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On the variational Torelli problem for complete intersections

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Introduction

Since Carlson and Griffiths [4] introduced the notion of the *infinitesimal variation of Hodge structure* (IVHS for short), several authors obtained the generic Torelli theorems for hypersurfaces of various varieties (e.g., [7], [10], [18], [8] and [13]) by showing variational Torelli ([5]). Roughly, their arguments have two common ingredients:

- (1) Interpretation of IVHS by the “Jacobian ring”.
- (2) Recovering the variety in question from the algebraic data of IVHS rewritten in terms of its Jacobian ring.

Thus the study of the Jacobian ring played an important role in the case of hypersurfaces. However, in higher codimensional case (e.g., complete intersections), we do not know much of the generic (or variational) Torelli problem, probably because we have no proper method to do even (1).

The purpose of this article is to develop a method for the step (1) in the case that varieties in question are defined by sections of ample vector bundles. More precisely, we work in the following situation. Let \mathcal{E} be an ample vector bundle on a nonsingular projective variety X , $\pi: Y = \mathbb{P}(\mathcal{E}) \rightarrow X$ the associated projective bundle, and \mathcal{L} the tautological line bundle such that $\pi_*\mathcal{L} \simeq \mathcal{E}$. Then we canonically have $H^0(Y, \mathcal{L}) \simeq H^0(X, \mathcal{E})$. Suppose that $\sigma \in H^0(X, \mathcal{E})$ defines an irreducible nonsingular subvariety Z of codimension $\text{rank } \mathcal{E}$. Then we can consider the hypersurface \tilde{Z} defined by $\tilde{\sigma} \in H^0(Y, \mathcal{L})$ corresponding to σ . We compare the IVHS's of Z and \tilde{Z} to see that they are isomorphic in some sense. Since the “Jacobian ring” of \tilde{Z} can be defined by means of the first prolongation bundle $\Sigma_{\mathcal{L}}$ of \mathcal{L} as in [10], we then consider it as that of Z and we can study its duality properties.

The above argument can be applied successfully to complete intersections of projective hypersurfaces. This gives us a hope in solving the generic Torelli problem. Of course, the essential difficulty is in the step (2). In the case of hypersurfaces, we have a strong tool for (2), the *Symmetrizer Lemma* invented by

Donagi [7]. In our case, however, it seems that we cannot apply it. We analyze the *infinitesimal Schottky relation* [3] to show our generic Torelli theorem which covers only a few types of complete intersections. We must wait for some new techniques to proceed further.

The plan of this article is as follows. In §1 and §2, we prepare the fundamental tools which we shall need later. Especially, the results due to Atiyah [1] and Morimoto [15] play an important role. In §3, we define the Jacobian ring and study its duality properties. As mentioned above, this is essentially Green's work [10]. In §4 and §5, we compare the IVHS of Z with that of \tilde{Z} . We also give a sufficient condition to interpret the IVHS in terms of the Jacobian ring (Proposition 4.7 and Lemma 5.2). In §6, we study the IVHS of a nonsingular complete intersection in \mathbb{P}^n and show that its algebraic part can be described in terms of the Jacobian ring (Theorem 6.1). Finally, in §7, we show the generic Torelli theorem (Theorem 7.1) for some types of complete intersections in \mathbb{P}^n .

We remark that Terasoma [20] showed the generic Torelli theorem for some complete intersections of projective hypersurfaces of the same degree along an analogous line. Ours includes his result as a particular case. We also remark that the infinitesimal Torelli problem for complete intersections was solved by Peters [16], Usui [21] and Flenner [9].

1. Cohomology of projective space bundles

1.1. Let \mathcal{E} be a holomorphic vector bundle of rank r on an n -dimensional compact complex manifold X . We denote by \mathcal{E}_x the fiber of $\mathcal{E} \rightarrow X$ over $x \in X$. We do not distinguish \mathcal{E} with the locally free sheaf of its sections. Consider the holomorphic fiber bundle

$$\pi: Y = \mathbb{P}(\mathcal{E}) \rightarrow X,$$

where the fiber $\pi^{-1}(x)$, $x \in X$, is the space $\mathbb{P}(\mathcal{E}_x)$ of 1-dimensional linear subspaces of the dual space \mathcal{E}_x^* of \mathcal{E}_x . We have the exact sequence of tangent bundles

$$0 \rightarrow T_{Y/X} \rightarrow T_Y \rightarrow \pi^*T_X \rightarrow 0, \quad (1)$$

where $T_{Y/X}$ is the relative tangent bundle. Let \mathcal{L} be the line bundle of Y whose dual \mathcal{L}^{-1} is the subbundle of $\pi^*\mathcal{E}^*$ given by

$$\mathcal{L}_y^{-1} = y \subset \mathcal{E}_{\pi(y)}^* \quad \text{for all } y \in Y = \mathbb{P}(\mathcal{E}).$$

As usual, we call \mathcal{L} the tautological line bundle. Then we have the Euler exact sequence,

$$0 \rightarrow \mathcal{O}_Y \rightarrow \pi^*\mathcal{E}^* \otimes \mathcal{L} \rightarrow T_{Y/X} \rightarrow 0, \tag{2}$$

induced by the canonical inclusion $\mathcal{L}^{-1} \rightarrow \pi^*\mathcal{E}^*$ (see, [19, p. 108]). By (1) and (2), we get

$$K_Y = \mathcal{L}^{-r} \otimes \pi^*(K_X \otimes \det \mathcal{E}).$$

The following is well-known.

LEMMA 1.2. *Let m be an integer.*

(1) *There is a canonical isomorphism*

$$R^i\pi_*(\mathcal{L}^m) \simeq \begin{cases} S^m\mathcal{E} & \text{if } m \geq 0, i = 0, \\ \det \mathcal{E}^* \otimes S^{-m-r}\mathcal{E}^* & \text{if } m \leq -r, i = r - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $S^m\mathcal{E}$ is m th symmetric tensor product of \mathcal{E} .

(2) *Let \mathcal{V} be any holomorphic vector bundle on X . Then,*

$$H^q(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^m) \simeq \begin{cases} H^q(X, \mathcal{V} \otimes S^m\mathcal{E}) & \text{if } m \geq 0, \\ H^{q-r+1}(X, \mathcal{V} \otimes \det \mathcal{E}^* \otimes S^{-m-r}\mathcal{E}^*) & \text{if } m \leq -r, \\ 0 & \text{otherwise.} \end{cases}$$

1.3. By Lemma 1.2, we have the natural isomorphism $H^0(X, \mathcal{E}) \simeq H^0(Y, \mathcal{L})$. For the later use, we describe this explicitly. Take a sufficiently small open subset U of X over which \mathcal{E} is trivial and let e_1, \dots, e_r be a frame of \mathcal{E} on U . If $\sigma \in H^0(X, \mathcal{E})$ is given locally by $\sigma = \sum_i \sigma_i e_i$, then the section $\tilde{\sigma} \in H^0(Y, \mathcal{L})$ corresponding to σ can be written as $\tilde{\sigma} = \sum_i \sigma_i e_i$, where we regard e_i 's as homogeneous fiber coordinates on $\mathbb{P}(\mathcal{E})|_U \simeq U \times \mathbb{P}^{r-1}$. We call $\tilde{\sigma}$ the *adjoint section* of σ . Let Z and \tilde{Z} be the zero varieties of σ and $\tilde{\sigma}$, respectively. We call \tilde{Z} the *adjoint hypersurface* of Z .

LEMMA 1.4. *Let \mathcal{V} be a holomorphic vector bundle on X and m an integer. Put $\Omega_{Y/X}^i = \wedge^i T_{Y/X}^*$. Then the group $H^q(Y, \Omega_Y^i \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V})$ vanishes provided that*

$$H^q(Y, \pi^*(\Omega_X^i \otimes \mathcal{V}) \otimes \Omega_{Y/X}^{p-i} \otimes \mathcal{L}^m) = 0$$

holds for any i satisfying $\max(0, p + 1 - r) \leq i \leq \min(n, p)$.

Proof. The dual sequence of (1) induces a filtration F^\cdot for Ω_Y^p :

$$\Omega_Y^p = F^0 \supset F^1 \supset \dots \supset F^p \supset F^{p+1} = 0,$$

whose successive quotients are given by

$$Gr_F^i(\Omega_Y^p) = F^i/F^{i+1} \simeq \Omega_{Y/X}^{p-i} \otimes \pi^*\Omega_X^i.$$

Tensoring $\mathcal{L}^m \otimes \pi^*\mathcal{V}$ with these, we get a spectral sequence

$$E_1^{i,j-i} = H^j(Y, \Omega_{Y/X}^{p-i} \otimes \mathcal{L}^m \otimes \pi^*(\Omega_X^i \otimes \mathcal{V})) \Rightarrow H^j(Y, \Omega_Y^p \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V}).$$

We note that $E_1^{i,j-i}$ vanishes unless $0 \leq i \leq n$, $0 \leq p-i \leq r-1$. Thus the assertion follows immediately from this spectral sequence. *q.e.d.*

LEMMA 1.5. *Let \mathcal{V} and m be as in Lemma 1.4. If one of the following conditions is satisfied, then $H^q(Y, \Omega_{Y/X}^p \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V})$ vanishes.*

- (1) $H^{q-j}(Y, \mathcal{L}^{m-p+j} \otimes \pi^*(\wedge^{p-j}\mathcal{E} \otimes \mathcal{V})) = 0$ for $0 \leq j \leq p$.
- (2) $H^{q+j-1}(Y, \mathcal{L}^{m-p-j} \otimes \pi^*(\wedge^{p+j}\mathcal{E} \otimes \mathcal{V})) = 0$ for $1 \leq j \leq r-p$.

Proof. From the dual sequence of (2), we get an exact sequence

$$0 \rightarrow \Omega_{Y/X}^v \rightarrow \pi^*(\wedge^v\mathcal{E}) \otimes \mathcal{L}^{-v} \rightarrow \Omega_{Y/X}^{v-1} \rightarrow 0$$

for each integer v with $1 \leq v \leq r-1$. Tensoring this with $\mathcal{L}^m \otimes \pi^*\mathcal{V}$, we get

$$\begin{aligned} 0 \rightarrow \Omega_{Y/X}^v \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V} &\rightarrow \pi^*(\wedge^v\mathcal{E} \otimes \mathcal{V}) \otimes \mathcal{L}^{m-v} \rightarrow \\ &\rightarrow \Omega_{Y/X}^{v-1} \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V} \rightarrow 0. \end{aligned} \tag{3}$$

Consider the exact sequence (3) for $v=p-i$, where $0 \leq i \leq p-1$. Then we find that $H^{q-i}(\Omega_{Y/X}^{p-i} \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V}) = 0$ if $H^{q-i}(\pi^*(\wedge^{p-i}\mathcal{E} \otimes \mathcal{V}) \otimes \mathcal{L}^{m-p+i}) = 0$ and $H^{q-i-1}(\Omega_{Y/X}^{p-i-1} \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V}) = 0$. Thus, by an inductive argument, we see that (1) is sufficient to imply $H^q(\Omega_{Y/X}^p \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V}) = 0$. Similarly, consider the exact sequence (3) for $v=p+i$, where $1 \leq i \leq r-1-p$. Then we find that $H^{q+i-1}(\Omega_{Y/X}^{p+i-1} \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V}) = 0$ if $H^{q+i-1}(\pi^*(\wedge^{p+i}\mathcal{E} \otimes \mathcal{V}) \otimes \mathcal{L}^{m-p-i}) = 0$ and $H^{q+i}(\Omega_{Y/X}^{p+i} \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V}) = 0$. Thus (2) is also sufficient. *q.e.d.*

From Lemmas 1.4 and 1.5, we get the following:

PROPOSITION 1.6. *Let \mathcal{V} be a holomorphic vector bundle on X and $m \in \mathbb{Z}$. Then $H^q(Y, \Omega_Y^p \otimes \mathcal{L}^m \otimes \pi^*\mathcal{V})$ vanishes provided that one of the following conditions is satisfied.*

- (1) $H^{q-j}(Y, \pi^*(\Omega_X^i \otimes \wedge^{p-i-j} \mathcal{E} \otimes \mathcal{V})) \otimes \mathcal{L}^{m-p+i+j} = 0$
for $\max(0, p+1-r) \leq i \leq \min(p, n)$ and $0 \leq j \leq p-i$.
- (2) $H^{q+j-1}(Y, \pi^*(\Omega_X^i \otimes \wedge^{p-i+j} \mathcal{E} \otimes \mathcal{V})) \otimes \mathcal{L}^{m-p+i-j} = 0$
for $\max(0, p+1-r) \leq i \leq \min(p, n)$ and $1 \leq j \leq r-p+i$.

2. Automorphism groups of principal bundles

2.1. Here we summarize the results due to Atiyah [1] and Morimoto [15]. Let X be as before an n -dimensional compact complex manifold and \mathcal{E} a holomorphic vector bundle of rank r on X . We denote by $\text{Aut}(X)$ the group of all holomorphic automorphisms of X . As is well-known, it is a complex Lie group. Let $\mathcal{P}\mathcal{E}^*$ be the principal holomorphic fiber bundle associated with the dual bundle $\pi_0: \mathcal{E}^* \rightarrow X$ of \mathcal{E} . We denote by $F(\mathcal{P}\mathcal{E}^*)$ the group of all fiber preserving holomorphic automorphisms of $\mathcal{P}\mathcal{E}^*$. Here we call a biholomorphic map f of $\mathcal{P}\mathcal{E}^*$ fiber preserving if it satisfies $f(x \cdot g) = f(x) \cdot g$ for all $x \in X$ and $g \in \text{GL}(r, \mathbb{C})$. By a result of Morimoto [15], the group $F(\mathcal{P}\mathcal{E}^*)$ is a complex Lie group, too. We can identify it with the group consisting of $f = (f_{\mathcal{E}^*}, f_X) \in \text{Aut}(\mathcal{E}^*) \times \text{Aut}(X)$ such that the diagram

$$\begin{array}{ccc}
 \mathcal{E}^* & \xrightarrow{f_{\mathcal{E}^*}} & \mathcal{E}^* \\
 \downarrow \pi_0 & & \downarrow \pi_0 \\
 X & \xrightarrow{f_X} & X
 \end{array}$$

commutes and $f_{\mathcal{E}^*}$ induces a linear isomorphism on each fiber of π_0 . Thus we get a homomorphism of Lie groups $\Pi_{\mathcal{E}^*}: F(\mathcal{P}\mathcal{E}^*) \rightarrow \text{Aut}(X)$ which sends $(f_{\mathcal{E}^*}, f_X)$ to f_X .

Let $T_{\mathcal{P}\mathcal{E}^*}$ denote the tangent bundle of $\mathcal{P}\mathcal{E}^*$. Since $\text{GL}(r, \mathbb{C})$ acts on $\mathcal{P}\mathcal{E}^*$, it also acts on $T_{\mathcal{P}\mathcal{E}^*}$. We put $\Sigma_{\mathcal{E}} = T_{\mathcal{P}\mathcal{E}^*}/\text{GL}(r, \mathbb{C})$ so that a point of it is a field of tangent vectors to $\mathcal{P}\mathcal{E}^*$, defined along one of its fiber, and invariant under the action of $\text{GL}(r, \mathbb{C})$. Then one can show that $\Sigma_{\mathcal{E}}$ has a natural vector bundle structure on X (see, [1, p. 187] where $\Sigma_{\mathcal{E}}$ is denoted by Q). The vector bundle associated to $\mathcal{P}\mathcal{E}^*$ by the adjoint representation of $\text{GL}(r, \mathbb{C})$ is isomorphic to $\text{End}(\mathcal{E}^*)$ and we get an exact sequence of vector bundles on X :

$$0 \rightarrow \text{End}(\mathcal{E}^*) \rightarrow \Sigma_{\mathcal{E}} \rightarrow T_X \rightarrow 0 \tag{4}$$

([1, Theorem 1]) whose extension class is known as the Atiyah class. The exact cohomology sequence derived from (4) is closely related to $\Pi_{\mathcal{E}^*}$. We in fact have

the following diagram (see, [15]):

$$\begin{array}{ccccccc}
 0 & \rightarrow & H^0(X, \text{End}(\mathcal{E}^*)) & \rightarrow & H^0(X, \Sigma_{\mathcal{E}}) & \rightarrow & H^0(X, T_X) \\
 & & \parallel & & \parallel & & \parallel \\
 0 & \rightarrow & \text{Lie}(\text{Ker}(\Pi_{\mathcal{E}^*})) & \rightarrow & \text{Lie}(F(\mathcal{P}\mathcal{E}^*)) & \rightarrow & \text{Lie}(\text{Aut}(X)),
 \end{array}$$

where, for a complex Lie group H , we denote by $\text{Lie}(H)$ the Lie algebra of H .

2.2. Let $J^1(\mathcal{E})$ denote the 1-jet bundle of \mathcal{E} . Then we see that $J^1(\mathcal{E})$ is a subbundle of $\Sigma_{\mathcal{E}}^* \otimes \mathcal{E}$. Let $\sigma \in H^0(X, \mathcal{E})$ and consider its 1-jet extension $j(\sigma) \in H^0(X, J^1(\mathcal{E}))$. Since

$$H^0(X, J^1(\mathcal{E})) \subset H^0(X, \Sigma_{\mathcal{E}}^* \otimes \mathcal{E}) \simeq H^0(X, \text{Hom}(\Sigma_{\mathcal{E}}, \mathcal{E})),$$

we get a homomorphism $\Sigma_{\mathcal{E}} \rightarrow \mathcal{E}$ defined by the contraction with $j(\sigma)$. We now describe it. Since the problem is local, we work in a sufficiently small open subset U of X over which \mathcal{E} is trivial. We take a system of local coordinates (x_1, \dots, x_n) on U and a frame (e_1, \dots, e_r) of $\mathcal{E}|_U$. We write $\sigma = \sum_{1 \leq i \leq r} \sigma_i(x)e_i$. If we consider (e_1, \dots, e_r) as a system of fiber coordinates on $\mathcal{E}^*|_U$, then

$$\{e_i \partial / \partial e_j (1 \leq i, j \leq r), \partial / \partial x_k (1 \leq k \leq n)\}$$

forms a local frame of $\Sigma_{\mathcal{E}}$. Thus a local section D of $\Sigma_{\mathcal{E}}$ can be expressed as

$$D = \sum_{1 \leq i, j \leq r} a_{ij}(x) e_i \partial / \partial e_j + \sum_{1 \leq k \leq n} b_k(x) \partial / \partial x_k.$$

Then we get

$$\begin{aligned}
 D \cdot j(\sigma) &= D(\sigma) \\
 &= \sum_{1 \leq i \leq r} \left(\sum_{1 \leq j \leq r} a_{ij}(x) \sigma_j(x) + \sum_{1 \leq k \leq n} b_k(x) \partial \sigma_i / \partial x_k \right) e_i.
 \end{aligned}$$

Thus, if σ is transversal to the zero section of \mathcal{E} , that is, the zero variety $Z = Z_{\sigma}$ of σ is nonsingular with codimension $\text{rank } \mathcal{E}$, then the map $\cdot j(\sigma): \Sigma_{\mathcal{E}} \rightarrow \mathcal{E}$ is surjective. We denote its kernel by $\Sigma_{\mathcal{E}} \langle -Z \rangle$ so that the sequence

$$0 \rightarrow \Sigma_{\mathcal{E}} \langle -Z \rangle \rightarrow \Sigma_{\mathcal{E}} \xrightarrow{j(\sigma)} \mathcal{E} \rightarrow 0 \tag{5}$$

is exact. Further, we have the following:

LEMMA 2.3. Assume that the zero variety Z of $\sigma \in H^0(X, \mathcal{E})$ is nonsingular and has codimension $r = \text{rank } \mathcal{E}$. Then the sequence

$$0 \rightarrow \text{Ker}(\mathcal{E} \otimes \mathcal{E}^* \xrightarrow{\sigma} \mathcal{E}) \rightarrow \Sigma_{\mathcal{E}} \langle -Z \rangle \rightarrow T_X(-\log Z) \rightarrow 0$$

is exact, where

- (1) the map $\mathcal{E} \otimes \mathcal{E}^* \xrightarrow{\sigma} \mathcal{E}$ sends $\varphi \in \mathcal{E}^* \otimes \mathcal{E} \simeq \text{End}(\mathcal{E})$ to $\varphi(\sigma) \in \mathcal{E}$, and
- (2) $T_X(-\log Z) = \{\theta \in T_X; \theta \cdot I_Z \subset I_Z\}$, where I_Z is the ideal sheaf of Z .

Proof. We employ the notation in 2.2. Since Z is nonsingular with codimension r , we can assume that $\sigma_i = x_i$ for all $1 \leq i \leq r$. Then we have

$$D \cdot j(\sigma) = \sum_{i=1}^r \left(\sum_{j=1}^r a_{ij}(x) \sigma_j + b_i(x) \right) e_i.$$

If D is a local section of $\Sigma_{\mathcal{E}} \langle -Z \rangle$, then $b_i(x) = -\sum_{1 \leq j \leq r} a_{ij}(x) x_j$ for any i . This means that $\theta = \sum b_i \partial / \partial x_i$ is a local section of $T_X(-\log Z)$. Moreover, if b 's are zero, then the linear map defined by the matrix (a_{ij}) sends σ to zero. Thus we get the sequence in the statement, and it is straightforward to check the exactness. *q.e.d.*

2.4. Let $Y = \mathbb{P}(\mathcal{E})$ be the projective bundle as in §1 and assume that the zero variety Z of $\sigma \in H^0(X, \mathcal{E})$ is nonsingular and has codimension r . By the description of the adjoint section $\tilde{\sigma}$ of σ in 1.3, we see that its zero variety \tilde{Z} is a nonsingular hypersurface of Y . Since \mathcal{L} is a line bundle, (4), (5) and Lemma 2.3 give the following exact sequences.

$$0 \rightarrow \mathcal{O}_Y \rightarrow \Sigma_{\mathcal{L}} \rightarrow T_Y \rightarrow 0. \tag{6}$$

$$0 \rightarrow T_Y(-\log \tilde{Z}) \rightarrow \Sigma_{\mathcal{L}} \xrightarrow{j(\tilde{\sigma})} \mathcal{L} \rightarrow 0. \tag{7}$$

These will play an important role in our consideration. We remark that the Atiyah class of (6) is $-2\pi\sqrt{-1}c_1(\mathcal{L})$.

LEMMA 2.5. With the above notation and assumptions, the following hold.

- (1) $R^q \pi_* \Sigma_{\mathcal{L}} \simeq \begin{cases} \Sigma_{\mathcal{E}}, & \text{if } q = 0, \\ 0, & \text{if } q > 0. \end{cases}$
- (2) $R^q \pi_* T_Y(-\log \tilde{Z}) \simeq \begin{cases} \Sigma_{\mathcal{E}} \langle -Z \rangle, & \text{if } q = 0, \\ 0, & \text{if } q > 0. \end{cases}$

(3) For any holomorphic vector bundle \mathcal{V} on X , the diagram

$$\begin{array}{ccc}
 H^0(Y, \Sigma_{\mathcal{L}} \otimes \pi^* \mathcal{V}) & \xrightarrow{j(\tilde{\sigma})} & H^0(Y, \mathcal{L} \otimes \pi^* \mathcal{V}) \\
 \uparrow \wr & & \uparrow \wr \\
 H^0(X, \Sigma_{\mathcal{E}} \otimes \mathcal{V}) & \xrightarrow{j(\sigma)} & H^0(X, \mathcal{E} \otimes \mathcal{V})
 \end{array}$$

commutes.

Proof. We show (1). Let q be a positive integer. Since $H^q(T_{\mathbb{P}^{r-1}}) = 0$, we have $R^q \pi_* T_{X/Y} = 0$. It follows from (1) that $R^q \pi_* T_Y = 0$, and thus we have $R^q \pi_* \Sigma_{\mathcal{L}} = 0$ by (6). We show that $\pi_* \Sigma_{\mathcal{L}} \simeq \Sigma_{\mathcal{E}}$. Since the problem is local, we work over a small open subset $U \subset X$ as in 2.2 and use the notation there. We cover the fiber \mathbb{P}^{r-1} by open subsets $W_i = \{e_i \neq 0\}$. Then, on $U \times W_i$, the local frame of T_Y is given by $\{\partial/\partial x_j, 1 \leq j \leq n, \partial/\partial(e_k/e_i), k \neq i\}$. We see from (6) that these together with $e_i \partial/\partial e_i$ form a local frame of $\Sigma_{\mathcal{L}}$, where we regard e_i as a frame of \mathcal{L} on $U \times W_i$. Thus $\tilde{D} \in H^0(\pi^{-1}(U), \Sigma_{\mathcal{L}})$ can be written as

$$\tilde{D} = \sum_{1 \leq i, j \leq r} a_{ij}(x) e_i \partial/\partial x_j + \sum_{1 \leq k \leq n} b_k(x) \partial/\partial x_k.$$

Thus we get $\pi_* \Sigma_{\mathcal{L}} \simeq \Sigma_{\mathcal{E}}$. We next show (2). It follows from (1) and (7) that $R^q \pi_* T_Y(-\log \tilde{Z}) = 0$ for $q \geq 2$. Further, we have the following commutative diagram by (1) and Lemma 1.2, 1):

$$\begin{array}{ccccccc}
 0 \rightarrow \pi_* T_Y(-\log \tilde{Z}) & \rightarrow & \pi_* \Sigma_{\mathcal{L}} & \rightarrow & \pi_* \mathcal{L} & \rightarrow & R^1 \pi_* T_Y(-\log \tilde{Z}) \\
 & & \uparrow & & \uparrow \wr & & \uparrow \wr \\
 0 \rightarrow \Sigma_{\mathcal{E}} \langle -Z \rangle & \rightarrow & \Sigma_{\mathcal{E}} & \rightarrow & \mathcal{E} & \rightarrow & 0.
 \end{array}$$

This implies that $R^1 \pi_* T_Y(-\log \tilde{Z}) = 0$ and $\pi_* T_Y(-\log \tilde{Z}) \simeq \Sigma_{\mathcal{E}} \langle -Z \rangle$. The assertion (3) follows from (1) and (2). *q.e.d.*

From Lemma 2.5, 1), we get $H^0(X, \Sigma_{\mathcal{E}}) \simeq H^0(Y, \Sigma_{\mathcal{L}})$. In view of 2.1, this implies that $\text{Lie}(F(\mathcal{P}\mathcal{E}^*)) \simeq \text{Lie}(F(\mathcal{P}\mathcal{L}^{-1}))$.

LEMMA 2.6. *With the above notation and assumptions, there is an isomorphism*

$$H^0(X, \mathcal{E}^* \otimes \mathcal{V}) \simeq H^0(Y, \Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1} \otimes \pi^* \mathcal{V})$$

for any holomorphic vector bundle \mathcal{V} on X such that the following diagram commutes.

$$\begin{array}{ccc}
 H^0(Y, \Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1} \otimes \pi^* \mathcal{V}) & \xrightarrow{j(\tilde{\sigma})} & H^0(Y, \pi^* \mathcal{V}) \\
 \wr & & \uparrow \wr \\
 H^0(X, \mathcal{E}^* \otimes \mathcal{V}) & \xrightarrow{\sigma} & H^0(X, \mathcal{V}).
 \end{array}$$

Proof. Tensoring (6) with $\pi^*\mathcal{V} \otimes \mathcal{L}^{-1}$, we get

$$0 \rightarrow \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \rightarrow \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes \Sigma_\varphi \rightarrow \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes T_Y \rightarrow 0.$$

From the derived cohomology exact sequence, we get $H^0(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes \Sigma_\varphi) \simeq H^0(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes T_Y)$ because we have $H^q(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1}) = 0$ for any q by Lemma 1.2. By the exact cohomology sequence derived from (1) tensored with $\pi^*\mathcal{V} \otimes \mathcal{L}^{-1}$, we have

$$H^0(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes T_{X/Y}) \simeq H^0(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes T_Y),$$

since $H^0(Y, \pi^*(\mathcal{V} \otimes T_X) \otimes \mathcal{L}^{-1}) = 0$ by Lemma 1.2. Similarly, by the cohomology exact sequence derived from (2) tensored with $\pi^*\mathcal{V} \otimes \mathcal{L}^{-1}$, we have $H^0(Y, \pi^*(\mathcal{E}^* \otimes \mathcal{V})) \simeq H^0(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes T_{X/Y})$. In summary, we get a homomorphism

$$\alpha: H^0(X, \mathcal{E}^* \otimes \mathcal{V}) \rightarrow H^0(X, \mathcal{V})$$

defined by the composition

$$\begin{aligned} H^0(X, \mathcal{E}^* \otimes \mathcal{V}) &\simeq H^0(Y, \pi^*(\mathcal{E}^* \otimes \mathcal{V})) \\ &\simeq H^0(Y, \pi^*\mathcal{V} \otimes \mathcal{L}^{-1} \otimes \Sigma_\varphi) \xrightarrow{j(\tilde{\sigma})} H^0(Y, \pi^*\mathcal{V}) \simeq H^0(X, \mathcal{V}). \end{aligned}$$

Now, it is easy to see that α is nothing but the map obtained by the contraction with σ . *q.e.d.*

3. The Jacobian rings

In this section, we recall the definition of the Jacobian ring due to Green [10] and study its duality properties. Similar computations are already found in [10] and [13].

3.1. In the rest of the paper, we freely use the notation in the preceding sections and fix the following set-up.

- (1) X is a projective manifold and $\dim X = n \geq 3$.
- (2) \mathcal{E} is a vector bundle of rank r on X , which is ample in the sense of Hartshorne [11] and satisfies $2 \leq r \leq n - 1$.
- (3) Y is the projective bundle $\mathbb{P}(\mathcal{E})$ associated with \mathcal{E} and $\pi: Y \rightarrow X$ is the projection map.

(4) There exists a section $\sigma \in H^0(X, \mathcal{E})$ whose zero variety $Z = Z_\sigma$ is irreducible nonsingular and has codimension r in X .

By (2), the tautological line bundle \mathcal{L} on $Y = \mathbb{P}(\mathcal{E})$ is ample ([11, Proposition (3.2)]). Thus the adjoint hypersurface \tilde{Z} of Z is an ample divisor. Further, it is nonsingular by (4). We denote by $\varpi: \tilde{Z} \rightarrow X$ the restriction of π to \tilde{Z} .

DEFINITION 3.2 ([10]). Let $\tilde{\sigma}$ be the adjoint section of σ in 3.1, (4). For any coherent \mathcal{O}_Y -module \mathcal{F} , we define the pseudo-Jacobian system $J_{\mathcal{F}}$ as the image of the map

$$H^0(Y, \mathcal{F} \otimes \Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1}) \xrightarrow{j(\tilde{\sigma})} H^0(Y, \mathcal{F})$$

derived from (7) tensored with $\mathcal{L}^{-1} \otimes \mathcal{F}$. In particular, if $\mathcal{F} = \mathcal{L}^a \otimes K_Y^b$, we set

$$J_{a,b} = J_{\mathcal{F}}, \quad R_{a,b} = H^0(Y, \mathcal{L}^a \otimes K_Y^b) / J_{a,b}.$$

We call $J = \bigoplus J_{a,b}$ and $R = \bigoplus R_{a,b}$ the *Jacobian ideal* and the *Jacobian ring* of $Z = Z_\sigma$, respectively.

3.3. From the dual sequence of (7), we can construct the following Koszul exact sequence as in [10]:

$$\begin{aligned} 0 \rightarrow \mathcal{L}^{-n-r} \rightarrow \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^{-n-r+1} \rightarrow \dots \rightarrow \\ \rightarrow \wedge^{n+r-1} \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^{-1} \rightarrow \wedge^{n+r} \Sigma_{\mathcal{L}}^* \rightarrow 0. \end{aligned} \tag{8}$$

Observe that we have

$$\begin{aligned} \text{Ker}\{ \wedge^{v+1} \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^{v+1-n-r} \rightarrow \wedge^{v+2} \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^{v+2-n-r} \} \\ \simeq \Omega_Y^v(\log \tilde{Z}) \otimes \mathcal{L}^{v-n-r}, \end{aligned}$$

where $\Omega_Y^v(\log \tilde{Z})$ is the sheaf of meromorphic v -forms on Y with at most logarithmic pole along \tilde{Z} . The exact sequence (8) tensored with $\mathcal{L}^{n+r-a} \otimes K_Y^{1-b}$ breaks into the short exact sequences

$$0 \rightarrow \mathcal{K}_{v-1} \rightarrow \wedge^v \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^{v-a} \otimes K_Y^{1-b} \rightarrow \mathcal{K}_v \rightarrow 0, \tag{9}$$

where we set

$$\mathcal{K}_v = \begin{cases} \mathcal{L}^{-a} \otimes K_Y^{1-b} & (v = 0), \\ \Omega_Y^v(\log \tilde{Z}) \otimes \mathcal{L}^{v-a} \otimes K_Y^{1-b} & (1 \leq v \leq n+r-2), \\ \wedge^{n+r} \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^{n+r-a} \otimes K_Y^{1-b} & (v = n+r-1). \end{cases}$$

Consider now the cohomology exact sequences derived from (9). Since we have $\wedge^{n+r}\Sigma_{\mathcal{F}}^* = K_Y$ from (6), we see $\wedge^{n+r-1}\Sigma_{\mathcal{F}}^* \simeq \Sigma_{\mathcal{F}} \otimes K_Y$. Thus

$$\text{Ker}\{H^{n+r-1}(Y, \mathcal{X}_0) \rightarrow H^{n+r-1}(Y, \Sigma_{\mathcal{F}}^* \otimes \mathcal{L}^{1-a} \otimes K_Y^{1-b})\} \simeq (R_{a,b})^*$$

by the Serre duality, and

$$\begin{aligned} & \text{Coker}\{H^0(Y, \wedge^{n+r}\Sigma_{\mathcal{F}}^* \otimes \mathcal{L}^{n+r-1-a} \otimes K_Y^{1-b}) \rightarrow H^0(Y, \mathcal{X}_{n+r-1})\} \\ & \simeq \text{Coker}\{H^0(Y, \Sigma_{\mathcal{F}} \otimes \mathcal{L}^{n+r-1-a} \otimes K_Y^{2-b}) \rightarrow H^0(Y, \mathcal{L}^{n+r-a} \otimes K_Y^{2-b})\} \\ & \simeq R_{n+r-a, 2-b}. \end{aligned}$$

We further have a homomorphism

$$d_{a,b}: R_{n+r-a, 2-b} \rightarrow (R_{a,b})^*$$

defined by the composition of the coboundary maps

$$H^s(Y, \mathcal{X}_{n+r-1-s}) \rightarrow H^{s+1}(Y, \mathcal{X}_{n+r-2-s}).$$

The following is straightforward.

PROPOSITION 3.4. *The map $d_{a,b}$ is injective if*

$$(1) \ H^{n+r-1-s}(Y, \wedge^s \Sigma_{\mathcal{F}}^* \otimes \mathcal{L}^{s-a} \otimes K_Y^{1-b}) = 0 \quad \text{for } 1 \leq s \leq n+r-2.$$

It is surjective if

$$(2) \ H^{n+r-s}(Y, \wedge^s \Sigma_{\mathcal{F}}^* \otimes \mathcal{L}^{s-a} \otimes K_Y^{1-b}) = 0 \quad \text{for } 2 \leq s \leq n+r-1.$$

COROLLARY 3.5. *The map $d_{a,b}$ is injective provided that the following conditions are satisfied for $1 \leq s \leq n+r-2$:*

$$\begin{aligned} (1) \ & H^{n+r-1-s}(Y, \Omega_Y^s \otimes \mathcal{L}^{s-a} \otimes K_Y^{1-b}) = 0. \\ (2) \ & H^{n+r-1-s}(Y, \Omega_Y^{s-1} \otimes \mathcal{L}^{s-a} \otimes K_Y^{1-b}) = 0. \end{aligned}$$

The map $d_{a,b}$ is surjective provided that the following conditions are satisfied for $2 \leq s \leq n+r-1$.

$$\begin{aligned} (3) \ & H^{n+r-s}(Y, \Omega_Y^s \otimes \mathcal{L}^{s-a} \otimes K_Y^{1-b}) = 0. \\ (4) \ & H^{n+r-s}(Y, \Omega_Y^{s-1} \otimes \mathcal{L}^{s-a} \otimes K_Y^{1-b}) = 0. \end{aligned}$$

Proof. From the dual sequence of (6), we have

$$0 \rightarrow \Omega_Y^j \rightarrow \wedge^j \Sigma_{\mathcal{F}}^* \rightarrow \Omega_Y^{j-1} \rightarrow 0 \tag{10}$$

for each $j \in \mathbb{N}$. Thus, for any vector bundle \mathcal{V} on Y , we have

$$H^i(Y, \wedge^j \Sigma_{\mathcal{L}}^* \otimes \mathcal{V}) = 0 \quad \text{if } H^i(Y, \Omega_Y^{j-1} \otimes \mathcal{V}) = H^i(Y, \Omega_Y^j \otimes \mathcal{V}) = 0.$$

This in particular implies the assertion by virtue of Proposition 3.4. *q.e.d.*

REMARKS 3.6. (1) The extension class of (10) is $2\pi\sqrt{-1}c_1(\mathcal{L})$. Thus the coboundary map $H^i(Y, \Omega_Y^{j-1}) \rightarrow H^{i+1}(Y, \Omega_Y^j)$ is (essentially) given by the cup-product with $c_1(\mathcal{L})$.

(2) We have the following commutative diagram by the Poincaré residue sequence and the dual sequences of (6), (7):

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{L}^{-1} & = & \mathcal{L}^{-1} & & \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow \Omega_Y^1 \rightarrow & \Sigma_{\mathcal{L}}^* & \rightarrow & \mathcal{O}_Y & \rightarrow & 0 & \\
 & \parallel & & \downarrow & & & \\
 0 \rightarrow \Omega_Y^1 \rightarrow \Omega_Y^1(\log \tilde{Z}) \rightarrow & \mathcal{O}_{\tilde{Z}} & \rightarrow & 0 & \rightarrow & 0 & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
 \end{array}$$

LEMMA 3.7. Assume that the following conditions are satisfied:

- (1) $H^{n+r-1-s}(Y, \wedge^s \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^s \otimes K_Y) = 0$ for $1 \leq s \leq n+r-2$.
- (2) $H^{n+r-s}(Y, \wedge^s \Sigma_{\mathcal{L}}^* \otimes \mathcal{L}^s \otimes K_Y) = 0$ for $2 \leq s \leq n+r-1$.

Then $R_{n+r,2} \simeq \mathbb{C}$ and $d_{a,b}$ is induced by the pairing

$$R_{a,b} \otimes R_{n+r-a,2-b} \rightarrow R_{n+r,2} \simeq \mathbb{C}.$$

Proof. By Proposition 3.4, the conditions (1) and (2) imply that $d_{0,0}: R_{n+r,2} \rightarrow (R_{0,0})^*$ is an isomorphism. We show $R_{0,0} \simeq \mathbb{C}$. For this purpose, it suffices to show that $H^0(Y, \Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1}) = 0$. Consider the cohomology exact sequence derived from (6) tensored with \mathcal{L}^{-1} . By Lemma 1.2, we have $H^0(Y, \mathcal{L}^{-1}) = 0$. On the other hand, since \mathcal{L} is ample, it follows from [14, Theorem 8] that $H^0(Y, T_Y \otimes \mathcal{L}^{-1}) = 0$. Thus we get $H^0(Y, \Sigma_{\mathcal{L}} \otimes \mathcal{L}^{-1}) = 0$. The rest is clear from the construction of $d_{a,b}$. *q.e.d.*

4. Hodge structures

DEFINITION 4.1. The inclusion $Z \hookrightarrow X$ induces the homomorphism

$H^*(X, \mathbb{Z}) \rightarrow H^*(Z, \mathbb{Z})$. We set

$$H_{\text{var}}^q(Z, \mathbb{Q}) = \text{Coker}\{H^q(X, \mathbb{Q}) \rightarrow H^q(Z, \mathbb{Q})\}$$

and call it the *variable part* of $H^q(Z, \mathbb{Q})$. Similarly, we define the variable part of $H^q(\tilde{Z}, \mathbb{Q})$ by

$$H_{\text{var}}^q(\tilde{Z}, \mathbb{Q}) = \text{Coker}\{H^q(Y, \mathbb{Q}) \rightarrow H^q(\tilde{Z}, \mathbb{Q})\}.$$

REMARK 4.2. We see that $H_{\text{var}}^q(\tilde{Z})$ is isomorphic to the usual primitive cohomology $H_{\text{prim}}^q(\tilde{Z})$ defined by means of the ample class $c_1(\mathcal{L}|_{\tilde{Z}})$. Thus it admits a polarized Hodge structure of weight q . On the other hand, it seems that $H_{\text{var}}^q(Z)$ is closely related to the primitive cohomology defined in [2] by means of the Chern classes $c_i(\mathcal{E}|_Z)$.

PROPOSITION 4.3 (cf. [20]). *There is a canonical isomorphism of the Hodge structures*

$$H_{\text{var}}^q(Z, \mathbb{C})(1-r) \simeq H_{\text{var}}^{q+2r-2}(\tilde{Z}, \mathbb{C}).$$

Proof. We consider the Leray spectral sequences

$$E_2^{i,j} = H^i(X, R^j\varpi_*\mathbb{C}) \Rightarrow H^{i+j}(\tilde{Z}, \mathbb{C}),$$

$$'E_2^{i,j} = H^i(X, R^j\pi_*\mathbb{C}) \Rightarrow H^{i+j}(Y, \mathbb{C}).$$

Since we have

$$\varpi^{-1}(x) = \begin{cases} \mathbb{P}^{r-1} & \text{if } x \in Z, \\ \mathbb{P}^{r-2} & \text{otherwise,} \end{cases}$$

we see that $R^j\varpi_*\mathbb{C}$ is given by

$$R^j\varpi_*\mathbb{C} = \begin{cases} \mathbb{C}_X & \text{if } j \text{ is even and } 0 \leq j < 2r - 2, \\ \mathbb{C}_Z & \text{if } j = 2r - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$R^j\pi_*\mathbb{C} = \begin{cases} \mathbb{C}_X & \text{if } j \text{ is even and } 0 \leq j \leq 2r - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus both spectral sequences degenerate at E_2 . Since we have $E_2^{i,j} = 'E_2^{i,j}$ unless $j = 2r - 2$, we get a homomorphism

$$E_2^{q,2r-2} / 'E_2^{q,2r-2} \simeq H_{\text{var}}^q(Z, \mathbb{C})(1-r) \rightarrow H_{\text{var}}^{q+2r-2}(\tilde{Z}, \mathbb{C}),$$

which is an isomorphism of Hodge structures.

4.4. We recall some results on the Hodge structure of ample hypersurfaces (see, [3] or [17]). The Poincaré residue operator gives an exact sequence of complexes

$$0 \rightarrow \Omega_Y \rightarrow \Omega_Y(\log \tilde{Z}) \rightarrow \Omega_{\tilde{Z}}^{-1} \rightarrow 0.$$

By a result of Deligne [6], the spectral sequence

$$E_1^{p,q} = H^q(Y, \Omega_Y^p(\log \tilde{Z})) \Rightarrow H^{p+q}(Y - \tilde{Z}, \mathbb{C})$$

degenerates at E_1 term. Thus the exact sequence of hypercohomology can be identified with the Gysin sequence (see, [17, p. 444]):

$$\begin{array}{ccccccc} \mathbb{H}^q(\Omega_Y) & \longrightarrow & \mathbb{H}^q(\Omega_Y(\log \tilde{Z})) & \longrightarrow & \mathbb{H}^q(\Omega_{\tilde{Z}}^{-1}) & \longrightarrow & \mathbb{H}^{q+1}(\Omega_Y) \\ \parallel & & \parallel & & \parallel & & \parallel \\ H^q(Y) & \xrightarrow{i^*} & H^q(Y - \tilde{Z}) & \xrightarrow{R} & H^{q-1}(\tilde{Z}) & \xrightarrow{g} & H^{q+1}(Y), \end{array}$$

where

- (1) i^* is the map induced by the inclusion $i: Y - \tilde{Z} \hookrightarrow Y$,
- (2) the residue map R is the dual of the “tube over cycle map”,
- (3) g is (essentially) the Gysin map,
- (4) the group $H^q(Y - \tilde{Z})$ has a mixed Hodge structure [6] with 2-stage weight filtration: $W_q = i^*H^q(Y)$ and $W_{q+1} = H^q(Y - \tilde{Z})$,
- (5) the maps i^* , R and g are morphisms of mixed Hodge structures.

LEMMA 4.5. *The homomorphism*

$$i^*: H^{n+r-1}(Y, \mathbb{C}) \rightarrow H^{n+r-1}(Y - \tilde{Z}, \mathbb{C})$$

is a zero map. Thus

$$H^{n+r-1}(Y - \tilde{Z}, \mathbb{C}) \simeq H_{\text{var}}^{n+r-2}(\tilde{Z}, \mathbb{C})(-1)$$

is an isomorphism of Hodge structures of pure weight $n + r$.

Proof. We show that the map of E_1 -terms

$$E_1^{p,q} = H^q(Y, \Omega_Y^p) \rightarrow 'E_1^{p,q} = H^q(Y - \tilde{Z}, \Omega_{Y-\tilde{Z}}^p)$$

of the Hodge-to-de Rham spectral sequence is a zero map for $p+q = n+r-1 = \dim Y$. Since $Y - \tilde{Z}$ is an affine variety, we have $'E_1^{p,q} = 0$ for $q > 0$. Since $r \geq 2$, we have

$$H^0(Y, K_Y) = H^0(Y, \mathcal{L}^{-r} \otimes \pi^*(K_X \otimes \det \mathcal{E})) = 0$$

by Lemma 1.2. Thus we get $E_1^{n+r-1,0} = H^0(Y, K_Y) = 0$. Thus i^* is a zero map. From this, we have

$$H^{n+r-1}(Y - \tilde{Z}) \simeq \text{Ker}\{g: H^{n+r-2}(\tilde{Z}) \rightarrow H^{n+r}(Y)\}.$$

By the Poincaré duality, we get the second assertion. *q.e.d.*

The above observations are summarized in the following:

PROPOSITION 4.6. *There are canonical isomorphisms*

$$H_{\text{var}}^{n-r}(Z, \mathbb{C})(-r) \simeq H_{\text{var}}^{n+r-2}(\tilde{Z}, \mathbb{C})(-1) \simeq H^{n+r-1}(Y - \tilde{Z}, \mathbb{C}).$$

of Hodge structures. In particular, for $r \leq p \leq n$,

$$H_{\text{var}}^{n-p}(Z, \Omega_Z^{p-r}) \simeq H_{\text{var}}^{n+r-1-p}(\tilde{Z}, \Omega_{\tilde{Z}}^{p-1}) \simeq H^{n+r-1-p}(Y, \Omega_Y^p(\log \tilde{Z})).$$

PROPOSITION 4.7. *Let R be the Jacobian ring of Z and fix p satisfying $r \leq p \leq n$. If the following conditions are satisfied, then $H_{\text{var}}^{n-p}(Z, \Omega_Z^{p-r}) \simeq R_{n+r-p,1}$.*

- (1) $H^{n+r-p-s}(Y, \wedge^{p+s} \Sigma_{\tilde{Z}}^* \otimes \mathcal{L}^s) = 0, \quad 1 \leq s \leq n+r-1-p.$
- (2) $H^{n+r-1-p-s}(Y, \wedge^{p+s} \Sigma_{\tilde{Z}}^* \otimes \mathcal{L}^s) = 0, \quad 1 \leq s \leq n+r-2-p.$

Proof. We consider the exact sequences (9) for $a=p, b=1$. In view of the derived cohomology exact sequences, the conditions (1) and (2) are sufficient to imply

$$H^{n+r-1-p-s}(Y, \Omega_Y^{p+s}(\log \tilde{Z}) \otimes \mathcal{L}^s) \simeq H^{n+r-p-s}(Y, \Omega_Y^{p+s-1}(\log \tilde{Z}) \otimes \mathcal{L}^{s-1})$$

for $1 \leq s \leq n+r-2-p$, and

$$R_{n+r-p,1} \simeq H^1(Y, \Omega_Y^{n+r-2}(\log \tilde{Z}) \otimes \mathcal{L}^{n+r-2-p}).$$

Thus we get

$$R_{n+r-p,1} \simeq H^{n+r-1-p}(Y, \Omega_Y^p(\log \tilde{Z})) \simeq H_{\text{var}}^{n-p}(Z, \Omega_Z^{-r}).$$

where the last isomorphism follows from Proposition 4.6. q.e.d.

COROLLARY 4.8. $H_{\text{var}}^{n-p}(Z, \Omega_Z^{-r}) \simeq R_{n+r-p,1}$ holds if the following conditions are satisfied.

- (1) $H^{n+r-p-s}(Y, \Omega_Y^{p+s} \otimes \mathcal{L}^s) = 0$ for $1 \leq s \leq n+r-1-p$.
- (2) $H^{n+r-p-s}(Y, \Omega_Y^{p+s-1} \otimes \mathcal{L}^s) = 0$ for $1 \leq s \leq n+r-1-p$.
- (3) $H^{n+r-1-p-s}(Y, \Omega_Y^{p+s} \otimes \mathcal{L}^s) = 0$ for $1 \leq s \leq n+r-2-p$.
- (4) $H^{n+r-1-p-s}(Y, \Omega_Y^{p+s-1} \otimes \mathcal{L}^s) = 0$ for $1 \leq s \leq n+r-2-p$.

Proof. See the proof of Corollary 3.5.

COROLLARY 4.9. Assume that \mathcal{E} is a direct sum of ample line bundles. Then $H_{\text{var}}^{n-p}(Z, \Omega_Z^{-r}) \simeq R_{n+r-p,1}$ holds if the following conditions are satisfied.

- (1) $H^{n-p-v}(X, \Omega_X^{p+v} \otimes S^v \mathcal{E} \otimes \det \mathcal{E}) = 0, 0 \leq v \leq n-p-1$.
- (2) $H^{n-p-v-1}(X, \Omega_X^{p+v+1} \otimes S^v \mathcal{E} \otimes \det \mathcal{E}) = 0, 0 \leq v \leq n-p-2$.
- (3) $H^{n-p-v-1}(X, \Omega_X^{p+v} \otimes S^v \mathcal{E} \otimes \det \mathcal{E}) = 0, 0 \leq v \leq n-p-2$.
- (4) $H^{n-p-v}(X, \Omega_X^{p+v} \otimes S^v \mathcal{E} \otimes \wedge^{r-1} \mathcal{E}) = 0, 0 \leq v \leq n-p-1$.

Proof. We first note that the condition (1) of Corollary 4.8 holds for any p by virtue of the vanishing theorem of Kodaira–Nakano, since we have assumed that \mathcal{L} is ample. We show that (3) and (4) imply 4.8, 4). From Proposition 1.6, we see that the following condition is sufficient to imply 4.8, 4):

$$H^{n+r-p-s+j-2}(Y, \pi^*(\Omega_X^i \otimes \wedge^{p+s-1-i+j} \mathcal{E} \otimes \mathcal{L}^{i-j-p+1})) = 0$$

for $\max(0, p+s-r) \leq i \leq \min(p+s-1, n)$,

$$1 \leq j \leq r-p-s+i+1. \tag{*}$$

Since $i-j-p+1 \geq s-r > -r$, we can assume $i-j-p+1 \geq 0$ by Lemma 1.2. Thus (*) is equivalent to

$$H^{n+r-p-s+j-2}(X, \Omega_X^i \otimes \wedge^{p+s-1-i+j} \mathcal{E} \otimes S^{i-j-p+1} \mathcal{E}) = 0$$

for $\max(0, p+s-r) \leq i \leq \min(p+s-1, n)$,

$$1 \leq j \leq \min(i-p+1, r-p-s+1+i). \tag{**}$$

Since $i \geq p+s-r$, we have

$$(n+r-p-s+j-2) + i \geq n+j-2.$$

Thus, since \mathcal{E} is a direct sum of ample line bundles, we see that (**) follows from the vanishing theorem of Kodaira–Nakano unless $(i, j) = (p + s - r, 1)$, $(p + s - r + 1, 1)$. Putting $v = s - r$, $s - r + 1$, respectively, we see that (3) and (4) are sufficient to imply 4.8, (4). Similarly, we can show that (1) (resp. (2)) implies (1) (resp (2)) of 4.8. *q.e.d.*

5. The period maps and the IVHS

5.1. Let \mathcal{U} be the open subset of $H^0(X, \mathcal{E})$ consisting of σ satisfying (4) of 3.1. Then we have a family $\mathcal{Z} = \{Z_\sigma\}_{\sigma \in \mathcal{U}}$ of nonsingular subvarieties defined by sections of \mathcal{E} and a period map

$$p: \mathcal{U} \rightarrow \Gamma \backslash \mathcal{D}, \quad \sigma \mapsto \text{the polarized Hodge structure on } H_{\text{var}}^{n-r}(Z_\sigma),$$

where \mathcal{D} is the corresponding Griffiths period domain and Γ is the monodromy group. The group $F(\mathcal{P}\mathcal{E}^*)$ acts naturally on $H^0(X, \mathcal{E})$ and \mathcal{U} is invariant with respect to this action. Since Z_σ and $Z_{\sigma'}$ should be isomorphic if σ and σ' are in the same $F(\mathcal{P}\mathcal{E}^*)$ -orbit, we are led to considering another period map

$$P: \mathcal{M} = \mathcal{U}/F(\mathcal{P}\mathcal{E}^*) \rightarrow \Gamma \backslash \mathcal{D}.$$

Our concern is to know whether P is “generally” injective. However, since \mathcal{M} does not necessarily have a good structure such as a quasi-projective variety, the meaning of “generic” is not clear at all. By a result of [5], we can find a dense open subset \mathcal{U}_0 of \mathcal{U} such that

- (1) the geometric quotient $\mathcal{M}_0 = \mathcal{U}_0/F(\mathcal{P}\mathcal{E}^*)$ exists,
- (2) \mathcal{M}_0 and each $F(\mathcal{P}\mathcal{E}^*)$ -orbit are smooth,
- (3) there exists a family of varieties in question over \mathcal{M}_0 , and
- (4) the period map

$$P_0: \mathcal{M}_0 \rightarrow \Gamma \backslash \mathcal{D}$$

is well-defined holomorphic map.

Thus we can say that P is generically injective if so is P_0 . Further, by Lemma 2.5, we can make the following identifications.

- (5) The tangent space at σ of the $F(\mathcal{P}\mathcal{E}^*)$ -orbit through σ is

$$\text{Im}\{H^0(X, \Sigma_\sigma) \rightarrow H^0(X, \mathcal{E})\} \simeq J_{1,0},$$

(6) the tangent space of \mathcal{M}_0 at $[\sigma]$ is

$$\text{Coker}\{H^0(X, \Sigma_{\mathcal{E}}) \rightarrow H^0(X, \mathcal{E})\} \simeq R_{1,0},$$

where $[\sigma] \in \mathcal{M}_0$ is the equivalence class of σ .

LEMMA 5.2. *Let $\rho: R_{1,0} \rightarrow H^1(Z, T_Z)$ be the Kodaira–Spencer map at $[\sigma] \in \mathcal{M}_0$.*

- (1) *ρ is injective provided that the following conditions are satisfied.*
 - (i) $H^{p-1}(X, \mathcal{E} \otimes \wedge^p \mathcal{E}^*) = 0$ for $2 \leq p \leq r$.
 - (ii) $H^p(X, T_X \otimes \wedge^p \mathcal{E}^*) = 0$ for $1 \leq p \leq r$.
- (2) *ρ is surjective provided that the following conditions are satisfied.*
 - (i) $H^p(X, \mathcal{E} \otimes \wedge^p \mathcal{E}^*) = 0$ for $1 \leq p \leq r$.
 - (ii) $H^{p+1}(X, T_X \otimes \wedge^p \mathcal{E}^*) = 0$ for $0 \leq p \leq r$.

Proof. We first show the assertion (1). By definition, we consider $R_{1,0}$ as a subspace of $H^1(X, \Sigma_{\mathcal{E}}\langle -Z \rangle)$. From the cohomology exact sequence derived from the exact sequence in Lemma 2.3, we see that

$$H^1(X, \Sigma_{\mathcal{E}}\langle -Z \rangle) \rightarrow H^1(X, T_X(-\log Z))$$

is injective if $H^1(X, \text{Ker}(\mathcal{E}^* \otimes \mathcal{E} \rightarrow \mathcal{E})) = 0$. We consider the Koszul resolution

$$0 \rightarrow \wedge^r \mathcal{E}^* \otimes \mathcal{E} \rightarrow \dots \rightarrow \wedge^2 \mathcal{E}^* \otimes \mathcal{E} \rightarrow \text{Ker}(\mathcal{E}^* \otimes \mathcal{E} \rightarrow \mathcal{E}) \rightarrow 0$$

and the spectral sequence

$$H^{p-q}(X, \wedge^p \mathcal{E}^* \otimes \mathcal{E}) \Rightarrow H^q(X, \text{Ker}(\mathcal{E}^* \otimes \mathcal{E} \rightarrow \mathcal{E})).$$

Thus (i) implies $H^1(X, \text{Ker}(\mathcal{E}^* \otimes \mathcal{E} \rightarrow \mathcal{E})) = 0$. From the Koszul exact sequence

$$0 \rightarrow \wedge^r \mathcal{E}^* \otimes T_X \rightarrow \dots \rightarrow \mathcal{E}^* \otimes T_X \rightarrow T_X(-Z) \rightarrow 0,$$

we see that (ii) is sufficient to imply $H^1(X, T_X(-Z)) = 0$. Thus the cohomology exact sequence derived from

$$0 \rightarrow T_X(-Z) \rightarrow T_X(-\log Z) \rightarrow T_Z \rightarrow 0$$

([12]) implies that $H^1(T_X(-\log Z)) \rightarrow H^1(T_Z)$ is injective. Thus we get (1). We next show (2). It follows from (5) that $R_{1,0} \simeq H^1(\Sigma_{\mathcal{E}}\langle -Z \rangle)$ holds if $H^1(\Sigma_{\mathcal{E}}) = 0$. This last follows from the conditions $H^1(\text{End}(\mathcal{E}^*)) = 0$ and $H^1(T_X) = 0$ by (4). The rest can be shown along an analogous line as in the proof of (1). *q.e.d.*

REMARK 5.3. We can explain the meaning of Lemma 5.2 in a slightly different way. Consider the diagram

$$\begin{array}{ccccc}
 H^0(X, T_X) & & & & \\
 \downarrow r_1 & & & & \\
 H^0(Z, T_{X|Z}) \rightarrow H^0(Z, N_{Z/X}) & \xrightarrow{\delta^*} & H^1(Z, T_Z) & & \\
 & \uparrow r_2 & & & \\
 & H^0(X, \mathcal{E}), & & &
 \end{array}$$

where the horizontal sequence arises from

$$0 \rightarrow T_Z \rightarrow T_{X|Z} \rightarrow N_{Z/X} \simeq \mathcal{E}|_Z \rightarrow 0$$

and r 's are restriction maps. As is well-known, the Kodaira–Spencer map for the family $\mathcal{X} \rightarrow \mathcal{U}$ in 5.1 at σ is given by $\delta^* \circ r_2$. The condition (1), (i) of Lemma 5.2 implies

$$\text{Coker}\{\cdot\sigma: H^0(X, \mathcal{E}^* \otimes \mathcal{E}) \rightarrow H^0(X, \mathcal{E})\} \simeq \text{Im}(r_2).$$

Similarly, the condition (1), (ii) of 5.2 implies that the restriction map r_1 is surjective. Since the map $H^0(Z, T_{X|Z}) \rightarrow H^0(Z, N_{Z/X})$ is given by the contraction with $j(\sigma)|_Z$, we see

$$T_\sigma \simeq \text{Coker}\{\cdot j(\sigma): H^0(X, \Sigma_\mathcal{E}) \rightarrow H^0(X, \mathcal{E})\} \simeq R_{1,0},$$

if we denote by T_σ the image of the Kodaira–Spencer map.

5.4. We recall here the definition of the infinitesimal variation of Hodge structure (IVHS for short). For the theory of IVHS, see [3].

The data $(H_Z, H^{p,q}, Q, T, \delta)$ is called an IVHS of weight k if

- (1) $(H_Z, H^{p,q}, Q)$ is a Hodge structure of weight k polarized by Q ,
- (2) T is a complex vector space and the linear map

$$\delta: T \rightarrow \bigoplus_{p+q=k} \text{Hom}_{\mathbb{C}}(H^{p,q}, H^{p-1,q+1})$$

satisfies the properties

- (i) $\delta(\xi_1)\delta(\xi_2) = \delta(\xi_2)\delta(\xi_1)$ for $\xi_1, \xi_2 \in T$,
- (ii) $Q(\delta(\xi)\phi, \psi) + Q(\phi, \delta(\xi)\psi) = 0$ for $\xi \in T, \phi \in H^{p,q}, \psi \in H^{q+1,p-1}$.

Returning to the situation we are interested in, let T_σ denote the image of the

Kodaira–Spencer map $\rho: T_\sigma(\mathcal{U}) \rightarrow H^1(Z_\sigma, T_{Z_\sigma})$. Then, the polarized Hodge structure on $H_{\text{var}}^{n-r}(Z)$ together with the linear map

$$\delta: T_\sigma \rightarrow \bigoplus_{p+q=n-r} \text{Hom}_{\mathbb{C}}(H_{\text{var}}^{p,q}(Z), H_{\text{var}}^{p+1,q-1}(Z))$$

induced by the cup product $T_Z \otimes \Omega_Z^p \rightarrow \Omega_Z^{p-1}$ gives an IVHS. This is nothing but the information coming from the differential at $[\sigma]$ of the period map P_0 . Proposition 4.7 and Lemma 5.2 show that the algebraic part of the IVHS can be interpreted in terms of the Jacobian ring, under the suitable conditions.

6. IVHS of complete intersections

We now restrict ourselves to nonsingular complete intersections in \mathbb{P}^n and study IVHS applying the results in the preceding sections.

We put $X = \mathbb{P}^n$ and $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}_X(d_i)$. We assume $d_i \geq 2$ for $1 \leq i \leq r$, and $2 \leq r \leq n-2$. Thus $Z = Z_\sigma$, $\sigma \in H^0(X, \mathcal{E})$, is an $(n-r)$ -dimensional nonsingular complete intersection of type (d_1, d_2, \dots, d_r) . We further set

$$d = \sum_{i=1}^r d_i, \quad d_{\max} = \max_{1 \leq i \leq r} (d_i), \quad d_{\min} = \min_{1 \leq i \leq r} (d_i).$$

The purpose of this section is to show the following:

THEOREM 6.1. *Let Z be as above and $R = \bigoplus R_{a,b}$ its Jacobian ring.*

- (1) $H_{\text{var}}^{n-p}(Z, \Omega_Z^{p-r}) \simeq R_{n+r-p,1}$ holds for any $r \leq p \leq n$.
- (2) If T_σ denotes the image of the Kodaira–Spencer map as in 5.4, then $R_{1,0} \simeq T_\sigma$. Further, $R_{1,0} \simeq H^1(Z, T_Z)$ holds unless Z is a K3 surface.
- (3) The cup-product map

$$T_\sigma \otimes H_{\text{var}}^{n-p}(Z, \Omega_Z^{p-r}) \rightarrow H_{\text{var}}^{n+1-p}(Z, \Omega_Z^{p-1-r})$$

can be identified with the multiplication map

$$R_{1,0} \otimes R_{n+r-p,1} \rightarrow R_{n+r+1-p,1}.$$

- (4) The cup-product pairing

$$H_{\text{var}}^{n-p}(Z, \Omega_Z^{p-r}) \otimes H_{\text{var}}^{p-r}(Z, \Omega_Z^{n-p}) \rightarrow H^{n-r}(Z, \Omega_Z^{n-r}) \simeq \mathbb{C}$$

can be identified with

$$R_{n+r-p,1} \otimes R_{p,1} \rightarrow R_{n+r,2} \simeq \mathbb{C},$$

except possibly in the case that Z is an odd dimensional complete intersection of type $(2, 2)$.

Proof. To see (1) and (2), apply Bott's theorem to Corollary 4.9 and Lemma 5.2. The assertion (3) follows from (1) and (2). The assertion (4) will be shown below. q.e.d.

LEMMA 6.2. $R_{n+r-p,1} \simeq (R_{p,1})^*$ for any $r \leq p \leq n$.

Proof. Without losing generality, we can assume $2p \leq n+r$. We show the map $d_{p,1}: R_{n+r-p,1} \rightarrow (R_{p,1})^*$ in 3.3 is an isomorphism. Since we saw in Theorem 6.1, (1) that $R_{n+r-p,1} \simeq H^{n+r-1-p}(Y, \Omega_Y^p(\log \tilde{Z}))$, it suffices to check the conditions in Proposition 3.4 for $s \leq p$ putting $(a, b) = (p, 1)$. We first assume $s < p$ and consider the conditions in Corollary 3.5. The condition (1) of 3.5 becomes

$$H^{n+r-1-s}(Y, \Omega_Y^s \otimes \mathcal{L}^{s-p}) = 0 \quad \text{for } 1 \leq s < p.$$

By Proposition 1.6, (1), it suffices to show

$$H^{n+r-1-s-j}(Y, \pi^*(\Omega_X^i \otimes \wedge^{s-i-j} \mathcal{E}) \otimes \mathcal{L}^{-p+i+j}) = 0,$$

for $\max(0, s+1-r) \leq i \leq s, 0 \leq j \leq s-i, 1 \leq s < p,$

which is equivalent to

$$H^{n-s-j}(X, \Omega_X^i \otimes \wedge^{r-s+i+j} \mathcal{E}^* \otimes S^{p-i-j-r} \mathcal{E}^*) = 0$$

for $\max(0, s+1-r) \leq i \leq s, 0 \leq j \leq s-i,$
 $i+j \leq p-r, 1 \leq s < p$

by Lemma 1.2. By the vanishing of Kodaira–Nakano, this is reduced to showing $H^{n-s}(X, \Omega_X^s \otimes \det \mathcal{E}^* \otimes S^{p-s-r} \mathcal{E}^*) = 0$ for $1 \leq s \leq p-r$. This last follows from Bott's theorem. We can check the other conditions in Corollary 3.5 similarly. (The condition (2) follows directly from the vanishing theorem of Kodaira–Nakano.) We next assume $s = p$ and consider the conditions in Proposition 3.4. We remark that $H^i(Y, \Omega_Y^j) = 0$ unless $i = j$, since Y is a projective space bundle on $X = \mathbb{P}^n$. Thus, by the proof of Corollary 3.5, we see that the group $H^q(Y, \wedge^p \Sigma_{\tilde{Z}}^*)$, where $q = n+r-1-p$ or $n+r-p$, vanishes except in the cases $2p = n+r-1, n+r$. If $2p = n+r-1$, then we have the commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 H^p(\Omega_Y^{p-1}(\log \tilde{Z}) \otimes \mathcal{L}^{-1}) & \rightarrow & H^p(\wedge^p \Sigma_{\tilde{Z}}^*) & \rightarrow & H^p(\Omega_Y^p(\log \tilde{Z})) \\
 & & \uparrow & \nearrow & \\
 & & H^p(\Omega_Y^p) & & \\
 & \nearrow & \uparrow & & \\
 H^{p-1}(\Omega_Y^{p-1}) & \longleftarrow & H^{p-1}(\Omega_Y^{p-1}) & &
 \end{array}$$

coming from (9), (10) and the Poincaré residue sequence. Thus $H^p(Y, \wedge^p \Sigma_{\mathcal{L}}^*)$ does not vanish. However, as we have seen in Lemma 4.5, $H^p(\Omega_Y^p) \rightarrow H^p(\Omega_Y^p(\log \tilde{Z}))$ is a zero map. Thus

$$H^p(Y, \Omega_Y^p(\log \tilde{Z})) \rightarrow H^{p+1}(Y, \Omega_Y^{p-1}(\log \tilde{Z}) \otimes \mathcal{L}^{-1})$$

is injective. If $2p = n + r$, then we have an exact sequence

$$0 \rightarrow H^{p-1}(\wedge^p \Sigma_{\mathcal{L}}^*) \rightarrow H^{p-1}(\Omega_Y^{p-1}) \rightarrow H^p(\Omega_Y^p) \rightarrow H^p(\wedge^p \Sigma_{\mathcal{L}}^*) \rightarrow 0.$$

Since we have $H^{p-1}(\Omega_Y^{p-1}) \simeq H^p(\Omega_Y^p)$ by the Hard Lefschetz theorem, we get $H^{p-1}(\wedge^p \Sigma_{\mathcal{L}}^*) = H^p(\wedge^p \Sigma_{\mathcal{L}}^*) = 0$. In any way, we see that $d_{p,1}$ is an isomorphism for any p with $2p \leq n + r$. q.e.d.

LEMMA 6.3. *With the above notation, $R_{n+r,2} \simeq \mathbb{C}$ holds unless Z is an odd dimensional complete intersection of type $(2, 2)$.*

Proof. We consider the conditions in Corollary 3.5. We only treat the condition (1), since the other cases are quite similar. We first consider the case $s \leq r$. By Proposition 1.6, (1), it suffices to check the following condition for $1 \leq s \leq r$:

$$H^{n+r-1-s-j}(Y, \pi^*(\Omega_X^i \otimes \wedge^{s-i-j} \mathcal{E} \otimes K_X \otimes \det \mathcal{E}) \otimes \mathcal{L}^{i+j-r}) = 0$$

for $\max(0, s + 1 - r) \leq i \leq s, \quad 0 \leq j \leq s - i$.

By Lemma 1.2, this is equivalent to the conditions

$$H^{n-s}(X, \wedge^s \mathcal{E} \otimes K_X) = 0, \quad 1 \leq s < r,$$

$$H^{n-1-r+i}(X, \Omega_X^i \otimes K_X \otimes \det \mathcal{E}) = 0, \quad 1 \leq i \leq r,$$

which follows from Bott's theorem. We next consider the case $s > r$. By Proposition 1.6, (2), the condition

$$H^{n+r-2-s+j}(Y, \pi^*(\Omega_X^i \otimes \wedge^{s-i+j} \mathcal{E} \otimes \det \mathcal{E} \otimes K_X) \otimes \mathcal{L}^{i-j-r}) = 0$$

for $s + 1 - r \leq i \leq \min(s, n), \quad 1 \leq j \leq r - s + i, \quad r < s \leq n + r - 2$

is sufficient. By Lemma 1.2, it suffices to check the following conditions for $r < s \leq n + r - 2$:

$$H^{n+r-2-s+j}(X, \Omega_X^i \otimes K_X \otimes \wedge^{s-i+j} \mathcal{E} \otimes \det \mathcal{E} \otimes S^{i-j-r} \mathcal{E}) = 0$$

for $s + 1 - r \leq i \leq \min(s, n), \quad 1 \leq j \leq r - s + i, \quad i - j \geq r$.

We remark that $0 < n + r - 2 - s + j < n$. Thus, in view of Bott's theorem, these do not vanish only when $n + r - 2 - s + j = i$ and the degree of a direct summand of

$$A = K_X \otimes \det \mathcal{E} \otimes \wedge^{2s+2-n-r} \mathcal{E} \otimes S^{n-2-s} \mathcal{E}$$

is 0. Since $d_v \geq 2$ for any $1 \leq v \leq r$, the degree of each direct summand of A is not less than $2s - n - 1$. Since $2s + 2 - n - r \geq 0$, we only have to consider the cases

- (a) $2s = n + 1, r = 3$ and $(d_1, d_2, d_3) = (2, 2, 2)$.
- (b) $2s = n, r = 2$ and $(d_1, d_2) = (2, 3)$.
- (c) $2s = n + 1, r = 2$ and $(d_1, d_2) = (2, 2)$.

If (a) or (b) is the case, then we have $i = j + s$ which contradicts the condition $i \leq s$. Thus only (c) is the exception. *q.e.d.*

7. Generic Torelli theorem

The following is our main result.

THEOREM 7.1. *The generic Torelli theorem holds for complete intersections of type (d_1, \dots, d_r) in \mathbb{P}^n provided that the following conditions are satisfied.*

- (1) $2d - 2n - 2 \geq d_{\max}$.
- (2) $d \leq \begin{cases} 2n - r & \text{if } n - r \text{ is even and } d_{\min} = 2 \\ 2n - r + 1 & \text{otherwise.} \end{cases}$

For the proof, we need some lemmas.

LEMMA 7.2. *Suppose that the canonical bundle K_Z of Z is ample, i.e., $d > n + 1$. Then $H^0(Z, K_Z^m) \simeq R_{mr,m}$ for any $m \in \mathbb{N}$.*

Proof. We set $\tilde{K} = K_X \otimes \det \mathcal{E}$. By Lemma 2.6, it suffices to show

$$H^0(Z, K_Z^m) \simeq \text{Coker}\{H^0(X, \mathcal{E}^* \otimes \tilde{K}^m) \xrightarrow{\sigma} H^0(X, \tilde{K}^m)\}.$$

We consider the Koszul resolution

$$0 \rightarrow \wedge^r \mathcal{E}^* \otimes \tilde{K}^m \rightarrow \dots \rightarrow \mathcal{E}^* \otimes \tilde{K}^m \rightarrow \tilde{K}^m(-Z) \rightarrow 0.$$

Since $H^i(X, \wedge^i \mathcal{E}^* \otimes \tilde{K}^m) = 0$ for $1 \leq i \leq r$, we see $H^1(X, \tilde{K}^m(-Z)) = 0$. Further, since $H^{i-1}(X, \wedge^i \mathcal{E}^* \otimes \tilde{K}^m) = 0$ for $2 \leq i \leq r$, we see that

$H^0(X, \mathcal{E}^* \otimes \tilde{K}^m) \rightarrow H^0(X, \tilde{K}^m(-Z))$ is surjective. Since $H^1(X, \tilde{K}^m) = 0$, it follows

$$\begin{aligned} H^0(Z, K_Z) &\simeq \text{Coker}\{H^0(X, \tilde{K}^m(-Z)) \rightarrow H^0(X, \tilde{K}^m)\} \\ &\simeq \text{Coker}\{H^0(X, \mathcal{E}^* \otimes \tilde{K}) \rightarrow H^0(X, \tilde{K}^m)\}. \end{aligned} \quad \text{q.e.d.}$$

The proof of the following lemma is easy and we left it for the reader.

LEMMA 7.3. *Suppose that the canonical bundle K_Z of Z is ample and consider the canonical map $\Phi_K: Z \rightarrow \mathbb{P}^{\dim|K_Z|}$. If $2d - 2n - 2 \geq d_{\max}$, then the homogeneous ideal of $\Phi_K(Z)$ is generated by quadrics.*

LEMMA 7.4. *Let Z be as above. The map*

$$d_{n-r,0}: R_{2r,2} \rightarrow (R_{n-r,0})^*$$

is injective if the condition (2) of Theorem 7.1 is satisfied.

Proof. We only have to check the conditions (1) and (2) of Corollary 3.5 putting $(a, b) = (n-r, 0)$. This can be carried out in a similar way as in the proof of Lemma 6.3, using 1.6, (1) (resp. (2)) if $s \leq n$ (resp. if $s > n$). Hence we omit it.

Proof of Theorem 7.1. We recover Z from the IVHS $(H_{\text{var}}^{n-r}(Z), T_\sigma, \delta)$. By Theorem 6.1, we are given the following data:

(1) The multiplication maps

$$m_{n-p}: R_{1,0} \otimes R_{n+r-p,1} \rightarrow R_{n+r+1-p,1} \quad \text{for } r < p \leq n.$$

(2) The perfect pairing

$$Q: R_{n+r-p,1} \otimes R_{p,1} \rightarrow R_{n+r,2} \simeq \mathbb{C} \quad \text{for } r \leq p \leq n.$$

Applying all the m_{n-p} in succession, we have

$$\underbrace{R_{1,0} \otimes \cdots \otimes R_{1,0}}_{n-r} \otimes R_{r,1} \rightarrow R_{n,1}.$$

Combining with the polarization Q , we get

$$\underbrace{R_{1,0} \otimes \cdots \otimes R_{1,0}}_{n-r} \otimes R_{r,1} \otimes R_{r,1} \rightarrow R_{n,1} \otimes R_{r,1} \rightarrow R_{n+r,2} \simeq \mathbb{C}.$$

Thus we get a bilinear map

$$S^{n-r}R_{1,0} \otimes S^2R_{r,1} \rightarrow R_{n+r,2} \simeq \mathbb{C}$$

or a linear map

$$\lambda: S^2R_{r,1} \rightarrow S^{n-r}(R_{1,0})^*.$$

Thus $\text{Ker}(\lambda)$ is an invariant of the IVHS and called the infinitesimal Schottky relation in [3]. Since λ is induced by the multiplication map, we get the following factorization of it:

$$\begin{array}{ccc} S^2R_{r,1} & \xrightarrow{\lambda} & S^{n-r}(R_{1,0})^* \\ \mu \downarrow & & \uparrow v \\ R_{2r,2} & \xrightarrow{d_{n-r,0}} & (R_{n-r,0})^*, \end{array}$$

where μ is the multiplication map and v is the dual of the multiplication map $S^{n-r}R_{1,0} \rightarrow R_{n-r,0}$. Since $S^{n-r}H^0(X, \mathcal{E}) \rightarrow H^0(X, S^{n-r}\mathcal{E})$ is surjective, we see that v is injective. Moreover, since $d_{n-r,0}$ is injective by Lemma 7.4, we conclude that $\text{Ker}(\lambda) \simeq \text{Ker}(\mu)$. By Lemma 7.2, μ is nothing but the natural map $S^2H^0(Z, K_Z) \rightarrow H^0(Z, K_Z^2)$. Thus $\text{Ker}(\mu)$ can be identified with the set of all quadrics through the canonical image of Z . In summary, we see that Z can be reconstructed from the infinitesimal Schottky relation $\text{Ker}(\lambda)$ by virtue of Lemma 7.3. This completes the proof of Theorem 7.1, because Variational Torelli implies Generic Torelli ([5]).

We list d_1, \dots, d_r satisfying the conditions of Theorem 7.1 when $n-r=2, 3$.

Table 7.5

$n-r$	$(d_1, \dots, d_r), d_1 \geq \dots \geq d_r$
2	(4, 3)
3	(6, 3), (5, 4), (4, 4), (6, 2, 2), (5, 3, 2), (4, 4, 2), (4, 3, 3), (3, 3, 3) (5, 2, 2, 2), (4, 3, 2, 2), (4, 2, 2, 2), (3, 3, 3, 2), (3, 3, 2, 2) (4, 2, 2, 2, 2), (3, 3, 2, 2, 2), (3, 2, 2, 2, 2), (2, 2, 2, 2, 2)

References

- [1] M. F. Atiyah: Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* 85 (1957), 181–207.
- [2] S. Bloch and D. Gieseker: The positivity of the Chern classes of an ample vector bundle, *Invent. Math.* 12 (1971), 112–117.
- [3] J. Carlson, M. Green, P. A. Griffiths and J. Harris: Infinitesimal variations of Hodge structures I, *Compositio Math.* 50 (1983), 109–205.
- [4] J. Carlson and P. A. Griffiths: Infinitesimal variations of Hodge structure and the global Torelli problem, *Journées de Géométrie Algébrique d’Angers 1979*, Sijthoff & Noordoff (1980), 51–76.
- [5] D. Cox, R. Donagi and L. Tu: Variational Torelli implies generic Torelli, *Invent. Math.* 88 (1987), 439–446.

- [6] P. Deligne: Theorie de Hodge II, *Publ. Math. IHES* 40 (1971), 5–57.
- [7] R. Donagi: Generic Torelli for projective hypersurfaces, *Compositio Math.* 50 (1983), 325–353.
- [8] R. Donagi and L. Tu: Generic Torelli for weighted hypersurfaces, *Math. Ann.* 276 (1987), 399–413.
- [9] H. Flenner: The infinitesimal Torelli problem for zero sets of sections of vector bundles, *Math. Z.* 193 (1986), 307–322.
- [10] M. Green: The period map for hypersurface sections of high degree of an arbitrary variety, *Compositio Math.* 55 (1985), 135–156.
- [11] R. Hartshorne: Ample vector bundles, *Publ. Math. IHES.* 29 (1966), 319–350.
- [12] Y. Kawamata: On deformations of compactifiable manifolds, *Math. Ann.* 235 (1978), 247–265.
- [13] K. Konno: Generic Torelli theorem for hypersurfaces of certain compact homogeneous Kähler manifolds, *Duke Math. J.* 59 (1989), 83–160.
- [14] S. Mori and H. Sumihiro: On Hartshorne’s conjecture, *J. Math. Kyoto Univ.* 18 (1978), 523–533.
- [15] A. Morimoto: Sur le groupe d’automorphismes d’un espace fibre principal analytique complexe, *Nagoya Math. J.* 13 (1958), 157–178.
- [16] C. Peters: The local Torelli theorem I: Complete intersections, *Math. Ann.* 217 (1975), 1–16.
- [17] C. Peters and J. H. M. Steenbrink: Infinitesimal variations of Hodge structure and the generic Torelli problem for projective hypersurfaces, in: *Classification of Algebraic and Analytic Manifolds*, Progress in Math. 39, Birkhäuser (1983), 399–464.
- [18] M.-H. Saito: Weak global Torelli theorem for certain weighted hypersurfaces, *Duke Math. J.* 53 (1986), 67–111.
- [19] B. Shiffman and A. J. Sommese: *Vanishing Theorems on Complex Manifolds*, Progress in Math. 56, Birkhäuser (1985).
- [20] T. Terasoma: Infinitesimal variation of Hodge structures and the weak global Torelli theorem for complete intersections, to appear in *Ann. of Math.*
- [21] S. Usui: Local Torelli theorem for non-singular complete intersections, *Japan J. Math.* 2-2 (1976), 411–418.