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0. Introduction

(0.1) This work grew out of an attempt to find a local analogue of a result of Griffiths. To state it we need some notation. Let Y be a hypersurface in an analytic space X. Then

$$\Omega_X^{\cdot}(*Y) = \bigcup_{k \geqslant 0} \Omega_X^{\cdot}(kY)$$

will denote the complex of sheaves of meromorphic differentials on X with poles of arbitrary order along Y.

DEFINITION. The *pole order filtration* $P^{\cdot}$ on $\Omega_X^{\cdot}(*Y)$ is given by

$$P^k\Omega_X^p(*Y) = \Omega_X^p((p-k+1)Y).$$

THEOREM [Gri]. *Let V be a smooth hypersurface in* $\mathbf{P}^n$. *Then*

(i) $H^n(\mathbf{P}^n - V; \mathbf{C}) \cong H^n_{DR}(\mathbf{P}^n; \Omega_{\mathbf{P}^n}^{\cdot}(*V))$,

(ii) *Under the isomorphism in* (i) *the Hodge filtration $F^{\cdot}$ on the* (*pure*) *Hodge structure $H^n(\mathbf{P}^n - V) \cong H^{n-1}(V)_{\text{prim}}$ is induced from the pole order filtration $P^{\cdot}$ on $\Omega_{\mathbf{P}}^{\cdot}(*V)$.*

The first part is a special case of the algebraic De Rham theorem of Grothendieck [Gro]:

If Y is a hypersurface in an analytic space X and the complement $U = X - Y$ is smooth, then

$$H^{\cdot}(U; \mathbf{C}) \cong \mathbf{H}^{\cdot}(X; \Omega_X^{\cdot}(*Y)),$$

and the hypercohomology can be replaced by the ordinary De Rham cohomology when Y is sufficiently positive.

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Part (ii) was extended by Deligne [DelI] to assert that the Hodge filtration on $H^{\cdot}(X - Y)$ is induced from the pole order filtration on $\Omega_X^{\cdot}(*Y)$ when X is a complete non-singular variety and Y is a divisor with normal crossings.

(0.2) Consider now an isolated hypersurface singularity $(Y, \{x_0\})$. Letting X denote a sufficiently small contractible Stein neighborhood of x_0 in the ambient space, we have a local version of Grothendieck's theorem:

$$H^{\cdot}(X - Y) \cong (\mathscr{H}^{\cdot}\Omega_X^{\cdot}(*Y))_{x_0}.$$

Here $\mathscr{H}^{\cdot}$ denotes De Rham cohomology sheaves, and $X - Y$ is what we call the *link* of the singularity. Several other objects going under this name have essentially the same cohomology, up to a shift of indices. In particular,

$$H^i(X - Y) \cong H^i_{\{x_0\}}(Y), \qquad i > 0.$$

The latter group can be equipped with a mixed Hodge structure by the original construction of Deligne [DelIII]. Thus the cohomology groups of the link carry the pole order and the Hodge filtrations. It is natural to ask whether there is a local analogue to the result of Griffiths, i.e. whether $P^{\cdot} = F^{\cdot}$ on the cohomology of the link. The connection with the situation studied by Griffiths is provided by homogeneous singularities, i.e. cones on smooth projective hypersurfaces.

(0.3) Our results are listed below in the order of their predictability. Assuming $\dim Y = n$, we restrict to the only interesting cohomology groups of the link, $H^n(X - Y)$ and $H^{n+1}(X - Y)$.

THEOREM.

(a) *Under the isomorphism* $H^{\cdot}(X - Y) \cong (\mathscr{H}^{\cdot}\Omega_X^{\cdot}(*Y))_{x_0}$ *we have* $P^{\cdot} = F^{\cdot}$ *on* $H^n(X - Y)$ *and* $H^{n+1}(X - Y)$ *in the quasi-homogeneous case.*
(b) *In general, we only have* $P^{\cdot} \subseteq F^{\cdot}$ *on* $H^n(X - Y)$ *and* $F^{\cdot} \subseteq P^{\cdot}$ *on* $H^{n+1}(X - Y)$.
(c) *On* $H^{n+1}(X - Y)$ $P^{\cdot}$ *is induced from the* third *filtration* $G^{\cdot}$ on the cohomology of the Milnor fiber.

REMARKS. (1) The *third* filtration $G^{\cdot}$ was introduced by Varchenko [V1,2] in conjunction with the Bernstein polynomial. The definition is given in (3.1).

(2) Concerning (b), in Section 4 we explain an example, suggested by Morihiko Saito, with $P^{\cdot} \neq F^{\cdot}$ on both cohomology groups.

(3) After this paper was completed the author received A. Dimca's preprint "Differential Forms and Hypersurface Singularities," in which he also obtains $P^{\cdot} \supseteq F^{\cdot}$ on $H^{n+1}(X - Y)$ and asks about the relationship between $P^{\cdot}$ and $F^{\cdot}$ on $H^n(X - Y)$. His methods are quite different from ours.

Some Notation

$H^k_{DR}(X; L^\cdot) = H^k(\Gamma(X; L^\cdot))$	De Rham cohomology of the sheaf complex $L^\cdot$
$\mathscr{H}^\cdot$	De Rham cohomology sheaves
$\mathbf{H}^\cdot$	hypercohomology

1. The link of an isolated singularity and its MHS

(1.1). Let Y be a contractible Stein space of dimension n and x_0 its only singular point. The complement $Y - \{x_0\}$ is known as the *link* of the singularity. Often, however, other topological objects are also called the link if they have the same cohomology groups. We will be lax about the terminology.

The first thing to notice is that because Y is contractible, the long exact cohomology sequence of the couple $(Y, Y - \{x_0\})$ implies

$$H^i(Y - \{x_0\}) \cong H^{i+1}_{\{x_0\}}(Y) (\stackrel{\text{def}}{=} H^{i+1}(Y, Y - \{x_0\})) \qquad \forall i > 0.$$

(The case of $i = 0$ will not interest us, and so we refrain from using augmented cohomology groups.) But Y may be extended to a *complete variety* Y', and by excision $H^\cdot_{\{x_0\}}(Y) \cong H^\cdot_{\{x_0\}}(Y')$. By the original construction of Deligne the latter is equipped with a canonical and functorial MHS. Thus the cohomology of the link carries a MHS independent of the choice of the representative $(Y, \{x_0\})$.

Our goal is to describe this MHS for a hypersurface singularity. Let us mention the relevant part of Milnor's topological analysis of the situation [Mi]. What we called Y will now be denoted X_0—the set cut out in a contractible Stein neighborhood X of $x_0 = 0 \in \mathbf{C}^{n+1}$ by $X_0 = \{f = 0\}$, with f—an analytic function on X taking values in a disc Δ centered at 0 in $\mathbf{C}$. For future use we introduce the *Milnor fiber* $X_t = f^{-1}(t)$, $t \in \Delta^* = \Delta - \{0\}$ and mention that, according to Milnor, X and Δ can be selected in such a way that $f: X \to \Delta$ is onto and $X^* = X - X_0$ fibers over Δ^* with all fibers X_t diffeomorphic to each other (the "Milnor fibration"). Milnor also shows that if B^{2n+2} is a sufficiently small ball in X centered at x_0, S^{2n+1}—its boundary, and $K = X_0 \cap S^{2n+1}$, then (X, X_0) is homeomorphic to the cone on (S^{2n+1}, K) with the vertex x_0. Here K is a (real, differentiable) *compact* manifold of dimension $2n - 1$, homotopy equivalent to $Y - \{x_0\}$. In fact, we have

$$H_i(Y - \{x_0\}) \overset{\text{homotopy equivalence}}{\cong} H_i(K) \overset{\text{Poincaré duality}}{\cong} H^{2n-1-i}(K)$$

$$\overset{\text{Alexander duality}}{\cong} H_{i+1}(S^{2n+1} - K) \overset{\text{homotopy equivalence}}{\cong} H_{i+1}(X^*)$$

for i in our range (as we shall see, only $i = n - 1$ and n are really interesting here). Since we usually work with homology and cohomology over a field, the Universal Coefficient Theorem provides the dual isomorphism $H^{i+1}(X^*) \cong H^i(Y - \{x_0\})$. In view of these isomorphisms X^* (and K, and even $S^{2n+1} - K$) can be called the link of the singularity (X_0, x_0).

(1.2) Let us establish a connection between the cohomology of the link and that of the Milnor fiber. First of all, we have the Wang sequence in topology:

$$\cdots \to H^k(X^*) \to H^k(X_t) \xrightarrow{T-I} H^k(X_t) \to H^{k+1}(X^*) \to \cdots.$$

Here T denotes the monodromy operator. By the key result of Milnor, X_t is homotopy equivalent to a bouquet of n-spheres, i.e. $H^k(X_t) \neq 0$ only for $k = 0$ and $k = n$. Consequently, the only interesting cohomology groups of the link are $H^n(X^*)$ and $H^{n+1}(X^*)$, which are the kernel and the cokernel, respectively, of the variation map $T - I\colon H^n(X_t) \to H^n(X_t)$.

Let X_∞ denote the *canonical Milnor fiber*, i.e. the total space of the pullback of the Milnor fibration to the universal cover of Δ^*. As each X_t is homotopy equivalent to X_∞, there is a canonical isomorphism between $H^n(X_t)$ and $H^n(X_\infty)$. The latter group carries the MHS first defined by Steenbrink [St]. However, the map $T - I$ need not be a morphism of Hodge structures. To correct this problem we recall the Jordan-Chevalley decomposition $T = T_s T_u$. Then

$$\begin{aligned} H^n(X^*) &= \ker(T - I) = H^n(X_\infty)^T \text{ (the invariants of } T) \\ &= ((H^n(X_\infty))^{T_s})^{T_u} = (H^n(X_\infty)_1)^{T_u}, \end{aligned}$$

where the subscript refers to the eigenspace of T_s with the eigenvalue one, as usual. Thus

$$H^n(X^*) = \ker(T_u - I)|_{H^n(X_\infty)_1};$$

similarly,

$$H^{n+1}(X^*) = \operatorname{coker}(T_u - I)|_{H^n(X_\infty)_1}.$$

But $T_u - I$ has the same kernel and cokernel as $N = \log T_u$, i.e. we end up with the exact sequence

$$(*) \quad 0 \to H^n(X^*) \to H^n(X_\infty)_1 \xrightarrow{N} H^n(X_\infty)_1 \to H^{n+1}(X^*) \to 0,$$

where the middle map is known to be an MHS endomorphism of type $(-1, -1)$ (cf. [St]).

(1.3) We claim that $(*)$ is a MHS exact sequence with the outlying terms carrying the MHS discussed in (1.1). Indeed, in [Du] Durfee constructed the "Mayer-Vietoris" mixed Hodge complex (MHC) computing the MHS on $H^\cdot(X_0 - x_0)$ as in (1.1). Navarro Aznar [NA] introduced another MHC for the same cohomology, the "localized" log-complex. In [D-H] the two MHC are shown to be equivalent up to a Tate twist. Finally, in (14.12) of [NA] Navarro Aznar gives a short exact sequence of MHC which

(a) induces the long exact hypercohomology sequence yielding $(*)$,
(b) presents the MHC in question as equivalent to the mapping cone of the MHC morphism inducing N on cohomology.

Thus the same sequence $(*)$ is also induced by the short exact sequence for the mapping cone of a MHC morphism. It is, therefore, an exact MHS sequence by (2.2) in [Du].

2. The pole order filtration and the Gauss-Manin system

(2.1) The reason for bringing out X^* is that its cohomology can be computed from the meromorphic De Rham complex on X, for X sufficiently small, by the local algebraic De Rham theorem of Grothendieck [Gro]:

$$H^\cdot(X^*; \mathbf{C}) \cong H^\cdot_{DR}(X; \Omega^\cdot_X(*X_0)) = \mathscr{H}^\cdot \Lambda^\cdot_{x_0},$$

where $\Lambda^\cdot = \Omega^\cdot_X(*X_0)$. As $\Lambda^\cdot$ is filtered by the order of the pole $P^\cdot$ (see Introduction), we have the induced filtration

$$P^\cdot H^\cdot(X^*; \mathbf{C}) = P^\cdot \mathscr{H}^\cdot \Lambda^\cdot_{x_0} = \mathrm{image}\{\mathscr{H}^\cdot P^\cdot \Lambda^\cdot_{x_0} \to \mathscr{H}^\cdot \Lambda^\cdot_{x_0}\}.$$

REMARK. Grothendieck's theorem says more:

$$\mathbf{H}^\cdot(X; \Omega^\cdot_X(*X_0)) \cong H^\cdot_{DR}(X; \Omega^\cdot_X(*X_0)),$$

i.e. the first hypercohomology exact sequence degenerates here. Now,

$$P^\cdot \mathbf{H}^\cdot(X; \Lambda^\cdot) \overset{\text{def}}{=} \mathrm{image}\{\mathbf{H}^\cdot(X; P^\cdot\Lambda^\cdot) \to \mathbf{H}^\cdot(X; \Lambda^\cdot)\},$$

and potentially this may be different from $P^\cdot \mathrm{H}^\cdot_{DR}(X; \Lambda^\cdot)$. However, X is assumed to be Stein, and all $P^k\Lambda^p$ are coherent (unlike Λ^p themselves, which are only quasi-coherent). Hence Cartan's Theorem B applies to insure the degeneration of the first hypercohomology spectral sequence for $P^k\Lambda^\cdot$, i.e.

$$\mathbf{H}^\cdot(X; P^k\Lambda^\cdot) = H^\cdot_{DR}(X; P^k\Lambda^\cdot),$$

and thus the isomorphism above is actually $P^\cdot$-filtered.

(2.2) We will prove Theorem (0.3) by bringing the pole order filtration into the picture in the context of the exact sequence (*). Let Γ be the graph of f, i.e. the image of the *smooth* embedding $j: X \to X \times \Delta$ with $j(x) = (x, f(x))$. Let $i: X \to X \times \Delta$ be the embedding identifying X with $X \times \{0\}$. Clearly $(X \times \{0\}) \cap \Gamma = X_0$. Here is a "picture" of $X \times \Delta$:

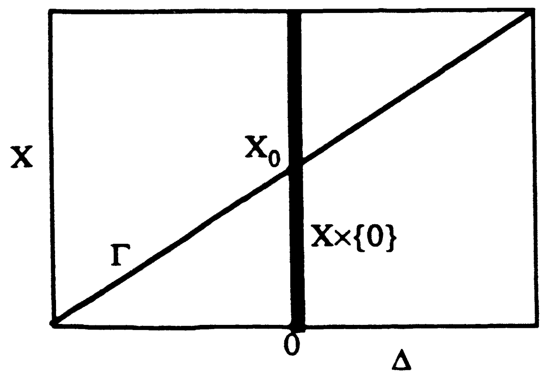

Fig. 1.

Put $K^\cdot = \Omega^\cdot_{X \times \Delta/\Delta}(*\Gamma)$ and let t denote the parameter in Δ, i.e. $\mathcal{O}^\cdot_{\Delta,0} = \mathbf{C}\{t\}$. Then $\Lambda^\cdot = \Omega^\cdot_X(*X_0) = i^*K^\cdot$, and we have the exact sequence of stalk complexes

$$0 \to K^\cdot_{(t_0,0)} \overset{\times t}{\to} K^\cdot_{(t_0,0)} \to \Lambda^\cdot_{x_0} \to 0$$

Taking the long exact cohomology sequence for $\mathcal{H}^\cdot$ and noticing that $\mathcal{H}^\cdot K^\cdot_{(x_0,0)} = \mathcal{H}^\cdot(p_*K^\cdot)_0$, where p is the projection: $X \times \Delta \to \Delta$, we get

$$\cdots \to \mathcal{H}^n\Lambda^\cdot_{x_0} \to \mathcal{H}^{n+1}(p_*K^\cdot)_0 \overset{\times t}{\to} \mathcal{H}^{n+1}(p_*K^\cdot)_0 \to \mathcal{H}^{n+1}\Lambda^\cdot_{x_0} \to \cdots$$

By Lemma (3.3) in [S-S] $\mathcal{H}^{n+1}(p_*K^\cdot)_0$ is (the stalk at 0 of) the *Gauss-Manin system*

$$\mathcal{G}_X = \int_f^{n+1} \mathcal{O}_X := \mathbf{R}^{n+1}f_*(j^{-1}\Omega^\cdot_{X \times \Delta/\Delta}(*\Gamma)/\Omega^\cdot_{X \times \Delta/\Delta}).$$

This is a coherent $\mathscr{D}$-module, where $\mathscr{D} = \mathscr{D}_\Delta$ is the sheaf of germs of linear differential operators on Δ with holomorphic coefficients. More about $\mathscr{G}_X$ later.

Quite generally, for each k $\mathscr{H}^{k+1}(p_* K^\cdot)_0 = \int_f^{k+1} \mathscr{O}_X$ is a coherent $\mathscr{D}$-module extending $\mathscr{O}_{\Delta^*}(\cup_{t\in\Delta^*} H^k(X_t))$ to Δ (cf. [Ph]), and we already know that $H^k(X_t) = 0$ for $0 < k < n$. So our sequence is just

$$(**)\quad 0 \to \mathscr{H}^n \Lambda^\cdot_{x_0} \to \mathscr{G}_{X,0} \overset{\times t}{\to} \mathscr{G}_{X,0} \to \mathscr{H}^{n+1}\Lambda^\cdot_{x_0} \to 0.$$

The outlying terms are, of course, $H^n(X^*)$ and $H^{n+1}(X^*)$, respectively.

(2.3) At this juncture let us explain the relevance of the Gauss-Manin system $\mathscr{G}_X$ to our problem of comparing the pole order filtration $P^\cdot$ with the Hodge filtration $F^\cdot$. The first point is that $\mathscr{G}_X$ is also filtered by $P^\cdot$ and $F^\cdot$, and the two agree up to a shift of indices: $P^p = F^{p+1}$. Here are the definitions. Following [Ph] or [S-S], introduce the ring $\Omega^\cdot_X[D]$ of polynomials in the indeterminate D with coefficients in the complex $\Omega^\cdot_X$. One has an isomorphism

$$\Omega^\cdot_X[D] \to j^{-1}\Omega^\cdot_{X\times\Delta/\Delta}(*\Gamma)/\Omega^\cdot_{X\times\Delta/\Delta}$$

given by $\omega D^k \mapsto [k!\omega/(f-t)^{k+1}]$. This map becomes an isomorphism of $\mathscr{D}$-complexes (i.e. an isomorphism of graded $\mathscr{D}$-modules with $\mathscr{D}$-equivariant differentials) if we define the action of $f^{-1}\mathscr{D}$ on $\Omega^\cdot_X[D]$ by $\partial_t \omega D^k = \omega D^{k+1}$, $t\omega D^k = f\omega D^k - k\omega D^{k-1}$, and equip the complex $\Omega^\cdot_X[D]$ with the differential $\mathbf{d}(\omega D^k) = d\omega D^k - df \wedge \omega D^{k+1}$. Thus

$$\mathscr{G}_X = \mathbf{R}^{n+1} f_* \Omega^\cdot_X[D] = \Omega^{n+1}_X[D]/\mathbf{d}\Omega^n_X[D].$$

Observe that $\Gamma \cap X \times \{t\} = X_t$ $\forall t \in \Delta$, and $\Omega^\cdot_X[D]|_{X_t} = (\Omega^\cdot_X(*X_t)/\Omega^\cdot_X)|_{X_t}$. Assuming $t \neq 0$, the latter complex, filtered by the order of the pole $P^\cdot$, is filtered quasi-isomorphic to $(\Omega^\cdot_{X_t}, F^\cdot)$, where $F^\cdot = \sigma_{\geqslant \cdot}$, the stupid filtration. Indeed, since each X_t is smooth for $t \neq 0$, the existence of the filtered quasi-isomorphism in question follows from the sheaf-theoretic version of the result of Griffiths quoted in the Introduction (see also [DeIII]). This motivates the following

DEFINITION. The *Hodge filtration* $F^\cdot$ on $\Omega^\cdot_X[D]$ is given by

$$F^{p-k-1}\Omega^p_X[D] = \bigoplus_{i=0}^{k} \Omega^p_X \cdot D^i.$$

From what has been said, up to a shift, this is the pole order: $F^p = P^{p-1}$.

DEFINITION. $F^p\mathscr{G}_X = \operatorname{image}\{\mathbf{R}^{n+1} f_*(F^p\Omega^\cdot_X[D]) \to \mathscr{G}_X\}$.

FACT[S-S]. $\mathscr{G}_X = \cup_{p \leqslant n} F^p\mathscr{G}_X$.

We must also mention that the smallest level of the Hodge filtration, $F^n\mathscr{G}_X$, is isomorphic to the *second Brieskorn module* $H'' = \Omega_X^{n+1}/df \wedge d\Omega_X^{n-1}$, and all other levels are related to it as $F^{n-k}\mathscr{G}_X = \partial_t^k F^n \mathscr{G}_X$.

(2.4) Let us now show the strictness of the outlying morphisms in (**) with respect to the pole order filtration $P^{\cdot}$. The surjection $\mathscr{G}_{X,0} \to \mathscr{H}^{n+1}\Lambda^{\cdot}_{x_0}$ is strictly compatible with $P^{\cdot}$. Indeed, being induced by the $P^{\cdot}$-compatible surjection $K^{\cdot} \to \Lambda^{\cdot}$, it respects $P^{\cdot}$. Now, any $\bar{\omega} \in \Lambda^{n+1}_{x_0}$ is automatically closed and is also the image of some $\omega \in K^{n+1}_{(x_0,0)}$, closed too. We may assume that if $\bar{\omega} \in P^k\Lambda^{n+1}$, then $\omega \in K^{n+1}$ as well. But then the same is true about the cohomology classes $[\bar{\omega}] \in \mathscr{H}^{n+1}\Lambda^{\cdot}_{x_0}$ and $[\omega] \in \mathscr{H}^{n+1}K^{\cdot}_{(x_0,0)} = \mathscr{G}_{X,0}$.

The injection $\delta\colon \mathscr{H}^n\Lambda^{\cdot}_{x_0} \to \mathscr{G}_{X,0}$ is also strictly compatible with $P^{\cdot}$. To see this we trace the definition of the connecting homomorphism δ on the cochain level (see Figure 2):

$$\begin{array}{ccccccccc}
 & & & & \omega & \mapsto & \bar{\omega} & & \\
 & & & & K^n & \to & \Lambda^n & \to & 0 \qquad d\bar{\omega} = 0 \\
\delta[\bar{\omega}] = [\eta] & & & & d\downarrow & & \downarrow d & & \\
 & 0 & \to & K^{n+1} & \xrightarrow{\times t} & K^{n+1} & \to & \Lambda^{n+1} & \to 0 \\
 & & & \eta & \mapsto & d\omega = t\eta & & &
\end{array}$$

Fig. 2.

The process is divided into three parts. The first, starting with $\bar{\omega}$—a cocycle in Λ^n—and choosing an ω in the preimage of $\bar{\omega}$ in K^n, can be performed so that ω has the same pole order filtration level as $\bar{\omega}$. The last step is also strict, since multiplication by t does not affect the order of the pole along Γ or the degree of the form. So we concentrate on the remaining step $d\colon \omega \mapsto d\omega$. Thinking of ω and $d\omega$ as forms in $\Omega^n_{X\times\Delta}(*\Gamma)$ (after all, we have the surjection coming from

$$\begin{array}{ccccccc}
 & K^{\cdot-1} & & K^{\cdot} & & & \\
 & \| & & \| & & & \\
0 \to & \Omega^{\cdot-1}_{X\times\Delta/\Delta}(*\Gamma) & \xrightarrow{\wedge dt} & \Omega^{\cdot}_{X\times\Delta}(*\Gamma) & \to & \Omega^{\cdot}_{X\times\Delta/\Delta}(*\Gamma) & \to 0),
\end{array}$$

we make use of the following property:

Let M be a complex manifold and $V \subset M$ a *smooth* hypersurface. *Locally*, if α is a closed form in $\Omega^{k+1}_M(*V)$ with a pole of order p such that $[\alpha] = 0$ in $\mathscr{H}^n\Omega^{\cdot}_M(*V)$, then there exists $\beta \in \Omega^k_M(*V)$ with a pole of order $p-1$ such that $\alpha = d\beta$. This fact is proved in [Gri], Lemma 10.9(i). So, if $t\eta = d\omega$, then also $t\eta = d\omega'$ with $\omega' \in K^n$ having one pole fewer than $t\eta$ (and η). Since $d\omega = d\omega'$, we see that the image $\bar{\omega}'$ of ω' in Λ^n is also a cocycle and agrees with $\bar{\omega}$ up to a coboundary. Hence $[\bar{\omega}] = [\bar{\omega}']$, and we could have started with $\bar{\omega}'$ in place of $\bar{\omega}$. Then $d\colon \omega' \mapsto d\omega'$ is strict for $P^{\cdot}$, and so is the composite map δ.

(2.5) We shall now relate the sequences (**) of (2.2) and (*) of (1.2).

LEMMA [S-S]. *For every $a \in \mathbf{C}$ the space*

$$C_a = \bigcup_{r>0} \ker(t\partial_t - a)^r \subset \mathscr{G}_{X,0}$$

is finite-dimensional, and $C_a = 0$ if $\exp(-2\pi i a)$ is not an eigenvalue of the monodromy T.

Note that $a \in \mathbf{Q}$ by the Monodromy Theorem.

DEFINITION [S-S]. The (decreasing) filtration $V^{\cdot}$ on $\mathscr{G}_{X,0}$ is given by the free $\mathcal{O}_{\Delta,0}$- submodules $V^a\mathscr{G}_{X,0}$ (respectively $V^{>a}\mathscr{G}_{X,0}$) generated by all C_b with $b \geqslant a$ (respectively $b > a$).

"Recall" also that

$$H^n(X_\infty)_1 \cong C_0 \cong \mathrm{Gr}_V^0\mathscr{G}_{X,0}.$$

Both C_0 and $\mathrm{Gr}_V^0\mathscr{G}_{X,0}$ acquire the filtrations $F^{\cdot}$ and $P^{\cdot}$ (identical up to a shift by one) from $\mathscr{G}_{X,0}$, one as a subspace, another as a quotient of a subspace. The key identification we need from [S-S] is

$$(H^n(X_\infty)_1, F^{\cdot}) \cong (\mathrm{Gr}_V^0\mathscr{G}_{X,0}, F^{\cdot}) \cong (\mathrm{Gr}_V^0\mathscr{G}_{X,0}, P^{\cdot+1}).$$

As we shall see below, C_0 and $\mathrm{Gr}_V^0\mathscr{G}_{X,0}$ are *not* isomorphic as *filtered* vector spaces.

Now, on $\mathscr{G}_{X,0}$ in general $\partial_t F^p = F^{p-1}$, i.e. $\partial_t F^p = P^p\, \forall p$. And under the isomorphism between $H^n(X_\infty)_1$ and $\mathrm{Gr}_V^0\mathscr{G}_{X,0}$ the endomorphism N becomes $-2\pi i t\partial_t$. Keeping in mind that $\times t$ and ∂_t are filtered endomorphisms of $(\mathscr{G}_{X,0}, V^{\cdot})$ of degrees 1 and -1, respectively, we have this join of two commutative diagrams:

$$\begin{array}{ccc}
(\mathscr{G}_{X,0}, F^{\cdot}) & \xrightarrow{\partial_t} (\mathscr{G}_{X,0}, P^{\cdot}) \xrightarrow{\times t} & (\mathscr{G}_{X,0}, P^{\cdot}) \\
\cup & & \cup \\
(C_0, F^{\cdot}) & \xrightarrow{t\partial_t} & (V^0\mathscr{G}_{X,0}, P^{\cdot}) \\
\downarrow & & \downarrow \\
(\mathrm{Gr}_V^0\mathscr{G}_{X,0}, F^{\cdot}) & & (\mathrm{Gr}_V^0\mathscr{G}_{X,0}, F^{\cdot})(-1) \\
\| & & \| \\
(H^n(X_\infty)_1, F^{\cdot}) & \xrightarrow{N} & (H^n(X_\infty)_1, F^{\cdot})(-1)
\end{array}$$

Here we omit the constant factors $-2\pi i$, and (-1) denotes the Tate twist.

(2.6) Using the invertibility of ∂_t on $\mathscr{G}_{X,0}$ we may finally compare the two exact sequences (*) and (**):

$$\begin{tikzcd}
0 \arrow[r] & (\mathscr{H}^n\Lambda^{\cdot}_{x_0}, P) \arrow[r] & (\mathscr{G}_{X,0}, P^{\cdot}) \arrow[r, "\times t"] & (\mathscr{G}_{X,0}, P^{\cdot}) \arrow[r] & (\mathscr{H}^{n+1}\Lambda^{\cdot}_{x_0}, P^{\cdot}) \arrow[r] & 0 \\
 & & (C_0, F^{\cdot}) \arrow[u, "\alpha"] \arrow[d] & (V^0\mathscr{G}, P^{\cdot}) \arrow[u, phantom, "\bigcup"] \arrow[ur, "\beta"'] \arrow[d] & & \\
0 \arrow[r] & (H^n(X^*), F^{\cdot}) \arrow[r] & (H^n(X_\infty)_1, F^{\cdot}) \arrow[r, "N"] & (H^n(X_\infty)_1(-1), F^{\cdot}) \arrow[r] & (H^{n+1}(X^*), F^{\cdot}) \arrow[r] & 0
\end{tikzcd}$$

Since C_0 contains $\ker t\partial_t$, it is clear that $(\mathscr{H}^n\Lambda^{\cdot}_{x_0}, P^{\cdot})$ is contained in the image of the monomorphism α. Thus we may think of $(\mathscr{H}^n\Lambda^{\cdot}_{x_0}, P^{\cdot})$ as $\ker t\partial^t \subset (C_0, F^{\cdot})$. Composing with the isomorphism $C_0 \to H^n(X_\infty)_1$, we get a *filtered* map:

$$(\mathscr{H}^n\Lambda^{\cdot}_{x_0}, P^{\cdot}) \to (H^n(X^*), F^{\cdot}).$$

This is an isomorphism of vector spaces compatible with the filtrations *though not necessarily strict*. We have established one half of part (b) of our Theorem (0.3).

To get the other half, observe that the composite map β induces a map

$$\begin{tikzcd}
\mathrm{Gr}^0_V\mathscr{G}/N \arrow[d, phantom, "\wr\|"] & H^{n+1}(X^*) \arrow[d, phantom, "\|\wr"] \\
(V^0\mathscr{G}/V^0\mathscr{G} \cap t\mathscr{G}, P^{\cdot}) \arrow[r] & (\mathscr{G}/t\mathscr{G}, P^{\cdot}),
\end{tikzcd}$$

which is an isomorphism of the underlying vector spaces and respects $P^{\cdot}$, but not necessarily strictly.

(2.7) Part (a) of Theorem (0.3) follows from the above proof of part (b). Indeed, when f is quasi-homogeneous one may identify $\mathscr{G}$ and $\mathrm{Gr}_V\mathscr{G}$, and $F^{\cdot}$ splits with respect to this decomposition (cf. [Sa]).

3. The third filtration

(3.1) "Recall" that the section s_ω of $\mathscr{G}$ defined by a differential form $\omega \in H''$ admits an asymptotic expansion

$$s_\omega = \sum_{\alpha,k} t^\alpha (\log t)^k A^\omega_{\alpha,k}$$

with $A^\omega_{\alpha,k}$—locally constant sections of $\mathscr{G}$ which can be identified with elements of $H^n(X_\infty)_{\exp(-2\pi i\alpha)}$. All numbers α are rational and greater than -1.

DEFINITION. The *third filtration* $G^{\cdot}$ on $H^n(X_\infty)$ is defined by

$$G^p = \operatorname{span}\{u \in H^n(X_\infty) | u = A^{\omega}_{\alpha,k} \text{ for some } \omega \in H'' \text{ with } \alpha - n + p \leqslant 0\}.$$

(3.2) LEMMA. *Multiplication by t in $\mathscr{G}$ maps C_α isomorphically onto $C_{\alpha+1}$ unless $a = -1$.*

Proof. First of all, $tC_\alpha \subset C_{\alpha+1}$. This follows immediately from the identity

$$(t\partial_t - a)^r t = t(t\partial_t - a + 1)^r \,\forall r,$$

which itself is a consequence of $\partial_t t = 1 + t\partial_t$. Now, thinking of elements of $H^n(X_\infty)$ as locally constant sections of $\mathscr{G}$, we have the following isomorphism (cf. [S-S]):

$$C_a \xrightarrow{\sim} H^n(X_\infty)_{\exp(-2\pi i a)}, \quad \text{by } v \mapsto t^{-a} \exp\left(N \frac{\log t}{2\pi i}\right) v$$

and its inverse

$$H^n(X_\infty)_{\exp(-2\pi i a)} \xrightarrow{\sim} C_a, \quad \text{by } w \mapsto t^a \exp\left(-N \frac{\log t}{2\pi i}\right) w.$$

This shows that for all a C_a and C_{a+1} have the same dimension. Finally, $\ker t$ is contained in C_{-1}; indeed, it is just $\partial_t(\ker t\partial_t)$, and $\ker t\partial_t \subset C_0$, while ∂_t maps C_{a+1} to C_a. Thus for all $a \neq -1$ the map

$$\times t : C_a \to C_{a+1}$$

is a monomorphism of two vector spaces of the same dimension, hence an isomorphism.

(3.3) Applying this lemma, we get the following description of the isomorphism

$$\mathscr{G}/t\mathscr{G} \cong H^n(X_\infty)_1/N:$$

Elements of $\mathscr{G}$ can be expanded asymptotically as

$$s = \sum_{\alpha,k} t^\alpha \left(\frac{\log t}{2\pi i}\right)^k u_{\alpha,k},$$

with each $u_{\alpha,k} \in H^n(X_\infty)_{\exp(-2\pi i \alpha)}$. It may also be assumed that $u_{\alpha,k} = N^k u_{\alpha,0}$. Then the isomorphism in question is induced by the map sending such a section to $u_{0,0}$ mod N.

(3.4) Now, any section s as above can be represented as $\partial_t^l s_\omega$ for some $\omega \in H''$. This implies $u_{0,0} \equiv A_{l,0}^\omega \pmod N$ in the decomposition

$$s_\omega = \sum_{\alpha,k} t^\alpha \left(\frac{\log t}{2\pi i}\right)^k A_{\alpha,k}^\omega.$$

Hence $u_{0,0} \in G^{n-l} H^n(X_\infty)_1/N$ (by the definition of the third filtration $G^\cdot$). Thus $\bar{u} = [u_{0,0} \bmod N] \in H^n(X_\infty)_1/N$ lies in G^{n-l} if it can be expressed by a form with $l+1$ poles, i.e. $P^{n-l} \subset G^{n-l}$.

The last argument is completely reversible, i.e. $P^\cdot = G^\cdot$ on $\mathscr{G}/t\mathscr{G} \cong H^n(X_\infty)_1/N$. This confirms part (c) of our Theorem (0.3).

4. An example

(4.1) In this section we give an example with $P^\cdot \subset F^\cdot$, $P^\cdot \neq F^\cdot$ on $H^n(X^*)$ and $F^\cdot \subset P^\cdot = G^\cdot$, $F^\cdot \neq P^\cdot$ on $H^{n+1}(X^*)$. This example was suggested by Morihiko Saito and is a modification of his earlier example, exhibiting a related phenomenon. Let us start with the latter:

$$f(x, y) = \frac{x^5}{5} + \frac{y^5}{5} + a\frac{x^3y^3}{3}.$$

Here a is a parameter introduced to show the behavior of various filtrations under deformations. For our purposes let us assume $a \neq 0$.

We shall write R for the ring $\mathbf{C}\{\{\partial_t^{-1}\}\}$. An R-basis for H_f'' is given by

$$\{w_{ij} = x^{i-1}y^{j-1}\,dx \wedge dy\}_{i,j=1,\ldots,4},$$

and the saturation $\tilde{H}'' = \Sigma_{i \geqslant 0}(\partial_t t)^i H''$ is

$$\tilde{H}'' = \sum_{(i,j)\neq(4,4)} Rw_{ij} + R\partial_t w_{44}.$$

Then Saito defines the following submodules:

$$M^0 = \sum_{j+1}^{3} Rw_{jj} + R\partial_t w_{44},$$

$$M^k = \sum_{j-i\equiv k \pmod 5} Rw_{ij} \quad \text{for } k = 1, \ldots, 4.$$

This gives a decomposition, compatible with the action of t and ∂_t^{-1}, of $\tilde{H}''$ into a direct sum of these submodules. Similar decompositions hold for other objects. In particular, if H_f stands for the cohomology of the Milnor fiber, then

$$H_f = \bigoplus_{k=0}^{4} H_f^k.$$

Let $u_1, \ldots, u_4$ be a basis of H_f^0 with $Tu_i = \exp(-2\pi\sqrt{-1}i/5)u_i$. Writing $\mathscr{L}$ for $V^{>-1}\mathscr{G}$ (with $t\mathscr{L} = V^{>0}\mathscr{G}$) Saito computes these asymptotics:

$$\begin{aligned}
w_{11} &\equiv t^{-3/5} \otimes u_2 - \tfrac{a}{3}t^{-2/5} \otimes u_3 \quad (\operatorname{mod} t\mathscr{L} + \mathbf{C}w_{22})\\
\partial_t w_{44} &\equiv t^{-2/5} \otimes u_3 \quad (\operatorname{mod} t\mathscr{L} + \mathbf{C}w_{22})\\
w_{22} &\equiv t^{-1/5} \otimes u_4 \quad (\operatorname{mod} t\mathscr{L})\\
w_{33} &\equiv 0 \quad (\operatorname{mod} t\mathscr{L}).
\end{aligned}$$

(4.2) Consider now the function

$$\varphi(x, y, z) = f(x, y) + g(z), \quad \text{where } g = z^5.$$

This is still quasi-homogeneous, i.e. $N = 0$, which means

$$H^1(X^*) \cong H^2(X_\infty)_1 \cong H^3(X^*)(1).$$

By Thom-Sebastiani (cf. [S-S], section 8) $\tilde{H}''_\varphi$ contains as direct factors $M^0 \wedge zdz$ and $M^0 \wedge z^2dz$. The first of these is described by the following asymptotics (where v_i denotes u_i tensored by the element of H_g corresponding to zdz):

$$\begin{aligned}
zdx \wedge dy \wedge dz = w_{11} \wedge zdz &\equiv t^{-1/5} \otimes v_2 - \tfrac{a}{3}v_3 (\operatorname{mod} t\mathscr{L})\\
(\partial_t w_{44}) \wedge zdz &\equiv v_3 (\operatorname{mod} t\mathscr{L})\\
w_{22} \wedge zdz &\equiv 0 (\operatorname{mod} t\mathscr{L})\\
w_{33} \wedge zdz &\equiv 0 (\operatorname{mod} t\mathscr{L}).
\end{aligned}$$

Thus v_3, which is an element of $H^2(X_\infty^\varphi)_1(-1) = H^3(X^*)$, lies in $P^3 = G^3$, but $v_3 \in F^2 - F^3$.

The second module $M^0 \wedge z^2dz$ is described similarly (but now v_i signifies u_i tensored by the element of H_g corresponding to z^2dz):

$$z^2dx \wedge dy \wedge dz = w_{11} \wedge z^2dz = v_2 - \tfrac{a}{3}t^{1/5} \otimes v_3 + \cdots,$$

where "…" stands for higher-order terms. The other three sections are all 0 (mod $t\mathscr{L}$). Thus here $v_2 \in F^2$ of $H^1(X^*) = H^2(X_\infty)_1 \cong \mathrm{Gr}_V^0\mathscr{G}$, but no element of $H''_\varphi \cap C_0$ $(=F^2\mathscr{G} \cap C_0 = P^2H^1(X^*))$ represents v_2.

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