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On the "Bruhat graph" of a Coxeter system

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Introduction

It is well-known that finite and affine Weyl groups play a central role in the structure and representation theory of complex semisimple Lie algebras, of algebraic groups and of finite groups with a BN-pair (see [C]). In recent years, important applications of more general Coxeter groups have been found to the study of Kac-Moody Lie algebras [K] and in topology [D], and they have been extensively studied in combinatorics [B].

In a number of these situations, the Bruhat order on the Coxeter group is of fundamental importance. For example, it is known that Bruhat order describes the closure patterns of Shubert cells for reductive algebraic groups over algebraically closed fields [J], and also inclusions among Verma modules for a complex semisimple Lie algebra [BGG].

This paper gives two applications of properties of reflection subgroups of Coxeter systems [Dy1, Dy2] to the study of Bruhat order. In Section 2, we answer a question of Bjorner [B, 4.7] by showing that only finitely many isomorphism types of posets of fixed length n occur as Bruhat intervals in finite Coxeter groups. In Section 3, we show that the pairs of elements x, y of a Coxeter group such that $x^{-1}y$ is a reflection are determined by Bruhat order as an abstract order (3.3). This supports the conjecture that the Kazhdan–Lusztig polynomial $P_{x,y}$ (see [KL]) depends only on the isomorphism type of the Bruhat interval $[x, y]$.

The key to both proofs is a certain directed graph (the "Bruhat graph," defined in (1.1)) associated to a Coxeter system. This graph determines the Bruhat order, but exhibits a kind of functoriality with respect to inclusions of reflection subgroups (1.4) not shown by Bruhat order.

1. The Bruhat graph of a Coxeter system

Let (W, R) be a finite Coxeter system and $l: W \to \mathbb{N}$ denote the corresponding length function. Let $\mathbf{T} = \bigcup_{w \in W} wRw^{-1}$ denote the set of reflections of (W, R) and for $w \in W$ write $N(w) = \{t \in T \mid l(tw) < l(w)\}$.

(1.1) DEFINITION. The Bruhat graph $\Omega_{(W,R)}$ of (W, R) is the directed graph with vertex set W and edge set

$$\begin{aligned} E_{(W,R)} &= \{(tw, w) \mid w \in W, t \in N(w)\} \\ &= \{(wt, w) \mid w \in W, t \in N(w^{-1})\}. \end{aligned}$$

For any subset A of W, define $\Omega_{(W,R)}(A)$ to be the full subgraph of $\Omega_{(W,R)}$ with vertex set A (i.e. the edge set of $\Omega_{(W,R)}(A)$ is

$$E_{(W,R)}(A) = \{(x, y) \in E_{(W,R)} \mid x \in A, y \in A\}).$$

(1.2) EXAMPLE. Let (W, R) be a dihedral Coxeter system and I be a closed Bruhat interval of length n in (W, R). Let Z be a set equipped with a function $f: Z \to \{0, 1, \ldots, n\}$ such that

$$\#(f^{-1}(i)) = \begin{cases} 1 & \text{if } i = 0 \quad \text{or } i = n; \\ 2 & \text{if } 1 \leqslant i \leqslant n - 1. \end{cases}$$

Then $\Omega_{(W,R)}(I)$ is isomorphic to the directed graph with vertex set Z and edge set

$$\{(x, y) \in Z \times Z \mid f(x) < f(y), f(y) - f(x) \text{ is odd}\}$$

(1.3) A subgroup W' of W such that $W' = \langle W' \cap T \rangle$ is called a reflection subgroup of (W, R). It is shown in [Dy2] that if W' is a reflection subgroup of (W, R) then $\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}$ is a set of Coxeter generators for W'. The following result is a reformulation of [Dy2, (3.3), (3.4)].

(1.4) THEOREM. *Let W' be a reflection subgroup of (W, R) and set $R' = \chi(W')$. Then*

(i) $\Omega_{(W',R')} = \Omega_{(W,R)}(W')$.
(ii) *For any $x \in W$, there exists a unique $x_0 \in W'x$ such that the map $W' \to W'x$ defined by $w \mapsto wx_0$ (for $w \in W'$) is an isomorphism of directed graphs $\Omega_{(W,R)}(W') \to \Omega_{(W,R)}(W'x)$.*

2. Bruhat intervals in finite Coxeter systems

Maintain the notation from §1 and let $\leqslant$ denote the Bruhat order on W. Recall that if $x, y \in W$, then $x \leqslant y$ iff there is a path $(x = x_0, x_1, \ldots, x_n = y)$ from x to y in $\Omega_{(W,R)}$ (that is $(x_{i-1}, x_i) \in E_{(W,R)}$ for $i = 1, \ldots, n$).

(2.1) PROPOSITION. *Let $I = [x, y] = \{z \in W \mid x \leqslant z \leqslant y\}$ be a closed (non-*

empty) Bruhat interval in (W, R) *with* $l(y)-l(x)=n$. *Then* $W'=\langle uv^{-1} \mid u, v \in I\rangle$ *is a reflection subgroup of* (W, R) *and* $\#(\chi(W')) \leqslant n$. *Moreover, I is isomorphic (as a poset) to a Bruhat interval in the Coxeter system* $(W', \chi(W'))$.

Proof. This is clear if $n \leqslant 1$ so assume $n \geqslant 2$. Let $(x_0, \ldots, x_n)$ be a path of length n in $\Omega_{(W,R)}$ with $x_0 = x$ and $x_n = y$ [B], and let W'' denote the reflection subgroup $W'' = \langle x_{i-1}x_i^{-1} (i=1, \ldots, n)\rangle$. Set $R''=\chi(W'')$ and let l'': $W'' \to \mathbb{N}$ be the length function of (W'', R''). Let z be the element of $W''x$ such that the map $w \mapsto wz (w \in W'')$ gives an isomorphism of directed graphs

$$\Omega_{(W,R)}(W'') \to \Omega_{(W,R)}(W''x).$$

Then $(x_0z^{-1}, \ldots, x_nz^{-1})$ is a path in $\Omega_{(W'',R'')}$ from xz^{-1} to yz^{-1}; in particular, $l''(yz^{-1})-l''(xz^{-1}) \geqslant n$. On the other hand, if $(y_0, \ldots, y_m)$ is a path in $\Omega_{(W'',R'')}$ from xz^{-1} to yz^{-1} then $(y_0z, \ldots, y_mz)$ is a path from x to y in $\Omega_{(W,R)}$. This implies that $l''(yz^{-1})-l''(xz^{-1}) \leqslant n$ and that, if I' denotes the Bruhat interval $[xz^{-1}, yz^{-1}]$ in the Bruhat order on (W'', R''), the map $f: w \mapsto wz$ is an injective, order-preserving map $I' \to I$. Note $n = l''(yz^{-1})-l''(xz^{-1})$. If $n=2$, then f is clearly an isomorphism, so assume $n \geqslant 3$.

Recall that the order complex of a finite poset X is the (abstract) simplicial complex with the totally ordered subsets of X as simplexes. Let Σ and Σ' denote the order complexes of $I \backslash \{x, y\}$ and $I' \backslash \{xz^{-1}, yz^{-1}\}$. By [BW, 5.4], Σ and Σ' are both combinatorial $(n-2)$-spheres. Now if $\sigma=(z_0, \ldots, z_k) \in \Sigma'$, we have $f(\sigma)=(f(z_0), \ldots, f(z_k)) \in \Sigma$; it follows that f induces an isomorphism of Σ' with a subcomplex $f(\Sigma')$ of Σ. Since Σ is a connected combinatorial $(n-2)$-manifold and $f(\Sigma')$ is a (non-empty) $(n-2)$-homogeneous boundaryless subcomplex of Σ, it follows that $f(\Sigma')=\Sigma$. This implies that $f: I' \to I$ is an isomorphism.

By [Dy2, (3.11)(*i*)], we have $\#(\chi(W'')) \leqslant n$. Now $W'' \subseteq W'$, and $W' \subseteq W''$ since for $u, v \in I'$ we have $f(u)f(v)^{-1} = uv^{-1}$. Hence $W' = W''$, completing the proof.

(2.2) COROLLARY. *For each* $n \in \mathbb{N}$, *there are only finitely many isomorphism types of Bruhat intervals of fixed length n occurring in finite Coxeter groups.*

Proof. By Proposition (2.1), a Bruhat interval of length n in a finite Coxeter group is isomorphic to a Bruhat interval of length n in a finite Coxeter system (W', R') of rank $\#(R') \leqslant n$. By the classification of finite Coxeter systems, there are only finitely many isomorphism types of such Coxeter systems (W', R'), and each contains only finitely many isomorphism types of Bruhat intervals of length n.

3. Bruhat order and the Bruhat graph

Maintain the notation from §1, but assume in addition that (W, R) is realized geometrically as a group of isometries of a real vector space V as in [De].

More specifically, assume that V is a real vector space with a symmetric bilinear form $(\cdot\,|\,\cdot)$ and that Π is a basis of V consisting of vectors α with $(\alpha\,|\,\alpha)=1$. Suppose that $R=\{r_\alpha\,|\,\alpha\in\Pi\}$, where for non-isotropic $\gamma\in V$, $r_\gamma\colon V\to V$ is the reflection in γ defined by $x\mapsto x-2(x\,|\,\gamma)/(\gamma\,|\,\gamma)\gamma(x\in V)$, and that $W=\langle R\rangle$. Assume that $\Phi=\Phi^+\cup(-\Phi^+)$ where $\Phi=W\Pi$ and $\Phi^+=\{\Sigma_{\alpha\in\Pi}c_\alpha\alpha\in\Phi\,|\,\text{all } c_\alpha\geqslant 0\}$ are the sets of roots and of positive roots respectively. Then (W,R) is a Coxeter system, and any Coxeter system is isomorphic to such a Coxeter system [De, §2].

Note that the map $\alpha\mapsto r_\alpha$ is a bijection $\Phi^+\to T$ and that for $w\in W$, one has

$$N(w) = \{r_\alpha\,|\,\alpha\in\Phi^+\cap w(-\Phi^+)\}.$$

A reflection subgroup W' of (W,R) is called dihedral if $\#(\chi(W'))=2$, or equivalently, if $W'=\langle t,t'\rangle$ for some $t,t'\in T(t\neq t')$[Dy2, (3.9)].

(3.1) LEMMA. *Suppose*

$$t_1,\ t_2,\ t_3,\ t_4\in T \quad \textit{and} \quad t_1t_2=t_3t_4\neq 1.$$

Then $W'=\langle t_1,t_2,t_3,t_4\rangle$ *is a dihedral reflection subgroup of* (W,R).

Proof. Conjugating by a suitable element of W, we may assume without loss of generality that $t_1\in R$. Write $t_i=r_{\alpha_i}(\alpha_i\in\Phi^+, i=1,\dots,4)$. By taking a geometric realization for a new Coxeter system containing W as a standard parabolic subgroup, one may assume that there exist simple roots β, $\gamma\in\Pi$ such that the matrix

$$\begin{pmatrix}(\beta\,|\,\alpha_1) & (\beta\,|\,\alpha_2)\\ (\gamma\,|\,\alpha_1) & (\gamma\,|\,\alpha_2)\end{pmatrix}$$

is non-singular. Now for $x\in V$, one has

$$x - r_{\alpha_1}r_{\alpha_2}(x) = 2(\alpha_2\,|\,x)\alpha_2 + 2(\alpha_1\,|\,x)(\alpha_1 - 2(\alpha_1\,|\,\alpha_2)\alpha_2).$$

Hence $(1-r_{\alpha_1}r_{\alpha_2})V=\mathbb{R}\alpha_1+\mathbb{R}\alpha_2$. But $(1-r_{\alpha_3}r_{\alpha_4})V\subseteq\mathbb{R}\alpha_3+\mathbb{R}\alpha_4$ from which $\mathbb{R}\alpha_1+\mathbb{R}\alpha_2=\mathbb{R}\alpha_3+\mathbb{R}\alpha_4$. Let $\Gamma\subseteq\Phi^+$ be defined by $\chi(W')=\{r_\alpha|\alpha\in\Gamma\}$. By [Dy2, (3.11)(*ii*)] we have $\Gamma\subseteq W'\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}\subseteq\mathbb{R}\alpha_1+\mathbb{R}\alpha_2$. By [Dy2, (4.4)] we have $(\gamma\,|\,\gamma')\leqslant 0$ for any distinct $\gamma,\gamma'\in\Gamma$; moreover, there is no non-trivial linear relation $\Sigma_{\gamma\in\Gamma}c_\gamma\gamma=0$ with all $c_\gamma\geqslant 0$ since $\Gamma\subseteq\Phi^+$. It follows that $\#(\Gamma)\leqslant 2$, hence W' is dihedral as claimed.

(3.2) REMARK. Let α, $\beta\in\Phi^+$ with $\alpha\neq\beta$ and set

$$W' = \langle\{r_\gamma\,|\,\gamma\in(\mathbb{R}\alpha+\mathbb{R}\beta)\cap\Phi^+\}.$$

An argument similar to the last part of the preceeding proof shows that W' is dihedral and that any dihedral reflection subgroup of (W, R) which contains $\langle r_\alpha, r_\beta \rangle$ is contained in W'. Thus, every dihedral reflection subgroup of W is contained in a unique maximal dihedral reflection subgroup.

(3.3) PROPOSITION. *If $I=[x, y]$ is a closed Bruhat interval in (W, R), then the isomorphism type of the directed graph $\Omega_{(W,R)}(I)$ is determined by the isomorphism type of the poset I.*

Proof. Let

$$E=\{(u, v)\in I\times I \mid u<v,\ [u, v]=\{u, v\}\} \quad \text{and} \quad X=\{0, 1, 3, 4, 6, 7\}.$$

We will show that $E_{(W,R)}(I)$ is the smallest subset $A \supseteq E$ of $I\times I$ such that if $v_i \in I (i\in X)$ and $(v_i, v_j)\in A$ whenever $i, j\in X$, $0<j-i<7$ and $j-i$ is odd, then $(v_0, v_7)\in A$.

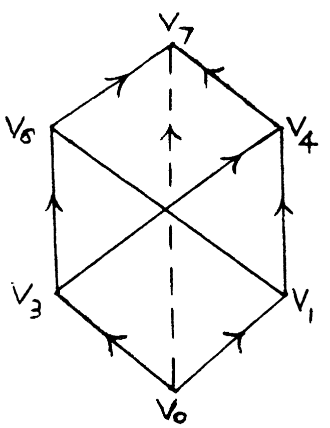

Fig. 1.

By Example (1.2) and Theorem (1.4), this follows from the following claims (i), (ii):

(i) if $t\in T$, $x\in W$ and $l(tx)<l(x)-1$ then there exists a path (x_0, x_1, x_2, x_3) in $E_{(W,R)}$ from tx to x such that $\langle x_1x_0^{-1}, x_2x_1^{-1}, x_3x_2^{-1}\rangle$ is dihedral,

(ii) if $v_i\in W (i\in X)$ and $(v_i, v_j)\in E_{(W,R)}$ whenever $i, j\in X$, $0<j-i<7$ and $j-i$ is odd, then $\langle v_iv_j^{-1}(i, j\in X)\rangle$ is dihedral.

The first claim may be proved by induction on $l(x)$ as follows. Note first that if $(u, v)\in E_{(W,R)}$ and $r\in R$, then $(ru, rv)\in E_{(W,R)}$ unless $v=ru$. Suppose that $t\in T$, $x\in W$ and $l(tx)<l(x)-1$. Choose $r\in R$ so that $rx<x$. If $rtx>tx$, then (tx, rtx, rx, x) is a suitable path from tx to x. Otherwise, $rtx<tx$ and by

induction there exists a path (x_0, x_1, x_2, x_3) in $E_{(W,R)}$ from rtx to rx with $\langle x_1x_0^{-1}, x_2x_1^{-1}, x_3x_2^{-1}\rangle$ dihedral. A suitable path from tx to x is then given by

$$\begin{cases} (tx, x_2, rx, x) & \text{if } rx_1 \in \{x_0, x_2\} \\ (tx, rx_1, rx_2, x) & \text{otherwise.} \end{cases}$$

To prove claim (ii), observe first that if W_1, W_2 are dihedral reflection subgroups of (W, R) and $W_1 \cap W_2$ contains a dihedral reflection subgroup W_3 of (W, R) then $\langle W_1, W_2\rangle$ is dihedral (for it is a reflection subgroup of the maximal dihedral reflection subgroup containing W_3).

Note that $W_1 = \langle v_1v_0^{-1}, v_0v_3^{-1}, v_1v_4^{-1}, v_4v_3^{-1}\rangle$ is dihedral by Lemma 3.1 since $1 \neq (v_1v_0^{-1})(v_0v_3^{-1}) = (v_1v_4^{-1})(v_4v_3^{-1})$. Similarly,

$$W_2 = \langle v_1v_0^{-1}, v_0v_3^{-1}, v_1v_6^{-1}, v_6v_3^{-1}\rangle \quad \text{and} \quad W_3 = \langle v_4v_1^{-1}, v_1v_6^{-1}, v_4v_7^{-1}, v_7v_6^{-1}\rangle$$

are dihedral. Since $W_1 \cap W_2 \supseteq \langle v_1v_0^{-1}, v_0v_3^{-1}\rangle$ we have $\langle W_1, W_2\rangle$ dihedral. Since $\langle W_1, W_2\rangle \cap W_3 \supseteq \langle v_4v_1^{-1}, v_1v_6^{-1}\rangle$ we have $\langle W_1, W_2, W_3\rangle$ dihedral.

(3.4) Let $[x, y]$ be a Bruhat interval in (W, R) and suppose that (x_0, x_1, x_2) is a path in $\Omega_{(W,R)}([x, y])$. Let W' be the maximal dihedral reflection subgroup containing $x_0x_1^{-1}$ and $x_1x_2^{-1}$, and set $R' = \chi(W')$. It follows from Lemma 3.1 that $W'x_0 \cap [x, y]$ is the smallest subset A of $[x, y]$ such that (i) $\{x_0, x_1, x_2\} \subseteq A$, (ii) if $w_1, w_2, w_3 \in A$ and $w \in [x, y]$ are such that $\{w_1, w_2\}$, $\{w_2, w_3\}$, $\{w_1, w\}$ and $\{w_3, w\}$ are edges of the undirected graph underlying $\Omega_{(W,R)}([x, y])$ and $w_1 \neq w_2$ then $w \in A$.

Now by Theorem 1.4, the full subgraph of $\Omega_{(W,R)}([x, y])$ on the vertex set A is isomorphic to $\Omega_{(W',R')}(B)$ for some (open, closed or half-open) interval B in the Bruhat order of (W', R'). In this way, one obtains conditions on the class of posets arising as Bruhat intervals in Coxeter groups. For example, a Bruhat interval with exactly two atoms (or coatoms) is isomorphic to a Bruhat interval in a dihedral group.

(3.5) For $x, y \in W$ define $R(x, y) = q^{-1/2(l(y)-l(x))}R_{x,y}$ where $R_{x,y}$ is defined in [KL]. Then $R(x, y)$ may be regarded as a polynomial in $\alpha = q^{1/2} - q^{-1/2}$. For $m \in N$, the coefficient of α^m in $R(x, y)$ is non-zero iff there is a path $(x_0, \ldots, x_m)$ of length m in $\Omega_{(W,R)}$ from x to y. Thus, Proposition 3.3 gives some support to the conjecture that $R(x, y)$ (and hence the Kazhdan–Lusztig polynomial $P_{x,y}$) depends only on the isomorphism type of the Bruhat interval $[x, y]$.

In [Dy1], we define an ordering $\preceq$ of the set T so that the coefficient of α^m in $R(x, y)$ is the number of paths $(x_0, \ldots, x_m)$ in $\Omega_{(W,R)}$ from x to y with $x_0^{-1}x_1 \preceq x_1^{-1}x_2 \preceq \cdots \preceq x_{m-1}^{-1}x_m$. Further connections between the Bruhat graph and the structure constants of the Hecke algebra are described in [Dy3, Dy4].

References

[BGG] Bernstein, I. N., Gelfand, I. M. and Gelfand, S. I., Structure of representations generated by highest weight vectors (English translation) *Funct. Anal. Appl.* 5 (1971) 1–8.

[B] Bjorner, A., Orderings of Coxeter groups, in "Combinatorics and algebra (Boulder, Colo. 1983)", *Contemp Math* 34, *Amer. Math. Soc.*, Providence, R. I (1984).

[BW] Bjorner, A. and Wachs, M., Bruhat order of Coxeter groups and shellability, *Adv. in Math.* 43 (1982), 87–100.

[C] Carter, R. W., Finite groups of Lie type, John Wiley & Sons, Chichester (1985).

[D] Davis, M.W., Groups generated by reflections and aspherical manifolds not covered by Euclidean space, *Ann. of Math.* 117(2) (1983) 293–324.

[De] Deodhar, V., On the root system of a Coxeter group, *Comm. Alg.* 10 (1982) 611–630.

[Dy1] Dyer, M., Hecke algebras and reflections in Coxeter groups, Ph.D. Thesis, University of Sydney (1987).

[Dy2] Dyer, M., Reflection subgroups of Coxeter systems, to appear in *J. of Alg.*

[Dy3] Dyer, M., Hecke algebras and shellings of Bruhat intervals, preprint.

[Dy4] Dyer, M., Hecke algebras and shellings of Bruhat intervals II; twisted Bruhat orders, preprint.

[J] Jantzen, J. C., Representations of algebraic groups, Pure and Applied Math. Vol. 131, Academic Press, Boston (1987).

[KL] Kazhdan, D. and Lusztig, G., Representations of Coxeter groups and Hecke algebras, *Invent. Math.* 53 (1979), 165–184.