

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 79, n° 1 (1991), p. 99-108

<http://www.numdam.org/item?id=CM_1991__79_1_99_0>

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Classification of \mathcal{A} -simple germs from k^n to k^2

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Received 11 December 1989; accepted 15 October 1990

Introduction and notation

There exist extensive classifications of map-germs from n -space ($n \geq 2$) into the plane up to contact equivalence (\mathcal{K} -equivalence), see for example [D, Da, G, M, W]. In the present paper we refine a small part of these \mathcal{K} -classifications by studying right-left equivalence classes (\mathcal{A} -classes) contained in certain \mathcal{K} -orbits. In particular we obtain a classification of \mathcal{A} -simple germs from (complex and real) n -space ($n \geq 2$) into the plane (the \mathcal{A} -simple germs of plane curves $\mathbf{C} \rightarrow \mathbf{C}^2$ have been classified in [BG]).

Let $f: k^n, 0 \rightarrow k^p, 0$ be a smooth map-germ (where $k = \mathbf{C}$ or \mathbf{R} , and where smooth means analytic in the former and C^∞ or analytic in the latter case). Let $\mathcal{A} = \text{Diff}(k^n, 0) \times \text{Diff}(k^p, 0)$ denote the group of right-left equivalences, which acts on the space of smooth terms f as follows: $(h, k) \cdot f = h \circ f \circ k^{-1}$, where $(h, k) \in \mathcal{A}$. Replacing the action on the left, i.e. the composition with elements of $\text{Diff}(k^p, 0)$, by composition with elements of $Gl(p, k)$ with entries in C_n (where $C_n =$ local ring of smooth function germs $k^n, 0 \rightarrow k, 0$) gives the group \mathcal{K} of contact equivalences. A \mathcal{G} -orbit U (where $\mathcal{G} = \mathcal{A}$ or \mathcal{K}) is said to be adjacent to another \mathcal{G} -orbit V , denoted by $U \rightarrow V$, if any representative f of U can be embedded in an unfolding $F(u, \bar{f}(u, x))$, where $\bar{f}(0, x) = f(x)$, such that the set $\{u, x\}$ for which $\bar{f}(u, x) \in V$ contains $u = x = 0$ in its closure. A \mathcal{G} -orbit U is said to be \mathcal{G} -simple if it is adjacent to only a finite number of other \mathcal{G} -orbits.

Let C_n and C_p denote the local rings of function-germs in source and target whose respective maximal ideals are m_n and m_p . Let θ_f denote the C_n -module of vector fields over f , and set $\theta_n = \theta(1_{k^n})$ and $\theta_p = \theta(1_{k^p})$. One can then define the homomorphisms

$$tf: \theta_n \rightarrow \theta_f, \quad tf(\psi) = Df \cdot \psi,$$

and

$$wf: \theta_p \rightarrow \theta_f, \quad wf(\phi) = \phi \circ f.$$

²Research partially supported by Vitae, CNPq and British Council.

The tangent space to the \mathcal{A} -orbit at f can be calculated to be $T\mathcal{A} \cdot f = tf(m_n \cdot \theta_n) + wf(m_p \cdot \theta_p)$, and the \mathcal{A} -codimension of f is $\text{cod}(\mathcal{A}, f) = \dim_k m_n \cdot \theta_n / T\mathcal{A} \cdot f$.

Let $J^k(n, p)$ denote the space of k th order Taylor polynomials without constant terms, and write $j^k f$ for the k -jet of f . The Lie group $\mathcal{A}^k := j^k(\mathcal{A})$ acts smoothly on $J^k(n, p)$, and we shall write $T\mathcal{A}^k \cdot f$ for the corresponding tangent space $T_{j^k f} \mathcal{A}^k \cdot j^k f$. A germ f is said to be k -determined if, for any g , $j^k f = j^k g$ implies that $f \sim g$. The calculation of \mathcal{A}^k -orbits (using Mather's Lemma [Ma IV, Lemma 3.1]) and of the determinacy degree of a germ f (using an estimate of duPlessis [dP, Corollary 3.9]) are the main tools that we use in the present classification. See [Rie] for further details on notation and techniques.

We thank the referee for his critical remarks concerning the presentation of results.

1. Classification of \mathcal{A} -simple Σ^1 -germs from $k^n, 0$ to $k^2, 0, n > 1$

We can assume that $f: k^n, 0 \rightarrow k^2, 0$ is of the form $f(x, y) = (x, f_2(x, y))$, and we denote by X, Y the coordinates in the target. The following splitting lemma is almost content-free but clarifies the subsequent discussion (see also [PW, Prop. 0.6]).

LEMMA 1.1. *Every map-germ $f: k^n, 0 \rightarrow k^2, 0, n > 1$, of corank 1 is \mathcal{A} -equivalent to a germ of the form*

$$h(x, y, z) = \left(x, g(x, y_1, \dots, y_m) + \sum_{i=1}^{n-m-1} \varepsilon z_i^2 \right),$$

where $g(0, y_1, \dots, y_m) \in m_n^3$ and $\varepsilon = \pm 1$ (for $k = \mathbf{C}$, $\varepsilon = 1$); and $\text{cod}(\mathcal{A}, h) = \text{cod}(\mathcal{A}, (x, g)) + n - m - 1$.

Proof. Any $j^k f(x, y, z) = (x, f_2(x, y, z))$ can be reduced to h by right-coordinate changes of the form $\bar{z}_i = \phi(x, y, z)$, where the bar denotes new coordinates and not complex conjugation. It is also clear that $T\mathcal{A}^k \cdot j^k f, k > 1$, contains all monomials containing powers of z_i except for $z_i \cdot \partial/\partial Y$, which implies the last statement of the Lemma. □

Hence, if we take $m = 1$ the lemma above reduces the classification of germs $f: k^n, 0 \rightarrow k^2, 0$ to the classification of germs $g: k^2, 0 \rightarrow k^2, 0$ of corank 1. The \mathcal{A} -simple germs of the plane of corank 1 have been classified by one of the authors in [Rie]. Next, consider the case $m = 2$.

LEMMA 1.2. *For $m = 2$ (or indeed for $m \geq 2$) there are no \mathcal{A} -simple germs $f(x, y) = (x, g(x, y_1, \dots, y_m))$, where $g(0, y_1, \dots, y_m) \in m_{m+1}^3$ as in Lemma 1.1.*

Proof. The 2-jet of f is, by the hypothesis on g , given by $j^2 f =$

$(x, ax^2 + bxy_1 + cxy_2)$, which can either be reduced to (x, xy_1) , provided that either b or c is nonzero, or else to $(x, 0)$. First, we show that the \mathcal{A}^3 -orbits over $j^2f = (x, xy_1)$ are at least uni-modal. Note that we can reduce any 3-jet over (x, xy_1) to $h := (x, xy_1 + ay_1^3 + by_1^2y_2 + cy_1y_2^2 + dy_2^3)$. There are exactly four generators, namely $wh(X, Y) - th(x, 0, 0)$, $th(0, y_1, 0) - th(x, 0, 0) + wh(X, 0)$, $th(0, 0, y_1)$, and $th(0, 0, y_2)$, for the subspace $V := k\{(0, y_1^i y_2^j), i + j = 3\}$ of $T\mathcal{A}^3 \cdot h$, leading to the following matrix of coefficients:

$$\begin{bmatrix} a & b & c & d \\ 3a & 2b & c & \\ b & 2c & 3d & \\ & b & 2c & 3d \end{bmatrix}$$

which doesn't have maximal rank (because $(\text{row } 4) - 3 \times (\text{row } 1) = -(\text{row } 2)$). It follows from Mather's lemma [Ma IV] that V is foliated by (at least) a 1-parameter family of \mathcal{A}^3 -orbits.

The \mathcal{A}^3 -orbits over $(x, 0)$ are adjacent to those over (x, xy_1) , hence they are also non-simple.

For $m > 2$ the \mathcal{A}^2 -orbits are still those of (x, xy_1) and $(x, 0)$, and the modality of the \mathcal{A}^3 -orbits over (x, xy_1) and $(x, 0)$ is clearly greater than or equal to one. Lemma 1.2 now follows. □

Using the results of [Rie] we get the following classification.

PROPOSITION 1.3. *An \mathcal{A} -simple map-germ $f: k^n, 0 \rightarrow k^2, 0$ ($n > 1$) of corank 1 is equivalent to one of*

Type	$f(x, y, z_1, \dots, z_{n-2}) =$	$\text{cod}(\mathcal{A}, f)$
1	(x, y)	0
2	$(x, y^2 + \sum \varepsilon z_i^2)$	$n - 1$
3	$(x, xy + y^3 + \sum \varepsilon z_i^2)$	n
4_k	$(x, y^3 + \varepsilon^{k-1} x^k y + \sum \varepsilon z_i^2), k > 1$	$n + k - 1$
5	$(x, xy + y^4 + \sum \varepsilon z_i^2)$	$n + 1$
6	$(x, xy + y^5 + \varepsilon y^7 + \sum \varepsilon z_i^2)$	$n + 2$
7	$(x, xy + y^5 + \sum \varepsilon z_i^2)$	$n + 3$
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1} + \sum \varepsilon z_i^2), k > 1$	$n + k$
12	$(x, xy^2 + y^5 + y^6 + \sum \varepsilon z_i^2)$	$n + 3$
13	$(x, xy^2 + y^5 + \varepsilon y^9 + \sum \varepsilon z_i^2)$	$n + 4$
14	$(x, xy^2 + y^5 + \sum \varepsilon z_i^2)$	$n + 5$
16	$(x, x^2 y + y^4 + \varepsilon y^5 + \sum \varepsilon z_i^2)$	$n + 3$
17	$(x, x^2 y + y^4 + \sum \varepsilon z_i^2)$	$n + 4$

(where $\varepsilon = \pm 1$ for $k = \mathbf{R}$, and $\varepsilon = 1$ for $k = \mathbf{C}$).

REMARK 1.4. A deformation of a germ as in 1.1 does not increase m . The \mathcal{A} -classes of 1.3 are hence simple for all n , because they can only be adjacent to other ($m = 1$)-germs.

2. Classification of \mathcal{A} -simple $\Sigma^2,0$ germs from $k^n, 0$ to $k^2, 0$

The main result in this section is the following classification.

PROPOSITION 2.1. Any simple germ $f: k^n, 0 \rightarrow k^2, 0$ (for $n > 1$) of corank 2 is \mathcal{A} -equivalent to some member of the following series of germs:

- (i) $k = \mathbf{R}: I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1}), \quad l \geq m \geq 1, \text{ or}$
 $II_{2,2}^{l,m} = (x^2 - y^2 + x^{2l+1}, xy), \quad l \geq 1;$
- (ii) $k = \mathbf{C}: I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1}), \quad l \geq m \geq 1.$

The \mathcal{A} -codimension of $I_{2,2}^{l,m}$ and $II_{2,2}^{l,m}$ are $l + m + 2$ and $2(l + 1)$ respectively.

To prove this statement we classify \mathcal{A} -orbits contained in \mathcal{K} -simple orbits of germs $f: k^n, 0 \rightarrow k^2, 0$ of corank 2. Such \mathcal{K} -simple germs have been classified in [D, M, Da]. In the present classification of \mathcal{A} -simple germs we can discard all \mathcal{K} -orbits adjacent to some \mathcal{K} -orbit that doesn't contain any \mathcal{A} -simple orbits.

2.1. Classification of \mathcal{A} -orbits in $\mathcal{K}(x^2, y^2)$, for $k = \mathbf{C}$ or \mathbf{R}

PROPOSITION 2.1.1. Any germ contained in the \mathcal{K} -orbit of (x^2, y^2) is \mathcal{A} -equivalent to some member of the series

$$I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1}), \quad l \geq m \geq 1.$$

The $I_{2,2}^{l,m}$ are $(2l + 1)$ -determined, and $\text{cod}(\mathcal{A}, I_{2,2}^{l,m}) = l + m + 2$.

Proof. Any k -jet $(x^2 + \sum a_{i,j} x^i y^j, y^2 + \sum b_{i,j} x^i y^j)$, where the $x^i y^j$ are of degree $k > 1$, can be reduced to $(x^2 + a_{0,k} y^k, y^2 + b_{k,0} x^k)$ by the right-coordinate change

$$(\bar{x}, \bar{y}) = (x - \frac{1}{2}(a_{k,0} x^{k-1} + \dots + a_{1,k-1} y^{k-1}), \quad y - \frac{1}{2}(b_{k-1,1} x^{k-1} + \dots + b_{0,k} y^{k-1}))$$

Now, suppose $k = 2l$: the left-coordinate changes $\bar{X} = X - a_{0,2l} Y^l$ and $\bar{Y} = Y - b_{2l,0} X^l$ give (x^2, y^2) , which is the single \mathcal{A}^{2l} -orbit over $j^{2l-1} f = (x^2, y^2)$. If $k = 2l + 1$, we have three \mathcal{A}^{2l+1} -orbits over $j^{2l} f = (x^2, y^2)$: (i) $(x^2 + y^{2l+1}, y^2 + x^{2l+1})$, (ii) $(x^2, y^2 + x^{2l+1})$, and (iii) (x^2, y^2) . By a result of du Plessis [dP, Example 3.18], $(x^2 + y^{2l+1}, y^2 + x^{2l+1})$ is $(2l + 1)$ -determined. Now, consider \mathcal{A}^k -orbits over $j^{2m+1} f = (x^2, y^2 + x^{2m+1})$. If $k = 2l > 2m + 1$, we find a single \mathcal{A}^{2l} -orbit $(x^2, y^2 + x^{2m+1})$, by the same coordinate changes as above; and if

$k = 2l + 1 > 2m + 1$, we can reduce to $(x^2 + a_{0,2l+1}y^{2l+1}, y^2 + x^{2m+1})$ leading to two \mathcal{A}^{2l+1} -orbits given by $a_{0,2l+1} = 0, 1$. Now, $I_{2,2}^{l,m}$ is $(2l + 1)$ -determined, again by [dP, Example 3.18]. (see also [BPW, Example 6.7]). Finally, we check that

$$k\{(x, 0), (y^{2i+1}, 0), (0, y), (0, x^{2j+1}) : l > i \in \mathbf{N}, m > j \in \mathbf{N}\}$$

forms a free basis for $m_n \cdot \theta_f / T\mathcal{A} \cdot f$, where $f = I_{2,2}^{l,m}$, which proves the proposition. □

2.2. Classification of \mathcal{A} -orbits contained in $\mathcal{K}(x^2 - y^2, xy)$ over \mathbf{R}

PROPOSITION 2.2.1. *Any germ contained in the \mathcal{K} -orbit of $(x^2 - y^2, xy)$ is \mathcal{A} -equivalent to some member of the series*

$$II_{2,2}^l = (x^2 - y^2 + x^{2l+1}, xy), \quad l \geq 1.$$

The $II_{2,2}^l$ are $(2l + 1)$ -determined, and $\text{cod}(\mathcal{A}, II_{2,2}^l) = 2(l + 1)$.

Proof. The calculations are entirely routine and we omit them. The $(2l + 1)$ -determinacy of $II_{2,2}^l$ and its codimension follow from 2.1, since the family $II_{2,2}^l$ is equivalent over \mathbf{C} to $I_{2,2}^l$.

2.3. Other \mathcal{K} -orbits of type $\Sigma^{2,0}$ do not contain \mathcal{A} -simple orbits

The equidimensional case ($n = p = 2$) and the non-equidimensional case ($n > 2, p = 2$) are considered respectively in Proposition 2.3.1 and Proposition 2.3.2.

PROPOSITION 2.3.1. *Let $f: (k^2, 0) \rightarrow (k^2, 0)$, ($k = \mathbf{R}, \mathbf{C}$) be an \mathcal{A} -finitely determined germ of type $\Sigma^{2,0}$. If f is \mathcal{A} -simple then the \mathcal{K} -orbit of f is of type $I_{2,2}$ or $II_{2,2}$.*

PROPOSITION 2.3.2. *Let $f: (k^n, 0) \rightarrow (k^2, 0)$, $n \geq 3, k = \mathbf{R}, \mathbf{C}$ be any \mathcal{A} -finitely determined germ of type $\Sigma^{2,0}$. Then f is non-simple.*

When $n = p = 2$, the \mathcal{K} -orbit of any Σ^2 germ not of type $I_{2,2}$ or $II_{2,2}$ is adjacent either to $I_{2,3}$ or IV_3 (see [L., Theorem 2.1], for the description of the adjacencies of real \mathcal{K} -orbits of types Σ^1 and $\Sigma^{2,0}$, and [G] for the complex case). Therefore, Proposition 2.3.1 will follow from Lemma 2.3.3 and 2.3.4 below, where we show that $I_{2,3}$ and IV_3 have no \mathcal{A} -simple orbits.

LEMMA 2.3.3. *The \mathcal{A} -orbits within $I_{2,3}$ are all non-simple.*

Proof. A germ f within $I_{2,3}$ is \mathcal{K} -equivalent to $(x^2 + y^3, xy)$ ([M, VI]). We show that there is no open \mathcal{A} -orbit within $\mathcal{K}(x^2 + y^3, xy)$, which will imply that this \mathcal{K} -orbit is filled up entirely with non-simple \mathcal{A} -orbits.

After some simple coordinate changes, we may assume that any \mathcal{A} -finitely determined germ in $I_{2,3}$ has the form:

$$(x^2 + y^3 + axy^2 + bxy^3 + cy^4 + \Phi(x, y), xy), \Phi \in m^4$$

One easily checks that $m^3\theta_f \subset T\mathcal{K} \cdot f$.

Now, the relevant relations in $T\mathcal{A}^4 \cdot f$ are given by $tf(x^i y^i, 0)$, $tf(0, x^i y^i)$, $i + j = 1, 2, 3$, $wf(X, 0)$, $wf(Y, 0)$, $wf(0, X)$, $wf(0, Y)$.

Thus there are only 22 generators for the vector subspace $m^2\theta_f/m^5\theta_f$, which has dimension 24. In particular, $T\mathcal{A}^4 \cdot f \not\subseteq m^3\theta_f/m^5\theta_f$, and the modality of the \mathcal{A} -orbit of f within $\mathcal{K}(x^2 + y^3, xy)$ is greater than one. \square

LEMMA 2.3.4. *The \mathcal{A} -orbits within IV_3 are all non-simple.*

Proof. The proof follows immediately, since $\mathcal{K}(x^2 + y^2, x^3)$ (type IV_3) is adjacent to $\mathcal{K}(x, y^6)$ (type \mathcal{A}_5) ([L, Theorem 2.1]), which in turn is entirely filled up with non-simple \mathcal{A} -classes (This follows from Proposition 1.3, but see [Rie] for details). \square

We consider now the case $n \geq 3$, $p = 2$.

It is well known that if $n \geq 4$, the \mathcal{K} -modality of a pair of quadrics is greater or equal to one ([W]). Therefore, we only have to consider the case $n = 3$, $p = 2$.

In the complex case, there is only one \mathcal{K} -orbit of type $\Sigma^{2,0}$, whose normal form is $(x^2 + y^2, y^2 + z^2)$.

PROPOSITION 2.3.5. *Any \mathcal{A} -orbit of a finitely determined germ within $\mathcal{K}(x^2 + y^2, y^2 + z^2)$ is at least 1-modal.*

Proof. Let $f: (\mathbf{C}^3, 0) \rightarrow (\mathbf{C}^2, 0)$ be any \mathcal{A} -finitely determined germ within $\mathcal{K}(x^2 + y^2, y^2 + z^2)$. Then, with simple coordinate changes, $j^3 f$ can be reduced to:

$$(x^2 + y^2 + ax^3 + cz^3, x^2 + z^2 + by^3)$$

As before, the result follows from the information given by $T\mathcal{K}^3 \cdot f$ and $T\mathcal{A}^3 \cdot f$:

- (i) $T\mathcal{K} \cdot f + m^4\theta_f \supset m^3\theta_f$.
- (ii) Inspecting $T\mathcal{A}^3 \cdot f$ we see that the elements of degree three are given by $tf(x, 0, 0)$, $tf(0, y, 0)$, $tf(0, 0, z)$, $\frac{1}{3}[tf(x, y, z) - 2wf(X, 0) - 2wf(0, Y)]$ and by $J(f) \cdot m_2$ (where $J(f)$ is the Jacobian ideal) $(\text{Mod } m^4\theta_f)$.

They generate the following subspace of $m^3/m^4\theta_f$:

$$\mathbf{C} \{ \text{all mixed terms of degree three, } (x^3, x^3), (y^3, 0), (0, z^3) \\ \text{and } (ax^3 + cz^3, by^3) \} (\text{Mod } m^4\theta_f).$$

Hence $T\mathcal{A}^3 \cdot f \not\cong m_2^3\theta_f$, and comparing with (i) we get the result. \square

REMARK 2.3.6. $\mathcal{K}(x^2 + y^2, y^2 + z^2)$ splits into various real orbits. These real \mathcal{K} -orbits do not have open \mathcal{A} -orbit either. In fact, if the condition

$$tf(m_2\theta_2) + f^*(m_2)\theta_f = tf(m_2\theta_2) + wf(m_2\theta_2)$$

were true for any such real germ, we should have:

$$tf_{\mathbb{C}}(m_2\theta_2) + f_{\mathbb{C}}^*(m_2)\theta_{f_{\mathbb{C}}} = tf_{\mathbb{C}}(m_2\theta_2) + wf_{\mathbb{C}}(m_2\theta_2),$$

where $f_{\mathbb{C}}$ is its complexification. This clearly contradicts the above lemma.

Proposition 2.3.2 will follow from the above discussion.

3. Adjacencies of \mathcal{A} -simple $\Sigma^{2,0}$ -germs $f: k^2, 0 \rightarrow k^2, 0$

The adjacencies between \mathcal{A} -simple Σ^1 -germs from \mathbb{C}^2 to \mathbb{C}^2 are shown in [Rie]. For corank 2 terms we have the following

PROPOSITION 3.1. Figures 1 and 2 show the adjacencies of \mathcal{A} -simple $\Sigma^{2,0}$ -germs $f: k^2, 0 \rightarrow k^2, 0$ for $k = \mathbb{C}$ and $k = \mathbb{R}$, respectively. (To denote \mathcal{A} -classes we use the notation of Propositions 1.3 and 2.1).

Proof. As in [Rie] we use three invariants $m(f)$, $c(f)$, and $d(f)$, which are upper-semicontinuous under deformation, to rule out certain adjacencies. Let Σ and Δ denote the critical set and the discriminant of f , which are both germs of plane curves, and let $\delta(C)$ denote the well-known δ -invariant of a germ of a plane curve C (see [Mi]). The three invariants of f can then be calculated as follows: $m(f) = \dim_k C_n/f^*m_p$, $c(f) = \dim_k C_n/I$ (where $I =$ ideal defined by the vanishing of 2×2 minors of $\begin{bmatrix} Df \\ \nabla|Df| \end{bmatrix}$), and $d(f) = \delta(\Delta) - \delta(\Sigma) - c(f)$. (For germs $f: \mathbb{C}^2$,

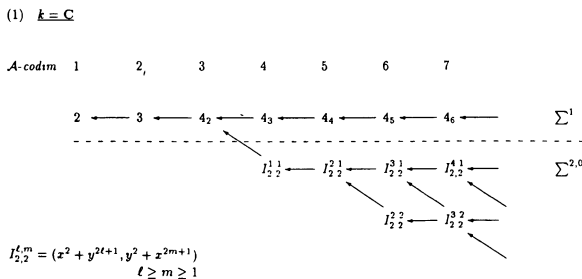


Fig. 1.

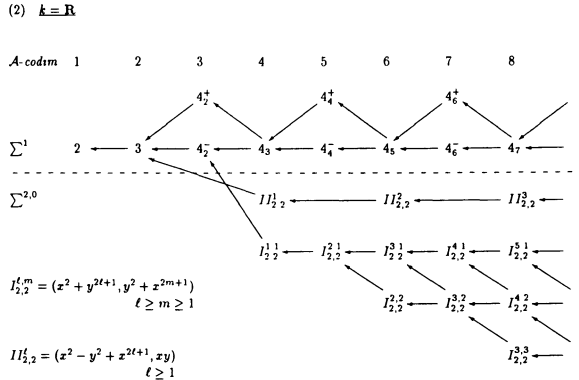


Fig. 2.

$0 \rightarrow \mathbb{C}^2, 0$ these invariants have the following geometrical meaning: $m(f)$ is the number of preimages of a target point off the discriminant Δ of f ; and $c(f)$ and $d(f)$ are the numbers of cusps and transverse fold crossings of a generic deformation of f).

For Σ^1 -germs these invariants have been calculated in [Rie] and for the \mathcal{A} -simple $\Sigma^{2,0}$ -germs of the classification in §2 we have the following

LEMMA 3.2. *The invariants m , c , and d associated with the members of the series of germs $I_{2,2}^{l,m}$ and $II_{2,2}^l$ have the values:*

$$m(I_{2,2}^{l,m}) = 4, \quad c(I_{2,2}^{l,m}) = 3, \quad d(I_{2,2}^{l,m}) = l + m;$$

and

$$m(II_{2,2}^l) = 4, \quad c(II_{2,2}^l) = 3, \quad d(II_{2,2}^l) = 2l.$$

These expressions also make sense for the “stems” of these series $I_{2,2}^{\infty} = (x^2, y^2)$, $I_{2,2}^{\infty,m} = (x^2, y^2 + x^{2m+1})$ and $II_{2,2}^{\infty} = (x^2 - y^2, xy)$.

Proof. These are just a trivial calculation. Note that the critical sets and the discriminants of the germs $I_{2,2}^{l,m}$ consist of two branches, so that $\delta(\Sigma)$ and $\delta(\Delta)$ are sums of δ -invariants of each branch and the (local) intersection numbers of the branch pairs. Also note that, as complex-analytic germs, $II_{2,2}^l \sim I_{2,2}^l$ (and that the dimensions of the relevant local algebras are not altered by complexifying).

Also notice that the Milnor number μ of the critical sets of the germs $I_{2,2}^{l,m}$ and $II_{2,2}^l$ is equal to one. The upper semicontinuity of these invariants and the adjacencies of \mathcal{A} -classes described in §2.3, together with the following lemma, conclude the proof of Proposition 3.1.

LEMMA 3.3

- (i) $k = \mathbf{C}$: the degenerate mono-germs in a versal deformation of $I_{2,2}^{1,1}$ are of type 4_2 ; and
- (ii) $k = \mathbf{R}$: the degenerate mono-germs in a versal deformation of $I_{2,2}^{1,1}$ are of type 4_2^- , and there are no degenerate mono-germs in a deformation of $II_{2,2}^1$.

Proof. We consider \mathcal{A} -versal unfoldings $F: k^d \times k^2, 0 \rightarrow k^d \times k^2, 0$, given by $F(u, x, y) = (u, f(u, x, y))$, where $f(0, x, y) = I_{2,2}^{1,1}$ or $II_{2,2}^1$. The set $B := \{u \in (k^d, 0) : c(f(u, 0, 0)) \geq 2\}$ gives all degenerate mono-germs in a deformation of $I_{2,2}^{1,1}$ or $II_{2,2}^1$, because $c \geq 2$ for any degenerate mono-germ of the plane and the origins in source and target are preserved under \mathcal{A} . Let $m_1(x, y), m_2(x, y)$, and $m_3(x, y)$ denote the determinants of the 2×2 minors of $\begin{bmatrix} Df(u, x, y) \\ \nabla |Df(u, x, y)| \end{bmatrix}$, where D is the differential of f with respect to x and y .

Now $c(f(u, 0, 0)) = \dim_k C_n / (m_1, m_2, m_3) \geq 2$ if and only if

$$m_1(0, 0) = m_2(0, 0) = m_3(0, 0) = 0$$

and the 2×2 minors of

$$\begin{bmatrix} \partial m_1(0, 0) / \partial x & \partial m_1(0, 0) / \partial y \\ \partial m_2(0, 0) / \partial x & \partial m_2(0, 0) / \partial y \\ \partial m_3(0, 0) / \partial x & \partial m_3(0, 0) / \partial y \end{bmatrix}$$

vanish. The six equations define an ideal I in $k[u_1, \dots, u_d]$.

First, consider the \mathcal{A} -versal deformation $f(u, x, y) = (u, x^2 + y^3 + u_1x + u_2y, y^2 + x^3 + u_3x + u_4y)$ of $I_{2,2}^{1,1}$. One calculates that $I = (u_1u_3 - u_4^2, u_1u_4 - u_2u_3, u_1^2 - u_2u_4, u_4(3u_1u_3 + 2u_2) + u_1(4u_1 + 3u_2^2), u_4(3u_3^2 + 4u_4) + u_1(2u_3 + 3u_2u_4), -(4u_1 + 3u_2^2)(3u_3^2 + 4u_4) + (3u_1u_3 + 2u_2)(2u_3 + 3u_2u_4))$, and, calculating a standard basis for I with respect to some lexicographical ordering of the variables in $k[u_1, \dots, u_4]$, one finds the following set of degenerate mono-germs: $B = \{u \in (k^4, 0) : u_1 = u_4 = u_2u_3 = 0\}$. Now, by direct coordinate changes, $f(0, u_2, 0, 0, x, y) \sim 4_2^-$ for $u_2 \in \mathbf{R} - \{0\}$ and $f(0, 0, u_3, 0, x, y) \sim 4_2^-$ for $u_3 \in \mathbf{R} - \{0\}$ (in the case $k = \mathbf{R}$), and $f(0, u_2, 0, 0, x, y) \sim f(0, 0, u_3, 0, x, y) \sim 4_2$ for $u_2, u_3 \in \mathbf{C} - \{0\}$ (for $k = \mathbf{C}$).

Finally, we consider the \mathcal{A} -versal deformation

$$f(u, x, y) = (x^2 - y^2 + x^3 + u_1x + u_2y, xy + u_3x + u_4y)$$

of the real germ $II_{2,2}^1$. Repeating the calculations above, one finds that $B = \{u \in (\mathbf{R}^4, 0) : u_1 = u_2 = u_3 = u_4 = 0\}$. Hence there are no degenerate mono-germs in a versal deformation of $II_{2,2}^1$, and the lemma follows.

Proof of Proposition 3.1; conclusion. Lemma 3.3 says that $I_{2,2}^{1,1}$ and $II_{2,2}^1$ are not adjacent to the Σ^1 -germ $(x, xy + y^4)$, which is the open \mathcal{A} -orbit in the \mathcal{K} -class A_3 . From the adjacencies in [Rie] of Σ^1 -germs it follows that none of the germs $I_{2,2}^{l,m}$ and $II_{2,2}^1$ is adjacent to some \mathcal{A} -orbit in A_3 . Finally, one checks that $II_{2,2}^1 \rightarrow 3$. \square

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