out the corresponding relations with the same 'quotients' as the analogous N-relations.

6. Application to the topological zeta function

Let K be a number field and R its ring of algebraic integers. For any maximal ideal $\mathfrak{p}$ of R, let $R_\mathfrak{p}$ and $K_\mathfrak{p}$ denote the completion of respectively R and K with respect to the $\mathfrak{p}$-adic absolute value. Let $|x|$ denote this absolute value for $x \in K_\mathfrak{p}$, and let q be the cardinality of the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

Let now ψ be a character of $R_\mathfrak{p}^\times$ of order d, and let $f(x) \in K[x]$, $x = (x_1, \ldots, x_{n+1})$. To these data one associates Igusa's local zeta function

$$Z_\psi(s) = \int_{R_\mathfrak{p}^{n+1}} \psi(\mathrm{ac}\, f(x))\, |f(x)|^s\, |\mathrm{d}x|,$$

where $|\mathrm{d}x|$ denotes the Haar measure so normalized that $R_\mathfrak{p}^{n+1}$ has measure one.

One can compute $Z_\psi(s)$ using an embedded resolution for $f = 0$ in $\mathbb{A}^{n+1}$. Let (X, h) be such a resolution, using now all notations of Section 2. We also set $S = \{1, \ldots, r\} \cup I$ and $E_l^{(r)} = Y_l^{(r)}$ for $l \in I$. Denef [D, Th. 2.2] proved the following formula.

THEOREM 6.1. *Let* ψ *be a character of* $R_\mathfrak{p}^\times$ *which is trivial on* $1 + \mathfrak{p}R_\mathfrak{p}$. *Then for almost all* $\mathfrak{p}$ (*i.e. for all except a finite number*) *we have*

$$Z_\psi(s) = q^{-(n+1)} \sum_{\substack{I\subset S\\ \forall i\in I: d|N_i}} C_{I,\psi} \prod_{i\in I} \frac{q-1}{q^{v_i+sN_i} - 1},$$

where $C_{I,\psi}(\in \mathbb{C})$ *depends on* ψ *and the points of* $\bigcap_{i\in I} E_i^{(r)}$.

Now to d and f we can also associate the topological zeta function

$$Z_{d,\mathrm{top}}(s) = \sum_{\substack{I\subset S\\ \forall i\in I: d|N_i}} \chi(\mathring{E}_I) \prod_{i\in I} \frac{1}{v_i + sN_i},$$

where $\mathring{E}_I = (\bigcap_{i\in I} E_i^{(r)})\setminus(\bigcup_{j\in S\setminus I} E_j^{(r)})$. Here for any scheme V of finite type over K we denote by $\chi(V)$ the Euler–Poincaré characteristic of $V(\mathbb{C})$ with respect to singular cohomology. This zeta function can be constructed as a limit of Igusa's local zeta functions and does not depend on the choice of the resolution (X, h) for f [DL, Th. 2.1.2].

If the monodromy conjecture [D, Conj. 4.3] is true, we expect the following. Fix $j\in\{1,\dots,r\}$. If $E_j^{(r)}$ is 'in general position' with respect to its numerical data, i.e. there is no $E_i^{(r)}$, $i\in S\setminus\{j\}$, intersecting $E_j^{(r)}$ with $\frac{\nu_i}{N_i}=\frac{\nu_j}{N_j}$, and if $\chi(\mathring{E}_{\{j\}})=0$, then the contribution of $E_j^{(r)}$ to the residue of the candidate-pole $-\frac{\nu_j}{N_j}$ for $Z_{d,\mathrm{top}}(s)$ is zero. We will give some examples of this situation, using the congruences of this paper.

We now suppose $n=2$, we fix one $E_j^{(r)}$ 'in general position' such that $d|N_j$ and we set (as in Section 4) $E=E_j^{(j)}$, $D=D_{j-1}$, $\Pi=g_{j-1}|_E\colon E\to D$, and $(N,\nu)=(N_j,\nu_j)$. Let R_d denote the contribution of $E_j^{(r)}$ to the residue of the candidate-pole $-\frac{\nu}{N}$ for $Z_{d,\mathrm{top}}(s)$.

EXAMPLE 6.2. Suppose $D\cong\mathbb{P}^1$, card $T=k\geqslant 3$, $d_1=d_2=1$, $E_1'\cap E_2'=\varnothing$, and $d_i=0$ for $i=3,\dots,k$ (Fig. 3). So $\chi(\mathring{E}_{\{j\}})=0$. Suppose also that $E_j^{(r)}\cong E$.

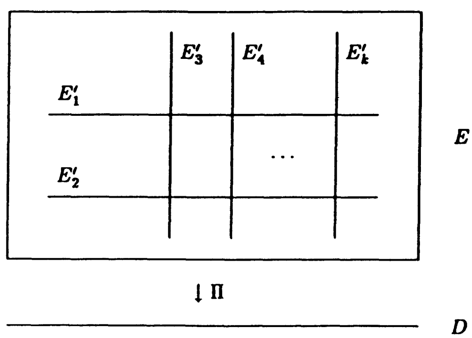

Fig. 3.

Let (N_i,ν_i) be associated to E_i' as in Proposition 4.2 and set $\alpha_i=\nu_i-\frac{\nu}{N}N_i$ for $i=1,\dots,k$.

When $d=1$ we have

$$R_d=\frac{2-(k-2)}{\alpha_1}+\frac{2-(k-2)}{\alpha_2}+\sum_{i=3}^{k}\frac{1}{\alpha_1\alpha_i}+\sum_{i=3}^{k}\frac{1}{\alpha_2\alpha_i}$$
$$=\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\cdot\left(4-k+\sum_{i=3}^{k}\frac{1}{\alpha_i}\right).$$

Now by [V, Th. 6.2] there is the relation $\alpha_1+\alpha_2=0$ between numerical data, implying that $R_d = 0$.

But if for example we should have $d \mid N_1$ and $d \nmid N_2$, then

$$R_d = \frac{4-k}{\alpha_1} + \sum_{\substack{i=3 \\ d\mid N_i}}^{k} \frac{1}{\alpha_1\alpha_i},$$

and there is no relation between numerical data available to make this expression zero. Now because of Theorem 4.3 this situation is impossible since congruence B1 states that

$$N_1 + N_2 \equiv 0 \pmod N,$$

implying that $d \mid N_1 \Leftrightarrow d \mid N_2$.

EXAMPLE 6.3. Suppose $D \cong \mathbb{P}^1$, card $T = 4$, $d_1 = d_2 = d_3 = 1$, $d_4 = 0$, and the curves $E'_1, \ldots, E'_4$ intersect as given by Fig. 4. So $\chi(\mathring{E}_{\{j\}}) = 0$. Suppose also that $E_j^{(r)}$ is obtained from E by one blowing-up with center P and exceptional curve E'_5 (Fig. 5).

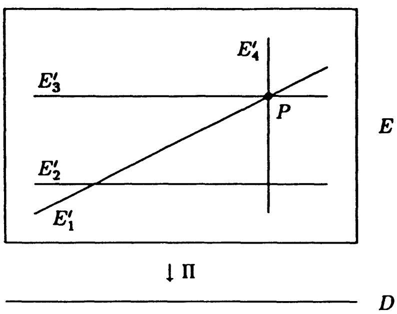

Fig. 4.

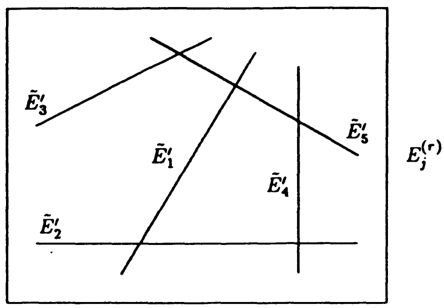

Fig. 5.

Let (N_i, ν_i) be associated to E'_i as in Proposition 4.2 or 3.1 and set $\alpha_i = \nu_i - \dfrac{\nu}{N} N_i$ for $i = 1, \ldots, 5$.

When $d = 1$ we have

$$\begin{aligned} R_d &= \frac{1}{\alpha_3} - \frac{1}{\alpha_5} + \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_5} + \frac{1}{\alpha_2\alpha_4} + \frac{1}{\alpha_3\alpha_5} + \frac{1}{\alpha_4\alpha_5} \\ &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_4}\right)\cdot\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_5}\right) + \left(\frac{1}{\alpha_3} - \frac{1}{\alpha_5} + \frac{1}{\alpha_3\alpha_5}\right). \end{aligned}$$

There are the relations

$$\begin{aligned}
&(\alpha_1-1)+(\alpha_2-1)+(\alpha_3-1)+2=0 && \text{by [V, Ex. 6.6b]},\\
&(\alpha_1-1)+(\alpha_4-1)=-2 && \text{by [V, Ex. 6.6c and 7.2]},\\
&\alpha_5=(\alpha_1-1)+(\alpha_3-1)+(\alpha_4-1)+2 && \text{by [V, Th. 4.4]},
\end{aligned}$$

implying that

$$\alpha_1+\alpha_4=0 \quad \text{and} \quad \alpha_5-\alpha_3+1=0.$$

So $R_d=0$.

Now if for example $d \nmid N_1$, $d \nmid N_2$, and $d \mid N_i$ for $i=3, 4, 5$; then we have

$$R_d=\frac{1}{\alpha_4\alpha_5}+\left(\frac{1}{\alpha_3}-\frac{1}{\alpha_5}+\frac{1}{\alpha_3\alpha_5}\right),$$

and we cannot make R_d zero by using the available relations. We will show that this situation, and more generally any 'problem situation', cannot occur.

By using [V, Ex. 7.2] we can compute the self-intersection number $\kappa_1=2$ and $\kappa_2=\kappa_3=0$. So Example 4.7 gives the relation

$$2N_1+2N_4=N(e+2m_1).$$

Now set $E_1'=C+b_1f$ in $\operatorname{Pic} E=\mathbb{Z}C\oplus\mathbb{Z}f$, where f is any fibre of Π and C corresponds to the invertible sheaf $\mathcal{O}(1)$ on E. Then $\kappa_1(=\deg E_1'^2)=e+2b_1$, which implies that e is even. We can thus derive the congruence

(i) $N_1+N_4\equiv 0 \pmod N$.

Moreover by congruence A of Theorem 3.3 we have

(ii) $N_5\equiv N_1+N_3+N_4 \pmod N$.

Because of (i) and (ii) we now see that

$$d\mid N_1 \Leftrightarrow d\mid N_4 \quad \text{and} \quad d\mid N_3 \Leftrightarrow d\mid N_5,$$

implying that $R_d=0$ for all d.

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