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# Abelian varieties and curves in $W_d(C)$

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## 1. Introduction

The questions dealt with in this paper were originally raised by Joe Silverman in [10]. A further impetus for studying them is given by the recent results of Faltings [1].

The starting point is the following. Suppose that $C' \to C$ is a nonconstant map of smooth algebraic curves. It is a classical observation in this situation that if $C'$ is hyperelliptic then $C$ must be as well. An immediate generalization of this is the statement that if $C'$ admits a map of degree $d$ or less to $\mathbf{P}^1$, then $C$ does as well. This is elementary: if $f \in K(C')$ is a rational function of degree $d$, then either its norm is a nonconstant rational function of degree $d$ or less on $C$; or else its norm is constant, in which case the norm of some translate $f - z_0$ will not be constant.

In [10], Silverman poses a similar problem: if in the above situation the curve $C'$ is bielliptic – that is, admits a map of degree 2 to an elliptic curve or $\mathbf{P}^1$ – does it follow that $C$ is as well? Silverman answers this affirmatively under the additional hypothesis that the genus $g = g(C) \geqslant 9$.

The most general question along these lines is this. Say a curve $C$ is of type $(d, h)$ if it admits a map of degree $d$ or less to a curve of genus $h$ or less. We may then make the

STATEMENT $S(d, h)$: If $C' \to C$ is a nonconstant map of smooth curves and $C'$ is of type $(d, h)$, then $C$ is of type $(d, h)$

and ask for which $(d, h)$ this holds. We may further refine the question by specifying the genus of the curve $C$: we thus have the

STATEMENT $S(d, h, g)$: Suppose $C' \to C$ is a nonconstant map of smooth curves with $C$ of genus $g$. If $C'$ is of type $(d, h)$, then $C$ is of type $(d, h)$.

As we remarked, this is known to hold in case $h = 0$. In the next case $h = 1$, Silverman in [10] gives some positive results: he shows that $S(2, 1, g)$ holds for $g \geqslant 9$. We have been able to extend this: we show in Theorem 1 below that $S(d, h)$ holds in general for $h = 1$ and $d = 2$ or 3, and that $S(4, 1, g)$ holds if $g \neq 7$. On the other hand, we see that the statement $S(d, h)$ is not true in general: by way of an example we construct a family of curves of genus 5 that are images of curves of type (3, 2) but are not of type (3, 2) themselves. It remains an interesting problem to determine for which values of $d$, $h$ and $g$ it does hold.

Our interest in these questions was greatly increased by the recent results of Faltings. Specifically, Faltings shows that if $A$ is an abelian variety defined over a number field $K$ and $X \subset A$ a subvariety not containing any translates of positive-dimensional sub-abelian varieties of $A$, then the set $X(K)$ of $K$-rational points of $X$ is finite. To apply this, suppose that $C$ is a curve that does not possess any linear series of degree $d$ or less (i.e., is not of type $(d, 0)$). Let $C^{(d)}$ be the $d$-th symmetric product of $C$, and $Pic^d(C)$ the variety of line bundles of degree $d$ on $C$ (this is isomorphic, though noncanonically, to the Jacobian $J(C)$ of $C$). It is then the case that $C^{(d)}$ embeds in $Pic^d(C)$ as the locus $W_d(C)$ of effective line bundles; applying Falting's result we see that if the subvariety $W_d(C) \subset Pic^d(C)$ contains no translates of abelian subvarieties of $Pic^d(C)$, then $C^{(d)}$ has only finitely many points defined over $K$.

We may reexpress this as follows. We consider not only the set $C(K)$ of points of $C$ rational over $K$, but the union $\Gamma_{C,d}(K)$ of all sets $C(L)$ for extensions $L$ of degree $d$ or less over $K$: that is, we set

$$\Gamma_{C,d}(K) = \{p \in C \mid [K(p): K] \leqslant d\}.$$

Since any point of $C$ whose field of definition has degree $d$ over $K$ gives rise to a point of $C^{(d)}$ defined over $K$ it follows in turn that under the hypotheses above – that is, if $C$ admits no map of degree $d$ or less to $\mathbf{P}^1$ and if the subvariety $W_d(C) \subset Pic^d(C)$ contains no translates of subabelian varieties of $Pic^d(C)$, then $C$ has only finitely many points defined over number fields $L$ of degree $d$ or less over $K$, i.e.,

$$\# \Gamma_{C,d}(K) < \infty.$$

Note that conversely if $C$ does admit a map of degree $d$ to $\mathbf{P}^1$ then there will be infinitely many points defined over extension fields of degree $d$ or less over $K$ (the inverse images of $K$-rational points of $\mathbf{P}^1$).

The only problem with this statement is that it seems a priori difficult to determine whether the subvarieties $W_d(C)$ contain abelian subvarieties of $Pic^d(C)$. Certainly one way in which it can happen that $W_d(C)$ contains an abelian subvariety of dimension $h$ is the following: if for some $n$ with $nh \leqslant d$ the curve $C$ admits a map of degree $n$ to a curve $B$ of genus $h$, then the Picard variety $Pic^h(B) \cong J(B)$ maps to the Picard variety $Pic^{nh}(C)$. Since any divisor class of degree $h$ on $B$ is effective, the image will be contained in the locus $W_{nh}(C)$ of effective divisor classes of degree $nh$ on $C$. (On the other hand, it will also be the case if $C$ is just the image of a curve $C'$ that admits a map of degree $n$ to a curve of genus $h$. It is for this reason that the statement $S(n, h)$ above is relevant.) This raises naturally the question of the correctness of the

STATEMENT $A(d, h, g)$: Suppose $C$ is a curve of genus $g$, and for some $d < g$ the locus $W_d(C)$ contains a sub-abelian variety of dimension $h$, then $C$ is the image of a curve $C'$ that admits a map of degree at most $d/h$ to a curve of genus $h$.

Here as in the previous question the answer is yes in some cases: it is apparent when $h = 1$, and we prove in Theorem 1 below that it holds for $h = 2$ and $d \leqslant 4$ if $g \geqslant 6$. At the same time, the answer in general is no – we have a counterexample to this below. It remains a relevant question for which values of $d, g$ and $h$ it may be true. (Note in particular that a positive answer to this question in general would imply that $W_d(C)$ could never contain an abelian subvariety of dimension strictly greater than $d/2$; we know of no counterexample to this assertion.)

The point of introducing the statements $S(d, h, g)$ and $A(d, h, g)$ is that, if true, they combine with Falting's theorem to give a simple and powerful statement about the sets $\Gamma_{C,d}(K)$: if $W_d(C)$ does contain a subtorus, and if the relevant cases of Statements $A(d, g, h)$ and $S(d, g, h)$ hold, then it follows that $C$ is of type $(n, h)$ for some $n$ and $h$ with $nh \leqslant d$. It would then follow that for any curve $C$ defined over a number field $K$, $\#\Gamma_{C,d}(L)$ will be infinite for some extension $L$ of $K$ if and only if $C$ is of type $(n, h)$ for some $n$ and $h$ with $nh \leqslant d$. (Note that one direction is clear: if $\pi: C \to B$ is a map of degree $n$ to a curve of genus $h$ then for some extension $L$ of $K$ the rank of $Pic^h(B)$ over $L$ will be positive and $C$ will similarly have infinitely many points $p$ with $[K(p): L] \leqslant d$.)

Of course, as we have indicated, neither of the statements $S(d, h, g)$ or $A(d, h, g)$ hold in general. Upon closer examination, however, we see that in order to establish the simplest possible statement along these lines we do not need to worry about $S(n, h)$ for all $h$. The reason is the fact that any curve of genus $g$ admits a map of degree $[g/2] + 1$ or less to $\mathbf{P}^1$. Thus, if the Picard variety $Pic^d(C)$ of a curve $C$ contains a translate of an abelian variety coming from a map of degree $n$ to a curve $B$ of genus $h$ with $nh \leqslant d$, and $h \geqslant 2$, then $B$ will be of type $(h, 0)$, and hence $C$ will be of type $(d, 0)$. The crucial case of the general question above about images of coverings of curves of low genus is the case $h = 1$, which is still very much open. It is similarly the case that we need only look at abelian subvarieties of $W_d(C)$ for $d < [g/2] + 1$, in which range there is no counterexample to the statement $A(d, g, h)$ above that any such sub-abelian variety comes from a correspondence with a curve of genus $h$. We may thus make the

CONJECTURE. If $C$ is a curve defined over the number field $K$, then

$$\#\Gamma_{C,d}(L) = \infty \text{ for some finite extension } L/K \Leftrightarrow C \text{ admits a map of degree } d \text{ or less to } \mathbf{P}^1 \text{ or an elliptic curve.}$$

Combining the results mentioned above, we have the main result of this paper: the

THEOREM 1. *The conjecture above holds when* $d=2$ *or* 3, *and when* $d=4$ *provided the genus of* $C$ *is not* 7.

REMARK. (1) Results related to the above have been obtained by many people, including Gross–Rohrlich [5], Hindry [7], Mazur [8] and others. The conjecture is also related to the generalized Mordell conjectures of Lang and Vojta (for example, in the case $d=2$ the Lang–Vojta conjectures say that a nonhyperelliptic curve possessing infinitely many points of degree 2 must admit a correspondence of bidegree $(2, m)$ with an elliptic curve, though they do not specify $m$).

The case $d=2$ was proved before by Harris and Silverman [6]. We give a slightly strengthened version here (see Theorem 3). Using methods similar to theirs, one can show the following amusing result: if our $C'$ maps with degree 2 to a *hyperelliptic* curve of genus $h$, then $C$ maps with degree 2 to a hyperelliptic (or rational) curve of genus at most $h$, with one exception which we cannot prove: $h=2$ and $g=3$.

(2) Vojta tries to attack this problem from another point of view, in [11, 12]. He assumes the existence of a map $f: C \to \mathbf{P}^1$ of low degree, and deduces that all but finitely many points of low degree over $K$ relate to this map: $K(p) \neq K(f(p))$. In general, the existence of such a map $f$ rules out the possibility of another map of low degree to a curve of low genus, assuming the genus of $C$ is large. In view of this, it turns out that in case of points of degree 2 and 3 his results give the same bounds as ours. In particular, on a trigonal curve over $K$ of genus at least 8, all but finitely many point of degree 3 over $K$ on $C$ map to rational points on $\mathbf{P}^1$ (this is sharp simply because there are curves of genus 7 which are trigonal and trielliptic). It would be interesting to have results similar to Vojta's for maps to an elliptic curve instead of $\mathbf{P}^1$.

## 2. Preliminary lemmas

Let $A$ be a complex abelian variety of dimension $a \geqslant 1$, and let $A \hookrightarrow W_d(C)$ be an embedding. Here $C$ is a smooth complex algebraic curve, and $W_d(C)$ is the variety of effective line bundles of degree $d$ over $C$.

We assume this embedding is minimal, that is, the line bundles given by $A$ do not have a common divisor: $A \not\subset p + W_{d-1}(C)\ \forall p \in C$. We also assume that $A \not\subset \Delta$, where $\Delta$ is the image of the big diagonal of $C^{(d)}$ in $W_d(C)$.

Note that $A$ is a coset of a subgroup in $Pic(C)$. If we write

$$A_k = \{\alpha_1 + \cdots + \alpha_k \mid \alpha_i \in A\}$$

then $A_k$ is a coset of the same subgroup, and thus $A_k \simeq A$.

For $\alpha \in A_k$ we write $L_\alpha$ for the associated line bundle and $D_\alpha$ for any effective divisor such that $\mathcal{O}(D_\alpha) = L_\alpha$. For any $\alpha \in Pic(C)$ we write $r(\alpha) = h^0(L_\alpha) - 1$.

The ideas of the proof of the main theorem are as follows:

1. We produce families of maps to projective spaces by taking sections of $L_\alpha$ for $\alpha \in A_2$ (Lemma 1).
2. In case the general such map is not birational onto the image, we reduce our problem to lower genus and an appropriately lower $d$. In the cases of our theorems, we actually get the required maps (Lemmas 2 and 3).
3. When these maps are birational, we use an estimate similar to Castelnuovo's bound, only stronger, to show that $g(C) \leqslant O(d^2/a)$.

LEMMA 1. *For any $\alpha \in A_2$ we have $r(\alpha) \geqslant a$.*

*Proof.* Let $\pi_d: C^{(d)} \to W_d(C)$ be the natural map, and let $\tilde{A} \subset C^{(d)}$ be the proper transform of $A$ under this map. Recall that the symmetrization map $C^{(d)} \times C^{(d)} \to C^{(2d)}$ is finite. Therefore $\tilde{A} \times \tilde{A} \to \tilde{A}_2$ is finite, where $\tilde{A}_2$ is the proper transform of $A_2$. So $\dim \tilde{A}_2 \geqslant 2a$, and the fibers of $\pi_2|_{\tilde{A}_2}: \tilde{A}_2 \to A_2$ have dimension at least $a$. Abel's theorem says that $r(\alpha) \geqslant a$ for all $\alpha \in A_2$. □

Note that the linear systems $|D_\alpha|$ obtained above are base point free. Special care is needed in case $a = 1$:

LEMMA 2. *Assume $a = 1$.*

1. *If the general point $p \in C$ belongs to exactly one $D_\alpha$ with $\alpha \in A$, such that $r(\alpha) = 0$, then there is a map of degree $d$ from $C$ to the elliptic curve $A$.*
2. *Assume that for the general $\alpha \in A_2$ we have $r(\alpha) = 1$. Let $\phi_\alpha: C \to \mathbf{P}^1$ be the map defined by the global sections of $L_\alpha$. Then $\phi_\alpha$ factors through a $d$-to-1 map to $A$.*

*Proof.* (1) is formal, and may be shown as follows: let $F: C \times C^{(d-1)} \to W_d(C)$ be the natural map, and let $C'$ be the normalization of the part of $F^{-1}(A)$ dominating $A$. Our minimality conditions mean that $C'$ is exactly a $d$-sheeted cover of $A$. On the other hand, the projection onto the first factor $\pi_1: C \times C^{(d-1)} \to C$ induces a map from $C'$ to $C$, the degree of which is the number of times a general point of $C$ belongs to a divisor $D_\alpha$. If this degree is 1, then $C \simeq C'$ and therefore $C$ admits a map to $A$ of degree $d$.

For (2), note that for any $\beta \in A$ we have $\alpha - \beta \in A$. Therefore $\alpha - \beta$ is effective, and thus $D_\beta$ imposes one condition on the linear system $|D_\alpha|$, so $D_\beta$ lies in a fiber of $\phi_\alpha$.

If the general fiber of $\phi_\alpha$ is written uniquely as a sum $D_\beta + D_{\beta'}$ where $\beta + \beta' = \alpha$, we are in case (1). Otherwise, for every $\alpha$ with $r(\alpha) = 1$ we have $\infty^1$ equations:

$$D_\beta + D_{\beta'} = D_\gamma + D_{\gamma'} \quad (\beta + \beta' = \gamma + \gamma' = \alpha). \tag{1}$$

Fixing $\beta$ and changing $\alpha$ (and thus $\beta'$) we see that since $D_\beta \cap D_\gamma \neq \emptyset$ the divisors $D_\gamma$ have a common divisor. Similarly for $D_{\gamma'}$. At least one of the two moves, and so the divisors of $A$ would have a common divisor, contradicting our assumption. □

The following lemma, together with the previous one, will establish the cases when the general $\phi_\alpha$ is not birational.

LEMMA 3. *Assume $a \geqslant 1$ and $r(\alpha) > 1$ for all $\alpha \in A_2$. If $\phi_\alpha: C \to \mathbf{P}^{r(\alpha)}$ is not birational for general $\alpha$, then either $A \subset W^1_{d'}(C)$ with $d' \leqslant d$, or $\phi_\alpha$ factors as:*

$$C \xrightarrow{\ \rho\ } C' \xrightarrow{\ \bar{\phi}_\alpha\ } \mathbf{P}^{r(\alpha)}$$

*and there is an imbedding $A \subset W_{d'}(C')$ where $d' = d/\deg \rho = \deg \bar{\phi}_\alpha$.*

*Proof.* Recall that the set of maps from $C$ to curves of positive genus (up to automorphisms) is discrete. If the general $\phi_\alpha$ map to rational curves of degree $m$, then their images must be rational normal curves (the linear series in question are complete) and we get an imbedding $A \subset W^1_{d'}(C)$, where $d' = d/m$. Otherwise, there is a generic image curve for the $\phi_\alpha$, call it $C'$. Let $p \in C'$ and let $q_1$, $q_2 \in \rho^{-1}(p)$. Suppose $q_1 \in D_\beta$ for some $\beta \in A$. We claim that $q_2 \in D_\beta$, which gives the lemma. If we let $\alpha$ vary in $A_2$ and set $\beta' = \alpha - \beta$, then $D_\beta + D_{\beta'}$ is a hyperplane section of $C \subset \mathbf{P}^r$ containing $q_1$. Therefore, since $\phi_\alpha$ factors through $C'$, also $q_2 \in D_\beta + D_{\beta'}$. But the divisors $D_{\beta'}$ do not have a common divisor, therefore $q_2 \in D_\beta$. This means that $A$ is a pull-back of an abelian variety from $W_{d'}(C')$. Again, since the linear series are complete, this pull-back is an isomorphism. □

We use the following classical lemma:

LEMMA 4. *Let $C \to \mathbf{P}^r$ be birational onto its image. Then for every $s < r$ there do not exist $\infty^s$ divisors of degree $s+1$, each spanning an $s-1$-plane.*

*Proof.* By a projection from a generic secant we reduce to the fact that a plane curve has finitely many singularities. □

Let $r_k = \min\{r(\alpha) \mid \alpha \in A_k\}$, that is, the general dimension of the complete linear series $|D_\alpha|$, $\alpha \in A_k$.

LEMMA 5. *Suppose $r_2 = a$. Then $\phi_\alpha$ is not birational.*

*Proof.* In case $a = 1$ this is trivial. Otherwise, the fibers of $\pi_2$ as in Lemma 1 are in general projective spaces of dimension $a$, which are surjected by the quotient of $A$ by an involution. In dimension $a > 1$ these are never rational. Therefore each divisor of $D_\alpha$ is represented in at least two ways as the sum of two divisors from $A$, and we get $\infty^a$ equations as in (1). Fixing $D_\beta$ again and letting $D_{\beta'}$ vary, and vice versa, we see that $A$ is generated by two subvarieties

$X_1 \subset W_{d_1}(C)$ and $X_2 \subset W_{d_2}(C)$, where $\dim(X_1) = a_1 > 0$ and $\dim(X_2) = a_2 > 0$ and $a_1 + a_2 \geqslant a$. In the target space of $\phi_\alpha$ we see that we get $\infty^{a_1}$ divisors, given by $X_1$, each spanning only an $a_1 - 1$-plane. By Lemma 4 the map is not birational onto the image. □

The following lemma uses the same kind of information for the next possible dimension:

LEMMA 6. *Suppose $r_2 = a + 1$, and suppose the general $\phi_\alpha$ is birational. Then for general points $p_1, \dots, p_a$ in $C$, and any $D_\beta$ with $\beta \in A$ such that $p_i < D_\beta$ there is another divisor $D_{\beta'}$ with $\beta' \in A$ so that* $\gcd(D_\beta, D_{\beta'}) = p_1 + \cdots + p_a$.

*Proof.* Now the fibre of $\pi_2$ is in general a projective space of dimension $a+1$, and the quotient of $A$ by an involution maps to it by a finite map. If the image is a linear space, we have a linear series to which we may apply the previous lemma. Otherwise, the image is of higher degree, in which case the line defined by general $a$ points of $C$ intersects this image several times. This means that the divisor $p_1 + \cdots + p_a$ lies on several of the hyperplanes defined by $A$, and therefore is in general contained in several divisors of $A$. If they all contain an extra point, we get $\infty^a$ intersections of hyperplanes containing $a+1$ points, contradicting Lemma 4. □

## 3. Number of conditions

We are left with the cases when $r(\alpha) > 1$ for all $\alpha \in A_2$ and $\phi_\alpha$ birational for the general $\alpha$. We continue and derive a strengthened Castelnuovo type bound on the genus of $C$. The argument is similar to the original argument of Castelnuovo's bound (see [2], Chapt. 3) and the generalized one by Accola [3]. The idea is to estimate the number of conditions a divisor $D_\beta$ for $\beta \in A$ imposes on the sections of a general $k$-fold sum $\alpha \in A_k$. The fact that we are working with cosets of subgroups plays an important role.

First, some observations. Since $\{D_\beta \mid \beta \in A\}$ have no common divisor, for all $p \in C$ the general $D_\beta$ does not contain $p$. As an immediate result we get:

LEMMA 7. $r_{k+1} - r_k \geqslant r_{k-1}$.

*Proof.* Let $\alpha \in A_{k+1}$ be a general point, and let $D$ be a general divisor coming from $A$. Let $D_\gamma$, $\gamma \in A$ be a general divisor, such that $\gcd(D, D_\gamma) = 0$. By the generality assumption, there are $r_k - r_{k-1}$ points in $D$ which impose independent conditions on sections of $L_{\alpha-\gamma}$. Multiplying by the canonical section of $\mathcal{O}(D_\gamma)$, which does not vanish on any point of $D$, certainly keeps this property. Hence the lemma. □

LEMMA 8.

1. *If for general $\alpha \in A_2$ the map $\phi_\alpha$ is birational onto its image then $r_3 \geqslant 2r_2$ and*

$r_{k+2} - r_k \geqslant \min(r_k - r_{k-2} + r_2, 2d)$ *for any* $k \geqslant 2$.
2. *If* $r_2 = a+1$ *then* $r_{k+1} - r_k \geqslant \min(ka+1, d)$.

*Proof.* The fact that $r_3 \geqslant 2r_2$ follows immediately from Lemma 7.

Let $D_\alpha = p_1 + \cdots + p_{2d}$ be a general divisor. Now, by the uniform position lemma (see [2]) if we take a general $\alpha' \in A_2$ then there is a divisor $D_{\alpha'}$ such that the common divisor with $D_\alpha$ is $p_1 + \cdots + p_{r_2}$.

Also, for a general $\gamma \in A_k$ we have a divisor $D_\gamma$ so that $\gcd(D_\gamma, D_\alpha) = p_{r_2+1} + \cdots + p_{r_2+r_k-r_{k-2}-1}$. But the order of the chosen points is unimportant. Therefore, for the general $\delta = \gamma + \alpha' \in A_{k+2}$ we have that $\alpha$ imposes at least $r_2 + r_k - r_{k-2}$ conditions on $|D_\delta|$.

For the second claim, notice that by Lemma 6, if $D_\beta = q_1 + \cdots + q_d$ is a general divisor corresponding to points of $A$, then there are divisors $D_{\beta_i}$ so that $gcd(D_\beta, D_{\beta_i}) = q_{(i-1)a+1} + \cdots + q_{ia}$. Summing up $k$ of these, and using Lemma 7 for an extra divisor, we get the inequality in (2). □

REMARK. Notice that we didn't make use, in the proof of first part of the lemma, of the fact that we have $\infty^a$ ways to choose $\alpha'$. For our results this turns out to be sufficient.

COROLLARY 1. *In the case of the lemma, we have* $r_3 \geqslant \min(2a+3, a+1+d)$, *and* $r_4 \geqslant \min(3a+6, r_3+d)$.

As a byproduct we get a theorem:

THEOREM 2. *Let* $A \subset W_d(C)$ *be an abelian variety of positive dimension* $a$. *Assume that for the general* $\alpha \in A_2$ *the map* $\phi_\alpha$ *is birational onto its image. Then*

$$g(C) \leqslant \binom{d}{2} + 1.$$

*Proof.* We know that $r_2 \geqslant 2$. If equality holds, we have $r_d \geqslant 2 + 3 + \cdots + d = \binom{d+1}{2} - 1$. But for $\alpha \in A_d$, $\deg\ \alpha = d^2$, so $2r(\alpha) > \deg \alpha$ for all $\alpha$, and by Clifford's theorem (see [2]) $\alpha$ is non-special, that is, $g(C) = \deg \alpha - r(\alpha) \leqslant \binom{d}{2} + 1$. Similarly, if $r_2 \geqslant 3$ one proves by induction, using Lemmas 8 and 7, that $r_k \geqslant \binom{k+1}{2} - 1$, and continues as before. □

## 4. Statement and proof of main theorems

THEOREM 3. *If* $A \subset W_2(C)$, *then either* $C$ *has genus at most* 2, *or* $C$ *is bielliptic. If* $g(C) > 3$ *then* $C$ *is not hyperelliptic.*

*Proof.* Lemma 2 settles the theorem when $r(\alpha) = 1$ for general $\alpha \in A_2$. If

$r(\alpha) > 1$, we have a family of $g_4^2$, which does not exist unless $g(C) \leqslant 2$, because for genus 3 a $g_4^2$ is the canonical series, and for higher genera it has to be twice a unique hyperelliptic series. For the last statement, a bi-elliptic hyperelliptic curve is of type $(2, 4)$ on a smooth quadric, and therefore of genus at most 3. □

THEOREM 4. *If $A \subset W_3(C)$ and $g(C) \geqslant 5$ then $C$ admits a map of degree at most 3 to a curve of genus 1. If* $\dim A \geqslant 2$ *then the genus of $C$ is at most 3. If $g(C) \geqslant 8$ then $C$ does not admit a $g_3^1$.*

*Proof.* Lemmas 2 and 3 settle the theorem for $\phi_\alpha$ not birational for general $\alpha \in A_2$. Corollary 1 shows that any other case has a $g_9^5$, and by Clifford's theorem ([2]) has genus at most 4, but these have a $g_3^1$ [2]. Similarly, if we take $a = 2$ we see that $g \leqslant 3$. □

THEOREM 5. *If $A \subset W_4(C)$ and $g(C) \geqslant 8$ then either $C$ admits a map of degree at most 4 to a curve of genus 1, or a map of degree 2 to a curve of genus 2. If* $\dim A \geqslant 2$ *and $g(C) \geqslant 6$ then $C$ is a double cover of a curve of genus 2.*

*Proof.* Again we may assume $\phi_\alpha$ is birational for general $\alpha \in A_2$. Corollary 1 and Clifford's theorem show that $g(C) \leqslant 7$.

Curves of genus at most 6 have a $g_4^1$. For the last statement, we see that if $a > 1$ then in fact $g \leqslant 5$. In the next section we show that there is a counterexample with $g = 5$. □

## 5. An example

We construct a 6 dimensional family of curves of genus 5, all having a curve of genus 2 in $W_3$, and none of them admits a map of degree 2 or 3 to curves of genus 0, 1 or 2. As a byproduct, we explain how a curve of genus 5 can possess an abelian surface in $W_4$ without being a double cover of a curve of genus 2. The construction is a special case of the tetragonal construction for Prym varieties, as in [4].

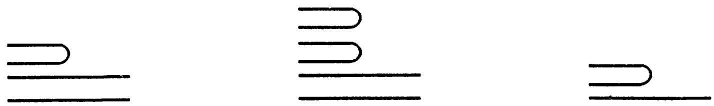
Let $f: C_2 \to \mathbf{P}^1$ be a map of degree 4, from a curve $C_2$ of genus 2 to $\mathbf{P}^1$. Assume $f$ has only simple ramifications.

Let $C' = \overline{C_2 \times_{\mathbf{P}^1} C_2 - \Delta}$ be the curve of pairs of distinct points in the fibres over $\mathbf{P}^1$. Let $C_5$ be the quotient of $C'$ by the symmetrization involution: $C_5 = \overline{(C_2)^{(2)}_{\mathbf{P}^1} - \Delta}$. $C_5$ is our curve of genus 5. Note that $C_5$ admits an involution that assigns to an unordered pair of distinct elements in a fiber, the residual pair in that fiber. The quotient is a curve of genus 3, $C_3$. We have the following commutative diagram:

$$\begin{array}{ccccc} C' & \xrightarrow{h} & C_5 & \xrightarrow{\pi} & C_3 \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \rho} \\ C_2 & & \xrightarrow{\quad f \quad} & & \mathbf{P}^1 \end{array}$$

where $\pi$ is unramified of degree 2, $\rho$ of degree 3 with 10 ramifications, and $f$ of degree 4 with 10 ramifications.

The corresponding ramification behavior of $C_2$, $C_5$ and $C_3$ over $\mathbf{P}^1$ is sketched below:



ramification in $C_2$ ramification in $C_5$ ramification in $C_3$

From this construction we see that the curves $C_5$ vary in at most 6 parameters: in fact the curves $C_3$ have only so many moduli. We show that the construction may be reversed, and that we really get 6 parameters.

Let $C_3$ be a nonhyperelliptic curve of genus 3. Let $\rho$ be any $g_3^1$ on the curve, with simple ramifications. Let $\pi$: $C_5 \to C_3$ be any connected unramified double cover. One checks that the monodromy of $C_5$ over $\mathbf{P}^1$ via the map $\rho \circ \pi$ is $S_4$. Now take the subvariety $D$ of the triple relative symmetric power $(C_5)^{(3)}_{\mathbf{P}^1}$ that does not map into a diagonal of $(C_3)^{(3)}_{\mathbf{P}^1}$. This subvariety is composed of two isomorphic components of genus 2, called $C_2$.

THEOREM 6. *The general $C_5$ in this family does not admit a map of degree at most* 3 *to a curve of genus at most* 2.

Let $\Lambda$ be the variety inside $\mathcal{M}_5$ described by our curves of genus 5, and let $D_{d,h}$ be the subset of $\Lambda$ of those curves that admit a map of degree d to a curve of genus h. We need to show: $\Lambda \neq \bigcup_{d \leqslant 3, h \leqslant 2} D_{d,h}$.

LEMMA 9. $\dim D_{3,2} \leqslant 5$.

*Proof.* The dimension of the variety of curves of genus 5 admitting a map of degree 3 to a curve of genus 2 is 5. □

LEMMA 10. $\dim D_{2,h} \leqslant 5$.

*Proof.* If an involution of $C_5$ commutes with $\pi$ then $C_3$ has automorphisms, and the dimension of such $C_3$ is 5. If they do not commute, the composition of the two involutions is of some order bigger than 2, and the dimension of the variety of curves of genus 5 admitting such an automorphism is again not more than 5 [2]. □

LEMMA 11. $\dim D_{3,0} \leqslant 5$.

REMARK. In fact, one can show that $D_{3,0}$ is empty.

*Proof.* We prove by specialization. Let $C_3$ be a nonhyperelliptic, bielliptic curve of genus 3, and $p$: $C_3 \to E$ the bielliptic map. Let $E' \to E$ be a two sheeted map of elliptic curves. Then $E' \times_E C_3$ is a bielliptic curve of genus 5 in our family. □

Now, a bielliptic curve of genus 5 does not admit a $g_3^1$. In fact, if $C_5$ has a map of degree 2 or 3 to $\mathbf{P}^1$, then as a cycle in $E' \times \mathbf{P}^1$ we have

$$[C_5]^2 = \text{degree of ramification} = 12 \text{ or } 14.$$

If $H = \pi_1^{-1}(p) + \pi_2^{-1}(q)$ is an ample divisor formed by fibers both ways, we have $H^2 = 2$ and $H.[C_5] = 4$ or 5. We get

$$(H.[C_5])^2 \leqslant (H.H)([C_5].[C_5])$$

which is a contradiction to the Hodge index theorem. □

LEMMA 12. *$h_* g^* Jac(C_2) \cap \pi^* Jac(C_3)$ is finite, and their sum is the whole $Jac(C_5)$, for general $C_5$. In other words, the two jacobians give subabelian varieties which are complementary up to isogeny.*

*Proof.* If $q \in C_2$ one checks explicitly that $\pi_* h_* g^*(q) = \rho^* f_*(q)$ (in fact, the big square in the commutative diagram is the normalization of a fiber square). This does not depend on $q$ because $f_*(q) \sim f^*(q_1)$ on $\mathbf{P}^1$.

If $C_5$ is general from $\Lambda$, then $\dim h_* g^* Jac(C_2) > 0$, otherwise $C_5$ has a $g_3^1$ (see Lemma 2).

If $C_2$ is of general moduli, it does not map to an elliptic curve, in which case $\dim h_* g^* Jac(C_2) \neq 1$. By semicontinuity, the dimension is 2 for general $C_2$. □

LEMMA 13. $\dim D_{d,1} \leqslant 5$.

*Proof.* In fact, if $C_5$ admits a map to an elliptic curve, then this elliptic curve maps to $Jac(C_5)$ by a nonconstant map. Projecting to $Jac(C_3)$ and to $Jac(C_2)$ we see that at least one of these jacobians is nonsimple. This again bounds the dimension of either $C_2$ or $C_3$. □

This finishes the verification of our theorem. □

COROLLARY 2. *There are curves $C_5$ of genus 5 such that $W_4(C_5)$ contains an abelian surface, but the curve $C_5$ does not map to any curve of genus 2.*

*Proof.* The Prym variety of the map $C_5 \to C_3$ has a translate which lies in $W_4$, namely the odd component of the inverse image of $K_{C_3}$ (see [9]). □

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