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On the Gauss maps of space curves in characteristic p, II

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0. Introduction

The purpose of this article is to study the extensions of function fields defined by Gauss maps of ordinary elliptic curves and curves of higher genus with a small number of cusps in projective spaces defined over an algebraically closed field of positive characteristic p.

For elliptic curves, we shall prove

THEOREM 0.1. *Let C be an elliptic curve, let $\imath$ be a morphism from C to a projective space $\mathbb{P}$, birational onto its image, and let C' be the normalization of the image of C under the Gauss map of C in $\mathbb{P}$ via $\imath$. If $\imath$ is unramified, then C' is also an elliptic curve. Furthermore, if C is ordinary, then the dual of the natural morphism $C \to C'$ as an isogeny of abelian varieties is a cyclic, étale cover.*
(See Section 2.)

We first note that the latter assertion is the main part of this result, and the former assertion is already known (see [5, Theorem 4.1]). But we shall show the former in a different way.

In the previous article [5], we gave a sufficient condition for a proper subfield K' of the function field $K(C)$ of an ordinary elliptic curve C to have an unramified $\imath$ from C to some $\mathbb{P}$, birational onto its image, such that the extension of function fields defined by the Gauss map coincides with $K(C)/K'$ (see [5, Theorem 5.1]). The main part of our Theorem 0.1 asserts that the condition given in [5] turns out a necessary condition as well. In fact, one can easily verify that the conclusion of Theorem 0.1 is equivalent to the assumption of [5, Theorem 5.1] with $K(C') = K'$ (see, for example, [6, Chapter 12] or [12, §§13–15]). We here note that it is known that a curve C has a $\imath$ with birational Gauss map if and only if the characteristic $p \neq 2$ (see, for example, [5, Corollary 2.3] or [2, Exposé XVII, §1.2 and Proposition 3.3]).

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Our main result for higher genus case is

THEOREM 0.2. *Let C be a curve of genus g, let ι be a morphism from C to $\mathbb{P}$, birational onto its image, let $\Omega^1_{C/\mathbb{P}}$ be the sheaf of relative differentials of C over $\mathbb{P}$ via ι, and let γ be the degree of $\Omega^1_{C/\mathbb{P}}$ as a torsion sheaf. If*

$$\gamma < 2g - 2,$$

then the Gauss map of C in $\mathbb{P}$ via ι has separable degree 1.
(See Section 3.)

We here note that if $\gamma < 2g - 2$, then $g \geqslant 2$ since $\gamma \geqslant 0$ by definition, and the following facts are well-known: in case of $p \neq 2$, γ is equal to the number of cusps of the image $\iota(C)$; and, in case of $p = 2$, γ is double the number of cusps (see, for example, [7, I, C] or [14, §3]). If we denote by r_m the m-rank of C in $\mathbb{P}$ via ι (see, for example, [14, §3]), then the assumption of Theorem 0.2 is equivalent to the condition,

$$2r_0 < r_1$$

because we have $r_1 = 2g - 2 + 2r_0 - \gamma$.

On the other hand, the conclusion above tells us that there are only a finite number of multiple tangents of C in $\mathbb{P}$ via ι, that is, lines in $\mathbb{P}$ tangent at two or more points to some smooth branches of $\iota(C)$.

Theorem 0.2 will give an improvement and another proof for the previous result [5, Corollary 4.4]. We shall give an example to show that the improvement is nontrivial (see Example 3.1).

Furthermore, we shall show that it is impossible to weaken the hypothesis on γ in Theorem 0.2. For any curve C' with the p-rank being not zero and for any positive integers s indivisible by p and l, taking a suitable curve C of genus g over C', we shall construct a morphism ι from C to some $\mathbb{P}$, birational onto its image, such that $C \to C'$ coincides with the natural morphism defined by the Gauss map as in Theorem 0.1, $\gamma = 2g - 2$, and the Gauss map has separable degree s and inseparable degree p^l (see Example 4.1).

1. Preliminaries

Let C be a smooth, connected, complete curve defined over an algebraically closed field k of positive characteristic p.

We use the same notations as in the previous article [5]. Recall that, for a morphism ι from C to a projective space $\mathbb{P}$, birational onto its image, we denote by $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \otimes_k \mathcal{O}_{\mathbb{G}} \to \mathcal{Q}$ the universal quotient bundle of the Grassmann manifold $\mathbb{G}$ consisting of 2-dimensional quotient spaces of $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, by $\mathcal{Q}_C$

the pull-back of $\mathcal{Q}$ to C by the Gauss map $C \to \mathbb{G}$ defined via ι, and by C' the normalization of the image of $C \to \mathbb{G}$. Assuming that $\iota(C)$ is not a line in $\mathbb{P}$, we have a natural, finite morphism of curves

$$C \to C'.$$

We denote by C_0 the section of the ruled surface $\mathbb{P}(\mathcal{Q}_C)$ over C consisting of the points of contact on $\mathbb{P}(\mathcal{Q}_C)$, and let

$$\mathcal{Q}_C \to \iota^*\mathcal{O}_{\mathbb{P}}(1)$$

be the natural quotient corresponding to C_0, as in [5, §3].

The property of $C \to C'$ and $\mathcal{Q}_C \to \iota^*\mathcal{O}_{\mathbb{P}}(1)$ used often in [5, Proof of Theorem 5.2] is stated in general as follows:

LEMMA 1.1. *If $C \to C'$ factors through a curve C'', and if $\mathcal{Q}_C \to \iota^*\mathcal{O}_{\mathbb{P}}(1)$ comes from a quotient of $\mathcal{Q}_{C''}$ as a pull-back, then $C \to C''$ is an isomorphism.*

Proof. Let f be the natural morphism $\mathbb{P}(\mathcal{Q}_C) \to \mathbb{P}(\mathcal{Q}_{C'})$ induced by the Gauss map, and let C_0'' be a section of $\mathbb{P}(\mathcal{Q}_{C''})$ over C'' associated to the quotient of $\mathcal{Q}_{C''}$ as above. Since $f|_{C_0}$ factors through C_0'' and is birational onto its image (see the proof of either [4, Lemma 1.4] or [5, Theorem 3.1 (1)$\Rightarrow$(2)]), $C_0 \to C_0''$ must be an isomorphism, which implies the result.

2. Ordinary elliptic curves

Assuming that C is an elliptic curve and ι is unramified, we have the following exact sequence

$$0 \to \iota^*\mathcal{O}_{\mathbb{P}}(1) \to \mathcal{Q}_C \to \iota^*\mathcal{O}_{\mathbb{P}}(1) \to 0 \qquad (\varepsilon)$$

(see, for example, [5, Lemma 3.2]).

LEMMA 2.1. *Assume that $C \to C'$ factors through a curve C''. If*

(1) *$\mathcal{Q}_{C''}$ is indecomposable of even degree when (ε) does not split, or*
(2) *$\mathcal{Q}_{C''}$ is isomorphic to $\mathcal{L}^{\oplus 2}$ for some line bundle $\mathcal{L}$ on C'' when (ε) splits,*

then $C \to C''$ is an isomorphism.

Proof. This result follows from Lemma 1.1 (see, for example, [1, Part II] or [3, Chapter IV, §4]).

Now, if C' were rational, then $\mathcal{Q}_{C'}$ would be decomposed and hence isomorphic to $\mathcal{L}^{\oplus 2}$ for some line bundle $\mathcal{L}$ on C'. It follows from Lemma 2.1 that $C \to C'$ is an isomorphism, and this is a contradiction. Therefore C' is an elliptic curve, and we have proved the former assertion of Theorem 0.1.

Next, assume moreover that C is ordinary, and let us prove the main part of Theorem 0.1. We may here assume that $C \to C'$ is not separable; otherwise, $C \to C'$ is an isomorphism (see [5, Corollary 2.2]), and there is nothing to prove.

Case: (ε) *does not split.*
It follows from Lemma 2.1 that if $C \to C'$ factors through a curve C'' and if $C \to C''$ is not an isomorphism, then $\mathcal{Q}_{C''}$ is indecomposable of odd degree, $C \to C''$ thus has even degree, and $C'' \to C'$ has odd degree. This implies that $C \to C'$ has inseparable degree 2 and the separable degree is odd, which completes the proof in this case (see, for example, [6, Chapter 12] or [12, §§13–15]).

REMARK. It can be shown that $C \to C'$ in this case is purely inseparable.

Case: (ε) *splits.*
If q denotes the inseparable degree of $C \to C'$, then one can factorize $C \to C'$ as a composition of a separable morphism $C^{(q)} \to C'$ with a Frobenius morphism $C \to C^{(q)}$ of degree q.

Since $\mathcal{Q}_C \simeq \imath^*\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$, it follows from [13, Theorem 2.16] that $\mathcal{Q}_{C^{(q)}}$ is decomposed. Clearly, $\mathcal{Q}_{C^{(q)}}$ has even degree.

I claim that $\mathcal{Q}_{C'}$ is also decomposed. Suppose the contrary. It is sufficient to show that $\mathcal{Q}_{C^{(q)}}$ is isomorphic to $\mathcal{L}^{\oplus 2}$ for some line bundle $\mathcal{L}$ on $C^{(q)}$, because, according to Lemma 2.1, this implies that $C \to C^{(q)}$ would be an isomorphism, which contradicts to our assumption on $C \to C'$. Now, if $\mathcal{Q}_{C'}$ has even degree, then we have a non-trivial extension

$$0 \to \mathcal{M} \to \mathcal{Q}_{C'} \to \mathcal{M} \to 0$$

with some line bundle $\mathcal{M}$ on C'. Since $\mathcal{Q}_{C^{(q)}}$ is decomposed, we see that $\mathcal{Q}_{C^{(q)}}$ must be isomorphic to $\mathcal{M}_{C^{(q)}}^{\oplus 2}$. If $\mathcal{Q}_{C'}$ has odd degree, then $C^{(q)} \to C'$ has even degree and it follows from Proposition 2.2 below that $\mathcal{Q}_{C^{(q)}}$ is of the form $\mathcal{L}^{\oplus 2}$. This proves the claim.

Now, one can write

$$\mathcal{Q}_{C'} = \mathcal{L}_1 \oplus \mathcal{L}_2$$

with some line bundles $\mathcal{L}_1, \mathcal{L}_2$ on C'. Obviously, $\mathcal{L}_1$ and $\mathcal{L}_2$ have same degree. Since $\mathcal{Q}_C \simeq \imath^*\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$, $\mathcal{L}_1 \otimes \mathcal{L}_2^{\vee}$ is contained in the kernel of the dual $\hat{C} \leftarrow \hat{C}'$.

Let K be a discrete subgroup scheme of $\hat{C}'$ generated by $\mathcal{L}_1 \otimes \mathcal{L}_2^{\vee}$, and let C'' be the dual of a quotient of $\hat{C}'$ by K. Then, $C \to C'$ factors through C'' since $\hat{C} \leftarrow \hat{C}'$ factors through $\hat{C}'/K$, and we have $\mathcal{L}_{1_{C''}} \simeq \mathcal{L}_{2_{C''}}$. By virtue of Lemma 2.1, we find that $C \to C''$ is an isomorphism. The kernel of $\hat{C} \leftarrow \hat{C}'$ is hence isomorphic to K as a finite group scheme. This completes the proof.

REMARK. Both cases in the proof actually occur. For the former, see [5, Theorem 5.2 and Remark at the end of §5]; and for the latter, see [5, Theorem 5.1].

Using the same technique as in [13], we prove

PROPOSITION 2.2. *Let $C \to C'$ be a separable finite morphism of arbitrary elliptic curves, and let $\mathscr{E}$ be an indecomposable vector bundle of rank 2 on C'. If $C \to C'$ has even degree, and if $\mathscr{E}$ has odd degree, then $\mathscr{E}_C$ is isomorphic to $\mathscr{L}^{\oplus 2}$ for some line bundle $\mathscr{L}$ on C.*

Proof. It is sufficient to consider the case when $C \to C'$ is separable of degree 2. In fact, by our assumption, one can factorize $C \to C'$ through a curve C'' such that $C'' \to C'$ is of degree 2 since the kernel of $C \to C'$ has a subgroup such that the quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Let us consider the following fibre product

$$\begin{array}{ccc} C \times_{C'} C & \xrightarrow{\tilde{f}} & C \\ \tilde{\pi}\downarrow & & \downarrow\pi \\ C & \xrightarrow{f} & C', \end{array}$$

where f is the given morphism, π is its copy, and $\tilde{f}$ and $\tilde{\pi}$ are the induced morphisms. Since $\mathscr{E}$ has odd degree, it follows from [13, Proposition 2.1] that, for a line bundle $\mathscr{L}$ of odd degree on C, we have

$$\mathscr{E} \simeq \pi_* \mathscr{L} \otimes \mathscr{M}$$

with some line bundle $\mathscr{M}$ on C'. It follows that

$$f^*\mathscr{E} \simeq f^*\pi_*\mathscr{L} \otimes f^*\mathscr{M} \simeq \tilde{\pi}_*\tilde{f}^*\mathscr{L} \otimes f^*\mathscr{M},$$

since f is a flat morphism. We see that $C \times_{C'} C$ is isomorphic to a disjoint union of two C's because $C \to C'$ is étale of degree 2. Hence, it follows that $\tilde{\pi}_*\tilde{f}^*\mathscr{L} \simeq \mathscr{L}^{\oplus 2}$, and we get the result.

3. Curves of higher genus with a small number of cusps

We first prove Theorem 0.2. Put

$$s(\mathscr{E}, \mathscr{L}) := 2 \deg \mathscr{L} - \deg \mathscr{E}$$

for a vector bundle $\mathscr{E}$ of rank 2 on a curve C and a quotient line bundle $\mathscr{L}$ of $\mathscr{E}$.

Then our assumption is equivalent to the condition,

$$s(\mathcal{Q}_C, \iota^*\mathcal{O}_{\mathbb{P}}(1)) < 0.$$

Now, let q be the inseparable degree of $C \to C'$. We have a commutative diagram

$$\begin{array}{ccc} C & \to & C^{(q)} \\ \downarrow & & \downarrow \\ C'^{(1/q)} & \to & C', \end{array}$$

where the horizontal arrows are Frobenius morphisms of degree q and the vertical arrows are both separable.

Since $s(\mathcal{Q}_C, \iota^*\mathcal{O}_{\mathbb{P}}(1)) < 0$ and $C \to C'^{(1/q)}$ is separable, by virtue of the theory of Galois descent, we find that the quotient $\mathcal{Q}_C \to \iota^*\mathcal{O}_{\mathbb{P}}(1)$ comes from a quotient of $\mathcal{Q}_{C'^{(1/q)}}$ (see, for example, [11, Proof of Proposition 3.2]). It follows from Lemma 1.1 that $C \to C'^{(1/q)}$ is an isomorphism, and hence $C \to C'$ has separable degree 1. This completes the proof.

Next, to show that Theorem 0.2 has a strictly wider range of application than [5, Corollary 4.4], we give this

EXAMPLE 3.1. We here show, for a certain curve C, there exists a morphism ι from C to some $\mathbb{P}$, birational onto its image, such that

(a) $\gamma < 2g - 2$,
(b) ι is *not* unramified, and
(c) the Gauss map is not separable,

in particular, $\iota(C)$ is not smooth. According to Theorem 0.2, the condition (a) implies that the Gauss map in this case has separable degree 1.

Let C be a curve with a Frobenius morphism $C \to C^{(p)}$ of degree p such that there is a line bundle $\mathscr{L}$ on $C^{(p)}$ satisfying

(0) $0 < \deg \mathscr{L}$,
(1) $\deg \mathscr{L} < (2g - 2)/p$, and
(2) $H^1(C, \mathscr{L}_C^\vee) \leftarrow H^1(C^{(p)}, \mathscr{L}^\vee)$ induced by $C \to C^{(p)}$ is not injective.

We note that such a curve C does exist. In fact, if Tango's invariant $n(C)$ of a curve C (see [16]) satisfies

$$0 < n(C) < (2g - 2)/p,$$

then C is the required curve, and examples in this case can be found in [16, §5].

Now, let

$$0 \to \mathcal{O}_{C^{(p)}} \to \mathcal{E} \to \mathcal{L} \to 0 \tag{ξ}$$

be an extension of $\mathcal{L}$ by $\mathcal{O}_{C^{(p)}}$ corresponding to a non-zero element ξ of $H^1(C^{(p)}, \mathcal{L}^\vee)$ killed by the Frobenius morphism. Since (ξ) is a non-trivial extension and the pull-back of (ξ) to C splits, we get a new quotient line bundle $\mathcal{O}_C$ of $\mathcal{E}_C$ Let

$$h\colon C \to \mathbb{P}(\mathcal{E})$$

be a morphism over $C^{(p)}$ associated to this $\mathcal{E}_C \to \mathcal{O}_C$. We see that $\mathcal{E}_C \to \mathcal{O}_C$ does not come from any quotient of $\mathcal{E}$ because we have $\deg \mathcal{L} \geqslant 0$. Therefore h is a birational onto its image, and $h(C)$ is purely inseparable of degree p over $C^{(p)}$ via the projection of $\mathbb{P}(\mathcal{E})$ over $C^{(p)}$ (see [5, Lemma 6.3, (2) $\Rightarrow$ (1)]).

Let ρ be an embedding of $\mathbb{P}(\mathcal{E})$ into a projective space $\mathbb{P}$ as a scroll, and let $\imath$ be a composition of ρ with h. We see, by construction, that $\imath\colon C \to \mathbb{P}$ is birational onto its image, the natural quotient $\mathcal{Q}_C \to \imath^*\mathcal{O}_{\mathbb{P}}(1)$ is isomorphic to our $\mathcal{E}_C \to \mathcal{O}_C$ up to tensoring a line bundle on C, and the Gauss map is purely inseparable of degree p (see [5, Lemma 3.3]), where we have used only (2) and $\deg \mathcal{L} \geqslant 0$.

In particular, $\imath$ has the required property (c). Moreover, $\imath$ enjoys the other properties, too. In fact, (a) and (b) are equivalent to (0) and (1), respectively, because we have

$$\deg \mathcal{L} = (2g - 2 - \gamma)/p.$$

REMARK. If there exists a line bundle $\mathcal{L}$ on $C^{(p)}$ satisfying (2) above and the condition

$$\deg \mathcal{L} = (2g - 2)/p$$

instead of (0) and (1), then C is called a *Tango-Raynaud curve* (see, for example, [5, §6] or [15]). In this case, the h constructed as above is a closed immersion, so is $\imath$ (see [5, Corollary 6.5]).

4. Curves of higher genus with infinitely many multiple tangents

Let us show that it is impossible to weaken the hypothesis on γ in our Theorem 0.2.

EXAMPLE 4.1. For any curve C' with the p-rank being not zero and for any positive integers s indivisible by p and l, taking a suitable curve C of genus g over

C', we here construct a morphism ι from C to a projective space $\mathbb{P}$, birational onto its image, such that

(a) $C \to C'$ coincides with the natural morphism defined by the Gauss map as in §1,
(b) $\gamma = 2g - 2$, and
(c) the Gauss map has separable degree s and inseparable degree p^l.

Let C' be a curve of genus $g' \geqslant 1$. Let s be a positive integer indivisible by p, and let $\mathscr{L}$ be a line bundle on C' such that $\mathscr{L}$ is a torsion element of order s in Pic C'. For a given positive integer l, consider the following diagram

$$\begin{array}{ccc} C & \to & C^{(p^l)} \\ & & \downarrow \\ & & C', \end{array}$$

where $C^{(p^l)} \to C'$ is an étale Galois cover with group $\mathbb{Z}/s\mathbb{Z}$ corresponding to $\mathscr{L}$ (see, for example, [10, Chapter III, §4]), and $C \to C^{(p^l)}$ is a Frobenius morphism of degree p^l.

Assume that the p-rank of the Jacobian of C' is not zero, and let $\mathscr{N}$ be a line bundle on C' such that $\mathscr{N}$ is a torsion element of order p^l in Pic C'. Then we put

$$\mathscr{E} := \mathcal{O}_{C'} \oplus \mathscr{L} \otimes \mathscr{N}.$$

Taking the pull-back to C, we have $\mathscr{E}_C \simeq \mathcal{O}_C^{\oplus 2}$. Let $\mathscr{E}_C \to \mathcal{O}_C$ be a quotient equal to the pull-back of neither $\mathscr{E} \to \mathcal{O}_{C'}$ nor $\mathscr{E} \to \mathscr{L} \otimes \mathscr{N}$. Then we see that if $C \to C'$ factors through a curve C'', and if $\mathscr{E}_C \to \mathcal{O}_C$ comes from a quotient line bundle of $\mathscr{E}_{C''}$, then $C \to C''$ is an isomorphism. In fact, by virtue of the choice of $\mathscr{E}_C \to \mathcal{O}_C$, we have $\mathscr{E}_{C''} \simeq \mathcal{O}_{C''}^{\oplus 2}$ in this case, and the result follows from Lemma 4.2 below.

Now, let $h: C \to \mathbb{P}(\mathscr{E})$ be a morphism over C' associated to this $\mathscr{E}_C \to \mathcal{O}_C$. We find that h is birational onto its image (see the part of [5, Proof of Theorem 5.1] showing the birationality of $f|_{C_0}$ and [5, Lemma 6.3, (2) $\Rightarrow$ (1)]).

As in Example 3.1, we obtain from h a morphism ι from C to some $\mathbb{P}$, birational onto its image, such that our $\mathscr{E}_C \to \mathcal{O}_C$ is isomorphic to the natural quotient $\mathscr{Q}_C \to \iota^* \mathcal{O}_{\mathbb{P}}(1)$ up to tensoring a line bundle on C, and our $C \to C'$ coincides with the natural morphism induced by the Gauss map. Hence, this ι has the required properties (a) and (c). Furthermore, ι enjoys (b) since we have $\mathscr{E}_C \simeq \mathcal{O}_C^{\oplus 2}$.

Since the following fact used above is well-known and easily verified, we omit the proof.

LEMMA 4.2. *Let $C \to C'$ be a finite morphism of arbitrary curves, and let $\mathscr{L}$ be a line bundle on C' such that $\mathscr{L}$ is torsion in* Pic C'. *If $\mathscr{L}_C$ is trivial, then the degree of $C \to C'$ is divisible by the order of $\mathscr{L}$ in* Pic C'.

Acknowledgements

After a first draft of this note was written, at the Conference on "Projective Varieties" (held in Trieste, Italy on 19–24 June 1989) Professor Steven L. Kleiman told me that, while looking for a new proof of [5, Corollary 4.4], he and Ragni Piene independently discovered Theorem 0.2 with a proof quite different from ours and their new proof yielded the equality, $2s(g-g')=(s-1)\gamma$, where s denotes the separable degree of the Gauss map and g' the genus of C', with the same notations as in §0.

However in May 1990, according to the referee, the equality turned out to be false and Kleiman asserts that the inequality,

$$2s(g-g') \leqslant (s-1)\gamma$$

holds.

I should like to express my thanks to the referee for his/her advice, especially by which my original proof for Theorem 0.1 could be simplified.

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