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On module categories with nilpotent infinite radical

OTTO KERNER¹ and ANDRZEJ SKOWROŃSKI²

¹*Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, Universitätsstr.1, 4000 Düsseldorf, Germany,* ²*Institute of Mathematics, Nicholas Copernicus University, Chopina 12, 87-100 Toruń, Poland*

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Introduction

Throughout the paper k denotes a fixed algebraically closed field. By an algebra A is meant an associative, finite-dimensional k -algebra with an identity, which we shall assume to be basic and connected. By an A -module is meant a finite dimensional right A -module. We shall denote by $\text{mod } A$ the category of finite-dimensional right A -modules, by $\text{rad}(\text{mod } A)$ the radical of $\text{mod } A$, and by $\text{rad}^\infty(\text{mod } A)$ the intersection of all powers $\text{rad}^i(\text{mod } A)$, $i \geq 0$, of $\text{rad}(\text{mod } A)$.

From the existence of Auslander–Reiten sequences in $\text{mod } A$ we know that $\text{rad}(\text{mod } A)$ is generated by the irreducible maps as a left and as a right ideal. We shall show here in (1.8) that an algebra A is representation-finite if and only if $\text{rad}^\infty(\text{mod } A) = 0$. On the other hand, it is well-known that, if A is a representation-infinite tame hereditary algebra, then $\text{rad}^\infty(\text{mod } A)$ is nilpotent. In contrast, if A is wild hereditary, then $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T -nilpotent. We are interested in studying the algebras A for which $\text{rad}^\infty(\text{mod } A)$ is left or right T -nilpotent (resp. nilpotent). We shall show that all such algebras are tame (see (1.7)).

The main result of the paper gives a complete description of standard selfinjective algebras A for which $\text{rad}^\infty(\text{mod } A)$ is nilpotent. Recall that an algebra A is called standard (cf. [S1]) if it admits a simply connected Galois covering. Moreover, an algebra A is called domestic [R2] if there exists a finite number of $k[X]$ - A -bimodules Q_i which are finitely generated free left $k[X]$ -modules satisfying the following condition: for each dimension d , all but a finite number of isomorphism classes of indecomposable A -modules of dimension d are of the form $V \otimes_{k[X]} Q_i$ for some i and some indecomposable finite-dimensional $k[X]$ -module V (here $k[X]$ denotes the polynomial algebra in one variable). The following theorem is the main result of this paper.

THEOREM. *Let A be a standard representation-infinite selfinjective algebra. The following conditions are equivalent*

- (i) $\text{rad}^\infty(\text{mod } A)$ is right T -nilpotent.
- (ii) $\text{rad}^\infty(\text{mod } A)$ is left T -nilpotent.

- (iii) $\text{rad}^\infty(\text{mod } A)$ is nilpotent.
- (iv) A is domestic.
- (v) A is isomorphic to an algebra \hat{B}/G where \hat{B} is the repetitive algebra of a representation-infinite tilted algebra B of Euclidean type and G is an admissible infinite cyclic group of k -linear automorphism of \hat{B} .

The equivalence of the last two conditions has been shown in [S1, (1.5)]. The assumption that A is selfinjective is essential for the equivalence of the remaining conditions. We shall show in (1.4) examples of standard domestic algebras A for which $\text{rad}^\infty(\text{mod } A)$ is not right or left T -nilpotent. On the other hand, the infinite radical $\text{rad}^\infty(\text{mod } A)$ is not right or left T -nilpotent for the known nondomestic tame algebras A . We conjecture that all algebras A with $\text{rad}^\infty(\text{mod } A)$ T -nilpotent are domestic.

In the paper we shall identify an algebra A with the associated finite category whose objects are formed by a complete set of its primitive orthogonal idempotents. Morphisms in $\text{mod } A$ are written on the opposite side of the scalars. We shall also agree to identify the vertices of the Auslander–Reiten quiver $\Gamma(A)$ of A with the corresponding indecomposable A -modules. We denote by τ and τ^{-1} the Auslander–Reiten translations $D\text{Tr}$ and $\text{Tr}D$, respectively. For a background used in this paper we refer the reader to [R3] and [S2].

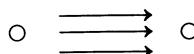
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1. Examples

We shall present here several examples with totally different behavior of the infinite radical. Recall that $\text{rad}^\infty(\text{mod } A)$ is called left (right) T -nilpotent if, for each sequence $(f_i)_{i \in \mathbb{N}}$ in $\text{rad}^\infty(\text{mod } A)$, there exists a natural number m such that $f_m \cdots f_1 = 0$ ($f_1 \cdots f_m = 0$).

(1.1) If A is representation-finite, then by the lemma of Harada and Sai, see [R1, (2.2)], $\text{rad}^\infty(\text{mod } A) = 0$.

(1.2) Let H be the path-algebra of the wild quiver



and let X be an indecomposable module of dimension-type $(1, 1)$. Then X is regular without nontrivial regular submodules and factormodules, thus each

nonzero map $f: U \rightarrow X$ with U regular is surjective and each nonzero map $g: X \rightarrow V$ with V regular is injective.

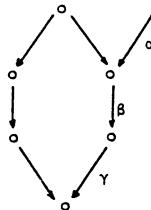
Since $\text{Ext}(X, X) \neq 0$, by the Auslander–Reiten formula there exist a nonzero map $f_0: X \rightarrow \tau X$, which is injective. Since A is hereditary τ is a left exact functor, thus all maps $f_i = \tau^i f: \tau^i X \rightarrow \tau^{i+1} X$ are injective for $i \geq 0$. Similarly for $j < 0$ all maps $\tau^j f: \tau^j X \rightarrow \tau^{j+1} X$ are surjective. Thus for all integers i and j with $j < i$ the composition $f_i \cdots f_j$ is nonzero. As all f_i are in $\text{rad}^\infty(\text{mod } A)$ the ideal $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T -nilpotent.

REMARK. Let A be wild hereditary and $(X_i)_{i \in \mathbb{Z}}$ be any family of indecomposable regular A -modules. Then one can show that there exist $n_i \in \mathbb{Z}$ and maps $f_i: \tau^{n_i} X_i \rightarrow \tau^{n_{i+1}} X_{i+1}$ with $f_i \in \text{rad}^\infty(\text{mod } A)$ such that for all integers j and l with $j < l$ the composition $f_l \cdots f_j$ is nonzero.

(1.3) Let A be any tilted algebra. Then it follows from [R3] and [K] that the following conditions are equivalent

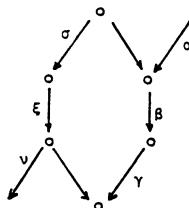
- (i) A is tame.
- (ii) A is domestic.
- (iii) $\text{rad}^\infty(\text{mod } A)$ is nilpotent.

(1.4) Let A be the algebra given by the quiver



bound by the relation $\alpha\beta\gamma = 0$. Thus A is a one-point extension of a hereditary algebra of type $\tilde{\mathbb{A}}_{3,3}$ by an indecomposable nonhomogenous regular module of regular length 2. Then by [R2] A is domestic but by Lemma (4.2) $\text{rad}^\infty(\text{mod } A)$ is not right T -nilpotent. Considering the new Auslander–Reiten component one can show that $\text{rad}^\infty(\text{mod } A)$ is left T -nilpotent.

If B is the algebra given by the quiver



bound by the relations $\alpha\beta\gamma=0$ and $\sigma\xi\nu=0$, then again B is domestic, but $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T -nilpotent.

We don't know any algebra A for which $\text{rad}^\infty(\text{mod } A)$ is not nilpotent but is T -nilpotent on both sides.

(1.5) Let A be a tubular algebra (in the sense of [R3]). We claim that $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T -nilpotent and $\text{rad}^\infty(\text{mod } A) \neq (\text{rad}^\infty(\text{mod } A))^2$ holds:

By [R3, (5.2)] we have

$$\Gamma(A) = \mathcal{P}_0 \vee \left(\bigvee_{\gamma \in \mathbb{Q}_0^\infty} \mathcal{T}_\gamma \right) \vee \mathcal{Q}_\infty$$

where \mathcal{P}_0 is a preprojective component, \mathcal{Q}_∞ is a preinjective component and for any $\gamma \in \mathbb{Q}_0^\infty$ \mathcal{T}_γ is a separating tubular family.

Let P be a projective module, E be an injective module with $\text{Hom}(P, E) \neq 0$ and let $f: P \rightarrow E$ be a nonzero map. Then f factors through \mathcal{T}_1 , that is, there exists a module $X \in \text{add}(\mathcal{T}_1)$ such that $f = h_0g_0$ with $g_0: P \rightarrow X$ and $h_0: X \rightarrow E$.

Let $1 < \gamma_1 < \gamma_2 < \dots$ be a strictly increasing sequence of rational numbers. Inductively we define the maps β_i and h_i by: h_0 factors through \mathcal{T}_{γ_1} , that is $h_0 = h_1\beta_1$ with $\beta_1: X \rightarrow X_1$, $h_1: X_1 \rightarrow E$ with $X_1 \in \text{add}(\mathcal{T}_{\gamma_1})$. For $i > 1$ there exists $X_i \in \text{add}(\mathcal{T}_{\gamma_i})$ such that $h_{i-1} = h_i\beta_i$ with $\beta_i: X_{i-1} \rightarrow X_i$ and $h_i: X_i \rightarrow E$.

For all $l \in \mathbb{N}$ we have

$$h_0 = h_l\beta_l \cdots \beta_1 \quad \text{with } \beta_i \in \text{rad}^\infty(\text{mod } A).$$

Dually, choosing a sequence

$$1 > \delta_1 > \delta_2 > \dots \text{ in } \mathbb{Q}_0^\infty$$

we get a family of maps $\alpha_i \in \text{rad}^\infty(\text{mod } A)$ and $g_i \in \text{rad}^\infty(\text{mod } A)$ such that

$$g_0 = \alpha_1 \dots \alpha_l g_l \quad \text{for all } l \in \mathbb{N}.$$

Therefore $\text{rad}^\infty(\text{mod } A)$ is not T -nilpotent on either side.

Finally, observe that, for $P \in \mathcal{P}_0$ and $Y \in \mathcal{T}_0$ with $\text{Hom}(P, Y) \neq 0$, a nonzero map $f: P \rightarrow Y$ is in $\text{rad}^\infty(\text{mod } A)$ but not in $(\text{rad}^\infty(\text{mod } A))^2$.

(1.6) Let A be a standard nondomestic selfinjective algebra of polynomial growth. Then by [S2, (1.5)] A is isomorphic to \hat{B}/G where \hat{B} is the repetitive algebra of a tubular algebra B and G an admissible infinite cyclic group of k -linear automorphisms of \hat{B} . Using the structure of $\text{mod } \hat{B}$, see [NS, §3] one can show, similarly as in §2, that $\text{rad}^\infty(\text{mod } A) = (\text{rad}^\infty(\text{mod } A))^2$.

(1.7) PROPOSITION. *Let A be an algebra such that $\text{rad}^\infty(\text{mod } A)$ is left or right T -nilpotent. Then A is tame.*

Proof. Suppose A is not tame. Then by [D] (see also [CB]), A is wild, that is, denoting by $k\langle X, Y \rangle$ the free algebra in two noncommuting variables X and Y , there exists a $k\langle X, Y \rangle$ — A -bimodule M , free and finitely generated as a $k\langle X, Y \rangle$ —module such that the functor $F = - \otimes M : \text{mod } k\langle X, Y \rangle \rightarrow \text{mod } A$ preserves indecomposability and isomorphism classes; in particular F is faithful.

Let now H be as in Example (1.2). There exists a family of maps $(f_i : X_i \rightarrow X_{i+1})_{i \in \mathbb{Z}}$ with $f_i \in \text{rad}^\infty(\text{mod } A)$ such that for all $j < i$ the composition $f_i \dots f_j$ is nonzero.

On the other hand, it is well known that there exists a full exact embedding $G : \text{mod } H \rightarrow \text{mod } k\langle X, Y \rangle$.

Applying the faithful functor $FG : \text{mod } H \rightarrow \text{mod } A$ we get a family $(FG(f_i))_{i \in \mathbb{Z}}$ of maps in $\text{rad}^\infty(\text{mod } A)$ with arbitrary long nonzero compositions. Thus $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T -nilpotent.

(1.8) COROLLARY. *Let A be an algebra. Then A is representation-finite if and only if $\text{rad}^\infty(\text{mod } A) = 0$.*

Proof. By (1.1) we have only to show that for representation-infinite algebras $\text{rad}^\infty(\text{mod } A) \neq 0$; so assume A is representation-infinite. For A wild by (1.7) $\text{rad}^\infty(\text{mod } A) \neq 0$. For A tame, by [CB] and the validity of the Brauer–Thrall II conjecture (for a proof, see, for instance [F]) there exist (infinitely many) stable tubes of rank 1. Let M be an indecomposable module from one of these tubes and P be an indecomposable projective module with $\text{Hom}(P, M) \neq 0$. Then P and M are in different components, and therefore $\text{Hom}(P, M) = \text{rad}^\infty(P, M)$.

2. Standard domestic selfinjective algebras

The aim of this section is to prove the following proposition which gives the implication (iv) \Rightarrow (iii) of the theorem.

(2.1) PROPOSITION. *Let A be a representation-infinite domestic standard selfinjective algebra. Then $\text{rad}^\infty(\text{mod } A)$ is nilpotent.*

Proof. We know from [S1, (1.5)] that $A \cong \hat{B}/G$ where B is a representation-infinite tilted algebra of Euclidean type $\Delta_B = \tilde{\Delta}_{p,q}, \tilde{D}_{r+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ with a complete slice in its preinjective component and G is an admissible infinite cyclic group of k -linear automorphism of \hat{B} . Let $F : \hat{B} \rightarrow \hat{B}/G = A$ be the canonical Galois covering with group G and $F_\lambda : \text{mod } \hat{B} \rightarrow \text{mod } A$ the associated push-down functor, see [BG]. By [ANS, (4.3)] we have

$$\Gamma(\hat{B}) = \bigvee_{p \in \mathbb{Z}} (\mathcal{X}_p \vee \mathcal{T}_p)$$

where the components \mathcal{X}_p and \mathcal{T}_p have the following properties: If v_B denotes the Nakayama-automorphism, we have $\mathcal{X}_{2q} = v_B^q(\mathcal{X}_0)$, $\mathcal{X}_{2q+1} = v_B^q(\mathcal{X}_1)$, $\mathcal{T}_{2q} = v_B^q(\mathcal{T}_0)$, $\mathcal{T}_{2q+1} = v_B^q(\mathcal{T}_1)$.

Here $\mathcal{X}_0(\mathcal{X}_1)$, respectively) denotes a component whose stable part is isomorphic to the translation quiver $\mathbb{Z}\Delta_B$, whereas $\mathcal{T}_0(\mathcal{T}_1)$ is a $\mathbb{P}_1(k)$ -family of quasi-tubes, whose stable part is a tubular $\mathbb{P}_1(k)$ -family of tubular type

$$n_B = (p, q), (2, 2, r), (2, 3, 3), (2, 3, 4) \quad \text{or} \quad (2, 3, 5)$$

respectively.

Recall that a translation-quiver \mathcal{T} is called a *quasi-tube* (see [S2]), if the full translation subquiver \mathcal{T}_s of \mathcal{T} consisting of all points of \mathcal{T} which are not projective-injective is a tube in the sense of [R3].

The components \mathcal{X}_p , \mathcal{T}_p ($p \in \mathbb{Z}$) are standard with $\text{Hom}(\mathcal{T}_p, \mathcal{X}_p) = 0$ and $\text{Hom}(\mathcal{X}_q \vee \mathcal{T}_q, \mathcal{X}_p \vee \mathcal{T}_p) = 0$ for $p, q \in \mathbb{Z}$, $q > p$.

Moreover, it was shown in [S1, (2.13)] that G is generated by an element g such that

$$g(\mathcal{X}_p) = \mathcal{X}_{p+m}, \quad g(\mathcal{T}_p) = \mathcal{T}_{p+m} \quad \text{for some } m > 0 \text{ and all } p \in \mathbb{Z}.$$

Further, [ANS, (4.3)] implies that $(\text{rad}^\infty(\text{mod } \hat{B}))^5 = 0$. Finally, [ANS, (4.3)] also states that \hat{B} is locally support-finite. Therefore, by [DS, (2.5)], the push-down functor $F_\lambda: \text{mod } \hat{B} \rightarrow \text{mod } \hat{B}/G$ is a Galois covering, so F_λ is dense and induces isomorphisms

$$\begin{aligned} \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\hat{B}}(M, {}^{g^n}N) &\rightarrow \text{Hom}_A(F_\lambda M, F_\lambda N) \\ \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\hat{B}}({}^{g^n}M, N) &\rightarrow \text{Hom}_A(F_\lambda M, F_\lambda N) \quad \text{for all } M, N \in \text{mod } \hat{B}. \end{aligned}$$

Let M and N be indecomposable \hat{B} -modules and $f: F_\lambda M \rightarrow F_\lambda N$ a nonzero map. Then there exists a (uniquely determined) family of maps $(h_n: M \rightarrow {}^{g^n}N)_{n \in \mathbb{Z}}$ in $\text{mod } \hat{B}$ such that $f = \Sigma F_\lambda(h_n)$.

We claim that for $f \in \text{rad}^\infty(F_\lambda M, F_\lambda N)$ all h_n are in $\text{rad}^\infty(M, {}^{g^n}N)$. It is clear if M and ${}^{g^n}N$ belong to different components of $\Gamma(\hat{B})$. Consider now the case that M and ${}^{g^n}N$ belong to the same component \mathcal{C} . Since $F_\lambda(N) = F_\lambda({}^{g^n}N)$ we may assume that M and N are in the same component \mathcal{C} . Then we have $f = \Sigma_{n \geq 0} F_\lambda(h_n)$ with $h_n \in \text{rad}^\infty(M, {}^{g^n}N)$ for all $n \geq 1$. We shall show that $h_0 = 0$. Suppose $h := h_0 \neq 0$. Then we get $F_\lambda(h) = f - \Sigma_{n \geq 1} F_\lambda(h_n) \in \text{rad}^\infty(F_\lambda M, F_\lambda N)$. Now there exists a natural number m such that $\text{rad}^m(M, N) = 0$. This is trivial if \mathcal{C} is one of the components \mathcal{X}_i and will be shown in (2.2) if \mathcal{C} is a quasi-tube. On the other hand, we have $F_\lambda(h) \in \text{rad}^\infty(F_\lambda M, F_\lambda N) \subset \text{rad}^m(F_\lambda M, F_\lambda N)$. Hence there

exist in \mathcal{C} paths of irreducible maps

$$M \xrightarrow{f_{1,i}} M_{1,i} \xrightarrow{f_{2,i}} \dots \xrightarrow{f_{m,i}} M_{m,i} = M_i, \quad i = 1 \dots r$$

and maps $u_i \in \text{Hom}_A(F_\lambda M_i, F_\lambda N)$ such that

$$F_\lambda(h) = \sum_{i=1}^r u_i F_\lambda(f_{m,i}) \dots F_\lambda(f_{1,i})$$

where all summands $u_i F_\lambda(f_{m,i}) \dots F_\lambda(f_{1,i})$ are nonzero.

Further, since $F_\lambda: \text{mod } \tilde{B} \rightarrow \text{mod } A$ is a Galois covering, for each i , there exist maps $v_{n,i}: M_i \rightarrow {}^{g^n}N$, $n \geq 0$ such that

$$u_i = \sum_{n \geq 0} F_\lambda(v_{n,i}).$$

So we have

$$\begin{aligned} F_\lambda(h) &= \sum_{i=1}^r \sum_{n \geq 0} F_\lambda(v_{n,i} f_{m,i} \dots f_{1,i}) \\ &= F_\lambda \left(\sum_{i=1}^r \sum_{n \geq 0} v_{n,i} f_{m,i} \dots f_{1,i} \right) \end{aligned}$$

with $v_{n,i} f_{m,i} \dots f_{1,i} \in \text{rad}^m(M, {}^{g^n}N)$

Since F_λ induces an isomorphism

$$\bigoplus_{n \geq 0} \text{Hom}(M, {}^{g^n}N) \rightarrow \text{Hom}(F_\lambda M, F_\lambda N)$$

we obtain

$$h = \sum_{i=1}^r \sum_{n \geq 0} v_{n,i} f_{m,i} \dots f_{1,i}$$

and consequently

$$h = \sum_{i=1}^r v_{0,i} f_{m,i} \dots f_{1,i}$$

By our choice of m the group $\text{rad}^m(M, N)$ is zero. Thus all summands $v_{0,i} f_{m,i} \dots f_{1,i}$ are zero and so h , a contradiction.

Let us finally show that $(\text{rad}^\infty(\text{mod } A))^5 = 0$.

Take a chain of maps between indecomposable A -modules

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} X_4 \xrightarrow{f_5} X_5$$

such that $f_i \in \text{rad}^\infty(X_{i-1}, X_i)$.

Let $X_i = F_\lambda(M_i)$ for some indecomposable \hat{B} -module M_i . For each i there exists a family of maps $f_{n,i} \in \text{rad}^\infty(M_{i-1}, {}^g M_i)$ such that $f_i = \sum_{n \in \mathbb{Z}} F_\lambda(f_{n,i})$.

Fix some family $(n_1, \dots, n_5) \in \mathbb{Z}^5$ and take

$$\begin{aligned} \varphi_1 &= f_{n_1,1}, \quad \varphi_2 = g^{n_1} f_{n_2,2}, \quad \varphi_3 = g^{n_1+n_2} f_{n_3,3}, \\ \varphi_4 &= g^{n_1+n_2+n_3} f_{n_4,4} \quad \text{and} \quad \varphi_5 = g^{n_1+n_2+n_3+n_4} f_{n_5,5} \end{aligned}$$

Then we have

$$\begin{aligned} F_\lambda(f_{n_5,5}) \circ \dots \circ F_\lambda(f_{n_1,1}) &= F_\lambda(\varphi_5) \circ \dots \circ F_\lambda(\varphi_1) \\ &= F_\lambda(\varphi_5 \dots \varphi_1). \end{aligned}$$

Since $\varphi_i \in \text{rad}^\infty(\text{mod } \hat{B})$ and $(\text{rad}^\infty(\text{mod } \hat{B}))^5 = 0$ we have $\varphi_5 \dots \varphi_1 = 0$ and therefore $f_5 \dots f_1 = 0$. This shows that $(\text{rad}^\infty(\text{mod } A))^5 = 0$.

(2.2) LEMMA. *Let A be a selfinjective algebra, \mathcal{T} be a standard quasi-tube in $\Gamma(A)$ and X, Y be indecomposables in \mathcal{T} . Then there exists a number m such that $\text{rad}^m(X, Y) = 0$.*

Proof. Let E_i ($i = 1, \dots, r$) be the indecomposable projective-injective modules in \mathcal{T} and take $p \in \mathbb{N}$ such that $\text{rad}^p(\text{End}(E_i)) = 0$ for all $i = 1, \dots, r$.

Notice that \mathcal{T}_s is a stable tube, since A is selfinjective. The modules $X_i = E_i/\text{Soc } E_i$ and $Y_i = \text{rad } E_i$ are in \mathcal{T}_s .

By the standardness of \mathcal{T} it is enough to show that all irreducible paths from X to Y of length at least m are zero.

First we consider the case X and Y not projective-injective. Since \mathcal{T}_s is a stable tube the notion of the regular length $l(M)$ of a point M in \mathcal{T}_s is defined. Let

$$l = \max\{l(X), l(Y), l(X_i), l(Y_i) \mid i = 1, \dots, r\}.$$

Let $\tilde{\mathcal{T}}$ be the universal covering of \mathcal{T} , see [BG]. Then the stable part $\tilde{\mathcal{T}}_s$ of $\tilde{\mathcal{T}}$ is of type $\mathbb{Z}A_\infty$ and we have

$$\begin{array}{ccc} \tilde{\mathcal{T}}_s & \rightarrow & \tilde{\mathcal{T}} \\ \downarrow & & \downarrow F \\ \mathcal{T}_s & \rightarrow & \tilde{\mathcal{T}} = \tilde{\mathcal{T}}/H \end{array}$$

where H is an infinite cyclic group $H = \langle h \rangle$ and F is the covering map, see [BG].

In the mesh-category $k(\tilde{\mathcal{T}}_s)$ every nonzero path between vertices X' , $Y' \in \tilde{\mathcal{T}}_s$ with $l(X') \leq l$ and $l(Y') \leq l$ has length at most $2l$.

Let $N > 2l$ and w be a nonzero irreducible path in $k(\tilde{\mathcal{T}})$ of length N from X' to Y' , where X' and Y' are in $\tilde{\mathcal{T}}_s$ with $l(X') \leq l$ and $l(Y') \leq l$. We claim that in the mesh-category $k(\tilde{\mathcal{T}})$ we have $w = \Sigma w_i$, where w_i are paths from X' to Y' of length N passing through at least $(N - 2l)/2(l + 1)$ projective-injective vertices. Indeed, in $k(\tilde{\mathcal{T}})$ we get $w = \Sigma v_i$, where each v_i is a path from X' to Y' passing through an injective-projective vertex $E(i)$, that is, $v_i = n_1 \alpha \beta n_2$ with

$$X' \xrightarrow{n_1} Y(i) \xrightarrow{\alpha} E(i) \xrightarrow{\beta} X(i) \xrightarrow{n_2} Y'$$

Since $l(X(i)) \leq l$, $l(Y(i)) \leq l$ we can repeat the argument for n_1 and n_2 , if necessary and get the assertion.

Take now $m = 4(l + 1)^2(p + 1)r$ and again X and Y in \mathcal{T}_s with $l(X) \leq l$, $l(Y) \leq l$. If w is an irreducible path of length $N \geq m$ in \mathcal{T} from X to Y , there exists a path \tilde{w} from ${}^1 X$ to ${}^{hi} Y$, for some i , in $\tilde{\mathcal{T}}$ such that $F(\tilde{w}) = w$.

In $k(\tilde{\mathcal{T}})$ we have $\tilde{w} = \Sigma w_i$, where each w_i passes at least through $r(p + 1)$ projective-injective vertices. But then $F(\tilde{w}) = w = \Sigma F(w_i)$ and each path $F(w_i)$ passes through at least $r(p + 1)$ projective-injective modules. Since there are only r projective-injective modules in \mathcal{T} , for each i , we find a fixed projective-injective module $E(i)$ such that $F(w_i)$ meets $E(i)$ at least $p + 1$ times. But $\text{rad}^r(\text{End}(E_i)) = 0$, so $F(w_i) = 0$.

If X or Y is projective-injective then there exists only one irreducible map $X \rightarrow X_1$ with source X and one irreducible map $Y_1 \rightarrow Y$ with target Y . Since X_1 or Y_1 are in \mathcal{T}_s , we can apply the first case.

3. Special biserial algebras

(3.1) Recall that a locally bounded category R is *special biserial*, if it is isomorphic to a bound quiver category $k\mathcal{Q}/I$, where the bound quiver (\mathcal{Q}, I) satisfies the following conditions:

- (1) The number of arrows in \mathcal{Q} with a prescribed source or target is at most two,
- (2) for any arrow α of \mathcal{Q} , there is at most one arrow β and one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I .

A triangular locally bounded category R is called *gentle* if it is isomorphic to $k\mathcal{Q}/I$, where the bound quiver (\mathcal{Q}, I) satisfies (1), (2) and the following two conditions:

- (3) I is generated by a set of paths of length two,

- (4) For any arrow α of \mathcal{Q} there is at most one arrow ξ and at most one arrow η such that $\alpha\xi$ and $\eta\alpha$ belong to I .

By a *tree category* we mean a locally bounded category whose ordinary quiver is a tree.

It was shown in [PS] that a special biserial selfinjective algebra A admits a Galois covering $R \rightarrow A = R/G$ where R is a simply connected locally bounded category $R = \hat{B}$, for some gentle tree B , and G is an admissible torsion-free group of k -linear automorphisms of R . Here, \hat{B} denotes the repetitive algebra of B (see [HW, S1]).

(3.2) PROPOSITION. Let R be a special biserial simply connected selfinjective locally bounded category, G an admissible group of k -linear automorphisms of R and assume additionally that $A = R/G$ is nondomestic. Then $\text{rad}^\infty(\text{mod } R)$ and $\text{rad}^\infty(\text{mod } A)$ are neither left nor right T -nilpotent.

Proof. We first consider $\text{rad}^\infty(\text{mod } R)$. From the proof of [ES, (2.4)] we see that R contains a full convex subcategory D of the following form

bound only by $\alpha_i \beta_i$, $\sigma_i \gamma_i$ ($i \in \mathbb{Z}$), where may be $v_i = x_i$. We may also assume that the number of objects on the line

$$x_0 \rightarrow \cdots \leftarrow x_1$$

is greater than $\dim_k A$.

By M_n , we denote the indecomposable R -module whose support is the line

$$v_0 \leftarrow \cdots \leftarrow y_0 \leftarrow x_0 \rightarrow z_0 \rightarrow \cdots \rightarrow v_n \leftarrow \cdots \leftarrow y_n \leftarrow x_n$$

It was shown in [ES, (2.4)] that the modules M_n , $n \geq 1$ belong to pairwise different Auslander–Reiten component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. Hence the canonical projections $f_n: M_{n+1} \rightarrow M_n$ are in $\text{rad}^\infty(M_{n+1}, M_n)$. Obviously for all $l \in \mathbb{N}$ $f_1 \circ \dots \circ f_l \neq 0$ holds, that is, $\text{rad}^\infty(\text{mod } R)$ is not right T -nilpotent.

Considering the sequence $(X_n)_{n \in \mathbb{N}}$ of indecomposable R -modules whose supports are the lines

$$v_{-n} \leftarrow \cdots \leftarrow y_{-n} \leftarrow x_{-n} \rightarrow \cdots \rightarrow v_0 \leftarrow \cdots \leftarrow x_0$$

we get a chain of canonical monomorphisms

$$X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3 \xrightarrow{g_3} \dots$$

Similarly, as for the modules M_n , we can also show that the modules X_n , $n \geq 1$, belong to pairwise different Auslander–Reiten components of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. Thus we have $g_i \in \text{rad}^\infty(X_n, X_{n+1})$ and therefore $\text{rad}^\infty(\text{mod } R)$ is not left T -nilpotent.

The assertion for $\text{rad}^\infty(\text{mod } A)$ then follows from the following lemma.

(3.3) LEMMA. *Let R be a locally bounded category and G be a group of k -linear automorphisms of R acting freely on the objects of R . Assume that $\text{rad}^\infty(\text{mod } R/G)$ is right (left) T -nilpotent. Then $\text{rad}^\infty(\text{mod } R)$ is right (left) T -nilpotent.*

Proof. If $\text{rad}^\infty(\text{mod } R)$ is not right T -nilpotent then there exists a chain

$$M_0 \xleftarrow{f_0} M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} \dots$$

in $\text{mod } R$ with $f_i \in \text{rad}^\infty(M_{i+1}, M_i)$ such that $f_0 \dots f_l \neq 0$ for all $l \geq 0$. Applying the push down functor $F_\lambda: \text{mod } R \rightarrow \text{mod } R/G$ we see that $F_\lambda(f_i) \in \text{rad}^\infty(F_\lambda M_{i+1}, F_\lambda M_i)$ with $0 \neq F_\lambda(f_0) \dots F_\lambda(f_l) = F_\lambda(f_0 \dots f_l)$ for all l .

Thus $\text{rad}^\infty(\text{mod } R/G)$ is not right T -nilpotent.

(3.4) The following is a direct consequence of (2.1), (3.1) and the structure theorem [S, (1.5)] for standard domestic selfinjective algebras.

COROLLARY. *Let A be a special biserial selfinjective algebra. Then A is domestic if and only if $\text{rad}^\infty(\text{mod } A)$ is nilpotent.*

4. Preparatory lemmas

We will present here several lemmas, needed for the remaining parts of the proof of our main result.

(4.1) LEMMA. *Let A be a category, B a full subcategory of A and assume that $\text{rad}^\infty(\text{mod } A)$ is right (left) T -nilpotent. Then $\text{rad}^\infty(\text{mod } B)$ is right (left) T -nilpotent.*

Proof. Let $E_0: \text{mod } A \rightarrow \text{mod } B$ be the restriction functor associated with the full embedding $E: B \rightarrow A$. Let $E_\lambda: \text{mod } B \rightarrow \text{mod } A$ be a left adjoint functor to E_0 such that $E_0 E_\lambda \cong \mathbb{1}_{\text{mod } B}$. Then the lemma follows from the fact of E_λ being faithful.

(4.2) LEMMA. *Let B be an algebra whose Auslander–Reiten quiver contains a tube \mathcal{T} . Assume that M is an indecomposable module in \mathcal{T} such that the full subcategory \mathcal{X} of the vector space category $\text{Hom}(M, \text{mod } B)$ consisting of the*

objects $\{\text{Hom}(M, X) \neq 0 \mid X \in \mathcal{T}\}$ has a cofinite, full, successor-closed subcategory of the form

$$\begin{array}{ccccccc} \text{Hom}(M, N_1) & \rightarrow & \text{Hom}(M, N_2) & \rightarrow & \text{Hom}(M, N_3) & \rightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Hom}(M, M_1) & \rightarrow & \text{Hom}(M, M_2) & \rightarrow & \text{Hom}(M, M_3) & \rightarrow & \cdots \end{array}$$

where the modules M_i, N_i are neither projective nor injective.

If $B[M]$ denotes the one-point extension of B by M , then $\text{rad}^\infty(\text{mod } B[M])$ is not right T -nilpotent.

Proof. Let X be a point from the mouth of \mathcal{T} such that there exists an infinite sectional path

$$\Sigma: \cdots \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots \rightarrow X.$$

This path meets the sectional path pointing to infinity

$$N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow \cdots$$

at the modules $N'_i = N_{p_0+ir}$ for some fixed p_0 and r .

Let $\psi_i: N'_{i+1} \rightarrow N'_i$ be the composition of the canonical maps lying on the sectional path Σ . Then we have $\psi_1 \circ \cdots \circ \psi_l \neq 0$ for all $l \geq 1$. We claim that $\psi_i: (0, N'_{i+1}, 0) \rightarrow (0, N'_i, 0)$ considered as map in $\text{mod } B[M]$ is in $\text{rad}^\infty(\text{mod } B[M])$ and thus $\text{rad}^\infty(\text{mod } B[M])$ is not right T -nilpotent.

Since $\text{Hom}(M, \psi_i): \text{Hom}(M, N'_{i+1}) \rightarrow \text{Hom}(M, N'_i)$ is zero, we get a nonzero map

$$(0, \psi_i): (k, N'_{i+1}, 1) \rightarrow (0, N'_i, 0).$$

Thus we have the following diagram of indecomposable modules

$$\begin{array}{ccc} (0, N'_{i+1}, 0) & \xrightarrow{\psi_i} & (0, N'_i, 0) \\ \downarrow \alpha_1 & & \uparrow (0, \psi_i) \\ (k, N'_{i+1} \oplus M_{p_0+ir+1}, \Delta) & \xrightarrow{\beta_1} & (k, N'_{i+1}, 1) \\ \downarrow \alpha_2 & & \uparrow \text{id} \\ (k, N'_{i+1} \oplus M_{p_0+ir+2}, \Delta) & \xrightarrow{\beta_2} & (k, N'_{i+1}, 1) \\ \downarrow \alpha_3 & & \uparrow \text{id} \\ (k, N'_{i+1} \oplus M_{p_0+ir+3}, \Delta) & \xrightarrow{\beta_3} & (k, N'_{i+1}, 1) \\ \downarrow \alpha_4 & & \uparrow \text{id} \\ \vdots & & \vdots \end{array}$$

where $\Delta: k \rightarrow \text{Hom}(M, N'_{i+1} \oplus M_{p_0 + ir + l})$ denotes the diagonal map, β_l the canonical projections, α_1 the canonical monomorphism and $\alpha_l (l \geq 2)$ the obvious maps induced by $M_{p_0 + ir + l} \rightarrow M_{p_0 + ir + l + 1}$. Clearly we have

$$\psi_i = (0, \psi_i) \circ \beta_l \circ \alpha_l \circ \dots \circ \alpha_1$$

for all $l \geq 1$, that is, $\psi_i \in \text{rad}^\infty(\text{mod } B[M])$.

(4.3) LEMMA. *Let B be an algebra whose Auslander–Reiten quiver contains (not necessarily different) tubes \mathcal{T}_1 and \mathcal{T}_2 . Assume M_1 is an indecomposable module in \mathcal{T}_1 and N_1 is an indecomposable module in \mathcal{T}_2 such that the full subcategory \mathcal{Y} of the vector space category $\text{Hom}(M_1 \oplus N_1, B\text{-mod})$ defined by the objects $\{\text{Hom}(M_1 \oplus N_1, Y) \neq 0 \mid Y \in \mathcal{T}_1 \cup \mathcal{T}_2\}$ is of the form*

$$\begin{aligned} \text{Hom}(M_1, M_1) &\rightarrow \text{Hom}(M_1, M_2) \rightarrow \text{Hom}(M_1, M_3) \rightarrow \dots \\ \text{Hom}(N_1, N_1) &\rightarrow \text{Hom}(N_1, N_2) \rightarrow \text{Hom}(N_1, N_3) \rightarrow \dots \end{aligned}$$

If $B[M_1 \oplus N_1]$ denotes the one-point extension of B by $M_1 \oplus N_1$, then $\text{rad}^\infty(\text{mod } B[M_1 \oplus N_1])$ is not right T -nilpotent.

Proof. Keeping the notations of (4.2) we analogously get a commutative diagram of indecomposable modules

$$\begin{array}{ccc} (0, N'_{i+1}, 0) & \xrightarrow{\psi_i} & (0, N'_i, 0) \\ \downarrow \alpha_1 & & \uparrow (0, \psi_i) \\ (k, N'_{i+1} \oplus M_1, \Delta) & \xrightarrow{\beta_1} & (k, N'_{i+1}, 1) \\ \downarrow \alpha_2 & & \uparrow \text{id} \\ (k, N'_{i+1} \oplus M_2, \Delta) & \xrightarrow{\beta_2} & (k, N'_{i+1}, 1) \\ \downarrow \alpha_3 & & \uparrow \text{id} \\ \vdots & & \vdots \end{array}$$

Again we have

$$\psi_i = (0, \psi_i) \beta_l \alpha_l \dots \alpha_1 \quad \text{for all } l \in \mathbb{N},$$

that is,

$$\psi_i \in \text{rad}^\infty(\text{mod } B[M_1 \oplus N_1])$$

with

$$\psi_1 \circ \cdots \psi_t \neq 0 \quad \text{for all } t \in \mathbb{N}.$$

(4.4) LEMMA. *Let $B = C[M]$ be a one-point extension of a tame concealed algebra C with extension vertex a_0 by a simple regular module M and let D be the category obtained from B by rooting the infinite branch L*

$$\begin{array}{ccccccccccccc} \circ & \rightarrow & \circ & \leftarrow & \circ & \rightarrow & \cdots & \rightarrow & \circ & \leftarrow & \circ & \rightarrow & \circ & \leftarrow & \cdots \\ a_0 & & a_1 & & a_2 & & & & a_{2i-1} & & a_{2i} & & a_{2i+1} & & \end{array}$$

at the vertex a_0 . Then $\text{rad}^\infty(\text{mod } D)$ is not right T -nilpotent.

Proof. Let \mathcal{T} be the tube of $\Gamma(B)$ containing the module M . Consider in \mathcal{T} the sectional paths

$$\Sigma: M = M_0 \xrightarrow{g_1} M_1 \xrightarrow{g_2} M_2 \rightarrow \cdots$$

starting at M and pointing to infinity and

$$\Omega: N_0 \xleftarrow{h_1} N_1 = M_1 \xleftarrow{h_2} N_2 \xleftarrow{h_3} N_3 \xleftarrow{\dots}$$

containing M_1 and pointing to the mouth of \mathcal{T} .

Then there is a sequence $1 = i_1 < i_2 < i_3 < \dots$ such that the modules $X_r = N_{i_r}$, $r \geq 1$ form the intersection of both paths Σ and Ω .

Denote by $f_r: X_{r+1} \rightarrow X_r$ the composition $f_r = h_{i_r+1} h_{i_r+2} \circ \cdots \circ h_{i_{r+1}}$. Obviously all f_r are surjective and thus $f_1 \circ \cdots \circ f_l \neq 0$ for all $l \in \mathbb{N}$.

We claim that $f_r \in \text{rad}_D^\infty(X_{r+1}, X_r)$ for all $r \geq 1$ which shows that $\text{rad}^\infty(\text{mod } D)$ is not right T -nilpotent.

Fix some $r \geq 1$. Let Y_i (Z_i , respectively) for $i \geq 1$ be the indecomposable D -module such that the restriction of Y_i (Z_i) to C is X_{r+1} (M , respectively) and the restriction of Y_i and Z_i to L is the unique indecomposable module whose support is the finite branch

$$a_0 \rightarrow a_1 \leftarrow \cdots \rightarrow a_{2i-1}.$$

For all $i \geq 1$ we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 \rightarrow & M & \xrightarrow{\alpha} & X_{r+1} & \xrightarrow{\beta} & U_r & \rightarrow 0 \\
 & \downarrow & & \downarrow \gamma_1 & & \downarrow \text{id} & \\
 0 \rightarrow & Z_1 & \longrightarrow & Y_1 & \xrightarrow{p_1} & U_r & \rightarrow 0 \\
 & \downarrow & & \downarrow \gamma_2 & & \downarrow \text{id} & \\
 0 \rightarrow & Z_2 & \longrightarrow & Y_2 & \xrightarrow{p_2} & U_r & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & \\
 0 \rightarrow & Z_{i-1} & \longrightarrow & Y_{i-1} & \xrightarrow{p_{i-1}} & U_r & \rightarrow 0 \\
 & \downarrow & & \downarrow \gamma_i & & \downarrow \text{id} & \\
 0 \rightarrow & Z_i & \longrightarrow & Y_i & \xrightarrow{p_i} & U_r & \rightarrow 0
 \end{array}$$

where $U_r = N_{i_{r+1}-1}$, $\beta = h_{i_{r+1}}$, α is the composition

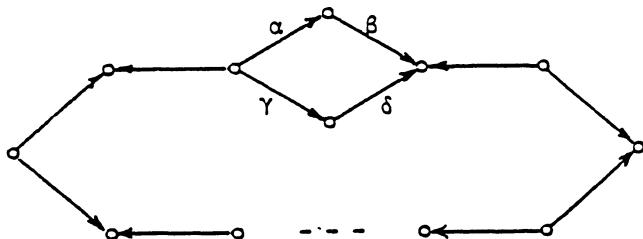
$$M \xrightarrow{g_1} M_1 \xrightarrow{g_2} M_2 \rightarrow \dots \xrightarrow{g_j} M_j = X_{r+1},$$

and the maps $p_i(\gamma_i)$ are the obvious epimorphisms (monomorphisms). Then for any $i > 1$ we have

$$\beta = p_i \gamma_i \dots \gamma_1,$$

that is, $\beta \in \text{rad}_D^\infty(X_{r+1}, U_r)$. But $f_r = f'_r \beta$ then also is contained in $\text{rad}^\infty(\text{mod } D)$ and we are done.

(4.5) LEMMA. *Let D be an algebra given by the quiver*



bound by $\alpha\beta = \alpha\gamma\delta$, $\alpha \in k^*$, and these are the only paths of length at least two. Then $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T -nilpotent.

Proof. Imitate the proof of [AS1, Lemma 3.3(III)] using (3.3), (4.1) and (4.2).

(4.6) LEMMA. *Let B an algebra given by the quiver*

$$\begin{array}{ccccccc} & a_1 & - & \cdots & - & a_t & \\ & \swarrow & & & & \searrow & \\ c_1 & - & c_2 & - & \cdots & - & c_{r-1} & - & c_r & = & a_{t+1} \\ & \searrow & & & & \swarrow & \\ & c_s & - & \cdots & - & c_{r+1} & \end{array}$$

bound only by zero-relations and such that the category C formed by the edges $c_1 - c_2 - \cdots - c_s - c_1$ is hereditary of type $\tilde{\mathbb{A}}_{s-1}$. Then $\text{rad}^\infty(\text{mod } \hat{B})$ is neither left nor right T -nilpotent.

Proof. Taking a suitable full subcategory of B , by (4.1), we may assume that the walks $c_1 - a_1 - \cdots - a_t - a_{t+1} = c_r$, $c_1 - c_2 - \cdots - c_r$ and $c_1 - c_s - \cdots - c_r$ have radical square zero.

We may also assume that B is gentle. Indeed, if one of the subcategories \mathcal{C} formed by the objects a_1, c_1, c_2, c_s or $a_t, c_{r-1}, c_r, c_{r+1}$ is not gentle, then $\text{mod } \hat{B}$ contains a subcategory $\text{mod } H$, where H is a wild hereditary algebra obtained from C by a one-point extension or coextension, and we are done.

If B is gentle, then $T(B)$ is special biserial and nondomestic (see [ANS, (5.2)]). Observe that there are Galois coverings $R \xrightarrow{F'} \hat{B}$, $\hat{B} \xrightarrow{F''} T(B)$ and $R \xrightarrow{F''F'} T(B)$ with R special biserial selfinjective and simply connected. Then, by (3.2), $\text{rad}^\infty(\text{mod } R)$ is neither left nor right T -nilpotent and consequently $\text{rad}^\infty(\text{mod } \hat{B})$ is neither left nor right T -nilpotent, by (3.3).

5. Repetitive algebras

For the proof of the implications (i) \Rightarrow (v) and (ii) \Rightarrow (v) we need the description of some representation-infinite algebra B for which $\text{rad}^\infty(\text{mod } \hat{B})$ is left or right T -nilpotent. Again \hat{B} denotes the repetitive algebra of B , see [HW, S1]. The Nakayama-automorphism of \hat{B} is denoted by v_B .

(5.1) PROPOSITION. *Let Λ be a finite category which contains a tame concealed full convex subcategory C . Assume that $\text{rad}^\infty(\text{mod } \hat{\Lambda})$ is right (left) T -nilpotent. Then Λ is a domestic branch enlargement of C (in the sense of [AS, (2.2)]) and $\hat{\Lambda} \cong \hat{B}$ for some representation-infinite tilted algebra B of Euclidean type.*

Proof. Assume that $\text{rad}^\infty(\text{mod } \hat{\Lambda})$ is right T -nilpotent. The proof is done by the following series of lemmas:

(5.2) LEMMA. Let $B=C[M]$ be a one-point extension of C which is a full subcategory of $\hat{\Lambda}$. Then M is a simple regular C -module.

Proof. Imitate the proof of [AS2, Lemma 4.4] using (1.7), (4.2) and (4.3).

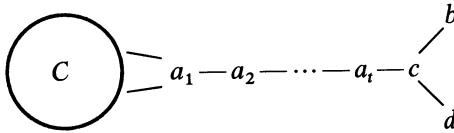
(5.3) LEMMA. Let $B=C[M]$ be a one-point extension of C by a simple regular C -module M . Then $A=B[M]$ is not a full subcategory of $\hat{\Lambda}$.

Proof. The assertion follows from (4.2).

(5.4) LEMMA. Let $B=C[M]$ be a one-point extension of C by a simple regular C -module M , with extension vertex a , and $D=B[X]$ be a one-point extension of B with extension vertex b . Suppose D is a full subcategory of Λ and let N be an indecomposable direct summand of X , containing the simple module $S(a)$ in its top. Then either $N \cong P(a)_B$ or $N \cong S(a)_B$ holds.

Proof. Follow the proof of [AS2, Lemma 4.6] using (5.2).

(5.5) LEMMA. Λ does not contain a full subcategory K of the form



where the full subcategory formed by the objects a_t, b, c, d is hereditary.

Proof. Suppose A contains a full subcategory of the above type. We can additionally assume that the walk $a_1—a_2—\cdots—a_t—c$ has radical-square zero. Applying, if necessary, the Nakayama-automorphism v_Λ on K , we may assume that a_1 is an extension vertex.

By (5.2) the largest C -submodule of $P(a_1)_K$ is a simple regular C -module. Thus, if $a_2 \rightarrow a_1$, by (5.4), we can assume that $S(a_1)$ is a direct summand of $\text{rad } P(a_2)$. Hence $\hat{\Lambda}$ contains a full subcategory \hat{L} , where L has the same form as K but in which the radical-square zero walk $a_1—a_2—\cdots—a_t$ is not bound and the edge $a_1—a_2$ is oriented as $a_1 \rightarrow a_2$.

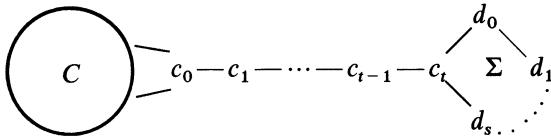
Since L is a full subcategory of $\hat{\Lambda}$, also $\text{rad}^\infty(\text{mod } L)$ is right T -nilpotent. Additionally we may assume that the edge $c—d$ is oriented as follows $c \leftarrow d$. Otherwise we apply an APR-tilting module to the vertex d , and the endomorphism algebra L' has the property that $\text{rad}^\infty(\text{mod } L')$ is right T -nilpotent if and only if $\text{rad}^\infty(\text{mod } L)$ is right T -nilpotent.

Let E be the full subcategory of L consisting of all objects of L except d . Then $L = E[P(c)_E]$ and E is a truncated branch extension of C (that is, a tubular extension in the sense of [R3, (4.7)]). Since $\text{rad}^\infty(\text{mod } E)$ is T -nilpotent by (1.5), (1.7), [NS, (2.1)] and [R3, (4.9)] E is a representation-infinite tilted algebra of Euclidean type with a complete slice in its preinjective component.

But the vector space category $\text{Hom}_E(P(c)_E, \text{mod } E)$ satisfies the conditions of

Lemma (4.2) and therefore $\text{rad}^\infty(\text{mod } L)$, and hence $\text{rad}^\infty(\text{mod } \hat{\Lambda})$ is not right T -nilpotent, a contradiction.

(5.6) LEMMA. Λ does not contain a full subcategory K of the form



where Σ is a noncommutative cycle.

Proof. Taking, if necessary, value of v_Λ on K we may assume that c_0 is an extension vertex. Then imitate the proof of [AS2, Lemma 4.8] using (4.4), (5.2), (5.4) and (5.5).

(5.7) LEMMA. Let a and b be two objects of Λ outside C each of them connected to C by an edge. Then any walk in Λ connecting a and b must intersect C .

Proof. Follow the proof of [AS2, Lemma 4.9] using (4.4), (5.2), (5.6) and, if necessary, the reflection v_Λ .

(5.8) LEMMA. Λ is a branch enlargement of C .

Proof. Repeat the proof of [AS2, Lemma 4.10] using reflection operators and (5.2)–(5.7).

(5.9) LEMMA. Λ is a domestic branch enlargement of C . Then we have $\hat{\Lambda} \cong \hat{B}$ for some representation-infinite tilted algebra B of Euclidean type with a complete slice in its preinjective component.

Proof. By [ANS, (2.6)] there exists a truncated branch extension B of C , obtained from Λ by a sequence of reflections, with the same tubular type, such that $\hat{\Lambda} \cong \hat{B}$.

Since $\text{rad}^\infty(\text{mod } \hat{\Lambda}) \cong \text{rad}^\infty(\text{mod } \hat{B})$ is right T -nilpotent by (1.5), (1.7) and [NS, (2.1)] B is a domestic truncated branch extension of C , and hence, by [R3, (4.9)], B is the required tilted algebra.

6. The proof of the theorem

The implications (iii) \Rightarrow (i) and (iii) \Rightarrow (ii) are obvious. The equivalence of (iv) and (v) has been shown in [S2, (1.5)]. The implication (iv) \Rightarrow (iii) was proved in Section 2.

We shall now prove that (i) \Rightarrow (v), the implication (ii) \Rightarrow (v) is dual.

Let A be a standard representation-infinite selfinjective algebra such that $\text{rad}^\infty(\text{mod } A)$ is right T -nilpotent. Then $A \cong R/G$ where R is a selfinjective simply

connected locally bounded category and G is an admissible group of k -linear automorphism of R .

We denote by v the Nakayama-automorphism of R : For each object x of R the object $v(x)$ is defined by the property that the top of $P(x)$ is the socle of $P(v(x))$, where $P(y)$ denotes the indecomposable projective R -module at y . A v -slice of R is a full subcategory of R which is connected and does not contain two objects from the same v -orbit.

Since $\text{rad}^\infty(\text{mod } A)$ is right T -nilpotent, by (3.3) and (1.7) also $\text{rad}^\infty(\text{mod } R)$ is right T -nilpotent and R is tame. We have to consider two cases:

(i) Every full finite subcategory of R is representation-finite. Then by [PS] R is a special biserial selfinjective category and, by (3.2), A is domestic. Consequently, by [S2, (1.5)] we have $A \cong \hat{B}/H$ where B is a representation-infinite tilted algebra of type $\hat{\mathbb{A}}_n$ and H is an admissible infinite cyclic group of k -linear automorphisms of \hat{B} .

(ii) R contains a representation-infinite full finite subcategory. Using the Lemmas (4.5) and (4.6) and the tameness of R we see, as in [S2, (4.3)], that R contains a tame concealed full convex subcategory C . Obviously C is a convex v -slice of R .

For an arbitrary convex v -slice D of R we denote by D^+ (resp. D^-) the full subcategory of R consisting of the objects of D and all objects x of R satisfying the following two conditions:

- (a) the v -orbit of x does not intersect D .
- (b) $R(x, y) \neq 0$ ($R(y, x) \neq 0$, respectively) for some objects y of D .

Similarly as in [S2, (4.4)], using (5.1) we see that D^+ , (D^- , respectively) is again a convex v -slice of R . Consider the following sequence of v -slices

$$C_0 \subset C_2 \subset C_3 \subset \dots$$

where $C_0 = C$, $C_{2n-1} = C_{2n-2}^+$, $C_{2n} = C_{2n-1}^-$ for $n \geq 1$ and let $B = \bigcup_{n \in \mathbb{N}} C_n$. Obviously B is a convex v -slice of R .

We claim that $R = \hat{B}$. Suppose \hat{B} is a proper subcategory of R . Since R is connected, then there is an object $x \in R$, $x \notin \hat{B}$ connected by an arrow to B . Then $v^r x$, for some $r \in \mathbb{Z}$, is connected by an arrow to B and, since $B = \bigcup_{n \in \mathbb{N}} C_n$, the object $v^r x$ belongs to B . Thus $R = \hat{B}$ holds.

For each $n \in \mathbb{N}$, \hat{C}_n is a full subcategory of $\hat{B} = R$ and so, by (4.1), $\text{rad}^\infty(\text{mod } \hat{C}_n)$ is right T -nilpotent. Hence C_n is a domestic branch enlargement of C , by (5.1). Moreover, by (4.4) B cannot contain an infinite line, that is, $B = C_n$ for some n . Finally, it follows from the construction of C_n that $B = C_n$ is in fact a truncated branch extension of C . Then B is a representation-infinite tilted algebra of Euclidean type \tilde{D}_n or \tilde{E}_p ($n \geq 4$, $p = 6, 7, 8$, \tilde{A}_n is excluded since R is simply connected) with a complete slice in its preprojective component.

By [S2, (2.13), (2.14)] G is an infinite cyclic group. Therefore we have proved that A has the required form $A \cong \widehat{B}/G$.

Note added in proof

Recently K. Bongartz informed us about a more general version of Corollary 1.8:

PROPOSITION. *Let R be an artinian ring, such that all indecomposable injective modules have finite length. Then R is representation-finite if and only if $\text{rad}^\infty(\text{mod } R) = 0$.*

The proof he gave, is surprisingly simple: If R is of finite type, as above the assertion follows from the Harada-Sai lemma.

If R is not representation-finite, by [1, 3.1] there exists an infinite sequence in $\text{mod } R$

$$U_0 \xrightarrow{f_0} U_1 \xrightarrow{f_1} U_2 \xrightarrow{f_2} \dots$$

with U_i indecomposable and f_i nontrivial monomorphisms, or an infinite sequence

$$V_0 \xleftarrow{g_0} V_1 \xleftarrow{g_1} V_2 \xleftarrow{g_2} \dots$$

with V_i indecomposable in $\text{mod } R$ and g_i nontrivial epimorphisms, for all i .

In the first case for the injective hull

$$\varepsilon: U_0 \rightarrow E(U_0)$$

we get

$$\varepsilon = \varepsilon_i f_i \cdots f_0,$$

for some $\varepsilon_i \in \text{Hom}(U_{i+1}, E(U_0))$, that is

$$\varepsilon \in \text{rad}^\infty(\text{mod } R).$$

In the latter case, analogously we consider the projective cover $\pi: P(V_0) \rightarrow V_0$.

Reference

- [1] M. Auslander: Representation theory of Artin algebras II. *Comm. Algebra* 1 (1974), 269–310.

References

- [AS1] I. Assem, A. Skowroński: On some classes of simply connected algebras. *Proc. London Math. Soc.* 56 (1988), 417–450.

- [AS2] I. Assem, A. Skowroński: Algebras with cycle-finite derived categories. *Math. Ann.* 280 (1988), 441–463.
- [ANS] I. Assem, J. Nehring, A. Skowroński: Domestic trivial extensions of simply connected algebras. *Tsukuba J. Math.* 13 (1989), 31–72.
- [BG] K. Bongartz, P. Gabriel: Covering spaces in representation theory. *Invent. Math.* 65 (1982), 331–378.
- [CB] W.W. Crawley-Boevey: On tame algebras and BOCSES. *Proc. London Math. Soc.* 56 (1988), 451–483.
- [D] Ju. A. Drozd: Tame and wild matrix problems. *Proc. ICRA II* (Ottawa 1979), Springer Lecture Notes in Mathematics 832 (1980), 242–258.
- [DS] P. Dowbor, A. Skowroński: Galois coverings of representation-infinite algebras. *Comment. Math. Helv.* 62 (1987), 311–337.
- [ES] K. Erdmann, A. Skowroński: On Auslander–Reiten components of blocks and selfinjective biserial algebras. To appear.
- [F] U. Fischbacher: Une nouvelle preuve d'une théorème de Nazarova et Roiter. *C. R. Acad. Sci. Paris*, t. 300, Série I, Nr. 9 (1985), 259–262.
- [HW] D. Hughes, J. Waschbüsch: Trivial extensions of tilted algebras. *Proc. London Math. Soc.* 46 (1983), 347–364.
- [K] O. Kerner: Tilting wild algebras. *J. London Math. Soc.* 39 (1989), 29–47.
- [NS] F. Nehring, A. Skowroński: Polynomial growth trivial extensions of simply connected algebras. *Fundamenta Math.* 132 (1989), 117–134.
- [PS] Z. Pogorzały, A. Skowroński: Selfinjective biserial standard algebras. To appear in *J. Algebra*.
- [R1] C.M. Ringel: Report on the Brauer–Thrall conjectures: Roiter's theorem and the theorem of Nazarova and Roiter. In *Representation theory I*, Springer Lecture Notes in Mathematics 831 (1980), 104–136.
- [R2] C.M. Ringel: Tame algebras. In *Representation theory I*, Springer Lecture Notes in Mathematics 831 (1980), 137–287.
- [R3] C.M. Ringel: Tame algebras and integral quadratic forms. Springer Lecture Notes in Mathematics 1099 (1984).
- [S1] A. Skowroński: Algebras of polynomial growth. In *Topics in Algebra*, Banach Center 177–199.
- [S2] A. Skowroński: Algebras of polynomial growth. In *Topics in Algebra*, Banach Center Publications, vol. 26, to appear.