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On $*$-representations of the Hopf $*$-algebra associated with the quantum group $U_q(n)$

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Abstract. The properties of $*$-representations of the Hopf $*$-algebra $\mathrm{Pol}(U_q(n))$ are investigated. We consider this Hopf $*$-algebra as a deformation of the algebra of polynomials on the group $U(n)$. The algebraic structure of $\mathrm{Pol}(U_q(n))$ is studied in some more detail in order to build $*$-representations of $\mathrm{Pol}(U_q(n))$ by means of a Verma module construction. The irreducible $*$-representations are classified. By use of these $*$-representations we can complete the Hopf $*$-algebra $\mathrm{Pol}(U_q(n))$ into a type I C^*-algebra, which is a quantum group in the sense of Woronowicz.

1. Introduction

Quantum groups were recently introduced by Drinfeld [4], Jimbo [7] and Woronowicz [21]. In Jimbo's approach, applicable to all root systems, quantum groups are deformations of the universal enveloping algebra of a semisimple Lie group and Drinfeld defines a quantum group as the spectrum of a Hopf algebra.

In this paper we will use the Woronowicz approach to quantum groups. A quantum group is to be considered as a deformation of the C^*-algebra of continuous functions on a compact group, see [20] and [22] for the quantum group $SU_q(n)$. For these quantum groups Woronowicz has obtained quantum analogues for the Haar measure, the Peter-Weyl theory and the Schur orthogonality relations, cf [21]. These tools permit us to do harmonic analysis on such quantum groups. There are, for instance, many quantum group theoretic interpretations of orthogonal polynomials of q-hypergeometric type on the quantum group $SU_q(2)$. Consult the survey article by Koornwinder [9] for further references on this subject.

We will start with a deformation of the algebra of polynomials on $U(n)$. This algebra, $\mathrm{Pol}(U_q(n))$, is a Hopf $*$-algebra and it will be introduced in section 2, together with a bialgebra $\mathscr{A}_q(n)$, which will be useful in some proofs. Our goal is to complete this algebra $\mathrm{Pol}(U_q(n))$ into a unital C^*-algebra in which $\mathrm{Pol}(U_q(n))$ is dense. This will be done in section 5 and the natural way to do this is to consider $*$-representations of $\mathrm{Pol}(U_q(n))$.

In section 3 we investigate the algebraic structure of $\mathrm{Pol}(U_q(n))$ in more detail.

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We will prove a Poincaré-Birkhoff-Witt theorem for $\mathcal{A}_q(n)$ and we will construct an explicit basis for $\mathrm{Pol}(U_q(n))$. See Yamane [23] and Rosso [14] for similar statements for the Jimbo quantum group. With these results at our disposal we can construct irreducible representations by means of a Verma module construction.

Irreducible and primary $*$-representations of $\mathrm{Pol}(U_q(n))$ will be considered in section 4. We derive some properties of these representations by induction with respect to n, which can be done by our main theorem 4.7. In Section 4 we also answer the question which of the irreducible representations constructed in section 3 are irreducible $*$-representations of $\mathrm{Pol}(U_q(n))$. Then it is proved that we obtain all irreducible $*$-representations in this way.

Finally, $\mathrm{Pol}(U_q(n))$ is completed into a unital C^*-algebra $C(U_q(n))$ in section 5. We prove that this C^*-algebra yields a quantum group in the sense of Woronowicz, so we have reached our goal. Our last theorem states that the C^*-algebra $C(U_q(n))$ is a type I C^*-algebra.

To end this section we state the definition of a bialgebra, a Hopf algebra (see Sweedler [17]) and a Hopf $*$-algebra. Let A be an algebra over $\mathbf{C}$ with unit I and denote the multiplication by $m: A \otimes A \to A$; $a \otimes b \mapsto ab$ and the unit by $\varepsilon: \mathbf{C} \to A$; $z \mapsto zI$. A is a *bialgebra* if unital algebra homomorphisms $\Phi: A \to A \otimes A$, $e: A \to \mathbf{C}$ exist, so that

$$(\Phi \otimes id) \circ \Phi = (id \otimes \Phi) \circ \Phi, \tag{1.1}$$

$$(e \otimes id) \circ \Phi = id = (id \otimes e) \circ \Phi.$$

Φ is the *comultiplication* and e is the *counit*. Property (1.1) is known as the *coassociativity*. The bialgebra A is a *Hopf algebra* if a linear map $\kappa: A \to A$, the *antipode*, exists, so that

$$(m \circ (\kappa \otimes id) \circ \Phi)(a) = e(a)I = (m \circ (id \otimes \kappa) \circ \Phi)(a) \quad \forall a \in A. \tag{1.2}$$

Finally, A is a *Hopf$*$-algebra* if an involutive antilinear mapping $*: A \to A$, $a \mapsto a^*$, exists, so that the comultiplication and the counit are $*$-homomorphisms and $\kappa \circ *$ is involutive,

$$\kappa \circ * \circ \kappa \circ * = id. \tag{1.3}$$

2. The bialgebra $\mathcal{A}_q$ and the Hopf $*$-algebra $\mathrm{Pol}(U_q(n))$

In this section we consider the bialgebra $\mathcal{A}_q$ and the Hopf $*$-algebra $\mathrm{Pol}(U_q(n))$. These algebras will play a fundamental role in this paper and therefore we will discuss their properties briefly in this section. We will also define the quantum determinant D. References for this section are Faddeev e.a. [5], Reshetikhin e.a. [13], Manin [11] and in particular Parshall and Wang [12].

Fix $0 < q < 1$. $\mathscr{A}_q$ or $\mathscr{A}_q(n)$ is the unital algebra generated by the matrix elements of $T = (t_{ij})_{i,j=1,\dots,n}$ subject to the relations

$$t_{ij}t_{il} = qt_{il}t_{ij},\ j < l;\ t_{ij}t_{kj} = qt_{kj}t_{ij},\ i < k; \tag{2.1}$$

$$t_{ij}t_{kl} = t_{kl}t_{ij},\ i > k,\ j < l;\ t_{ij}t_{kl} - t_{kl}t_{ij} = (q - q^{-1})t_{il}t_{kj},\ i < k,\ j < l. \tag{2.2}$$

These relations can be equivalently written as

$$\sum_{k,l=1}^{n} R_{ij}^{kl} t_{km} t_{lp} = \sum_{k,l=1}^{n} R_{kl}^{mp} t_{ik} t_{jl},\quad \forall i, j, m, p = 1, \dots, n, \tag{2.3}$$

where R is a $n^2 \times n^2$ matrix defined by

$$R_{ii}^{ii} = q^{-1};\ R_{ij}^{ji} = 1,\ i \neq j; \tag{2.4}$$

$$R_{ij}^{ij} = q^{-1} - q,\ i > j;\ R_{ij}^{kl} = 0,\ \text{otherwise}. \tag{2.5}$$

Then R is a *Yang-Baxter operator*, i.e. R satisfies $R_{(12)}R_{(23)}R_{(12)} = R_{(23)}R_{(12)}R_{(23)}$, where

$$R_{(12)}, R_{(23)}: (\mathbf{C}^n)^{\otimes 3} \to (\mathbf{C}^n)^{\otimes 3} \tag{2.6}$$

are defined by $R_{(12)} = R \otimes id$ and $R_{(23)} = id \otimes R$. This can be verified by direct computation.

There exist unique unital algebra homomorphisms $\Phi: \mathscr{A}_q \to \mathscr{A}_q \otimes \mathscr{A}_q$ and $e: \mathscr{A}_q \to \mathbf{C}$ so that $e(t_{ij}) = \delta_{ij}$ and $\Phi(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$. Then $\mathscr{A}_q$ becomes a bialgebra with comultiplication Φ and counit e. The matrix T is *multiplicative*, see Manin [11, 2.6], which means that T is a corepresentation of $\mathscr{A}_q$.

Next we consider a special element of the algebra $\mathscr{A}_q$, the *quantum determinant D*, defined by

$$D = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)}, \tag{2.7}$$

where S_n denotes the permutation group of $\{1, \dots, n\}$ and $l(\sigma)$ is the length of the permutation σ. One also has the following expressions for D, which one can prove by use of symmetries for $\mathscr{A}_q$.

$$D = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{\sigma(1),1} \cdots t_{\sigma(n),n},\quad D = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} t_{n\sigma(n)} \cdots t_{1\sigma(1)}. \tag{2.8}$$

Define the *quantum minor* D^{ij} to be the $(n-1) \times (n-1)$ quantum determinant of the matrix obtained from T by deleting the ith row and jth column. So

$$D^{ij} = \sum_{\substack{\sigma \in S_{n-1} \\ \sigma:\{1,\dots,\hat{i},\dots,n\} \to \{1,\dots,\hat{j},\dots,n\}}} (-q)^{l(\sigma)} t_{1\sigma(1)} \cdots t_{i-1,\sigma(i-1)} t_{i+1,\sigma(i+1)} \cdots t_{n\sigma(n)}. \tag{2.9}$$

Now we can develop the quantum determinant along a row or along a column; this yields the following four equations.

$$\delta_{ij}D = \sum_{k=1}^{n} (-q)^{k-j} t_{ik} D^{jk}, \quad \delta_{ij}D = \sum_{k=1}^{n} (-q)^{i-k} D^{ki} t_{kj}, \tag{2.10}$$

$$\delta_{ij}D = \sum_{k=1}^{n} (-q)^{j-k} D^{jk} t_{ik}, \quad \delta_{ij}D = \sum_{k=1}^{n} (-q)^{k-i} t_{kj} D^{ki}. \tag{2.11}$$

From these developments of D we see that $D \in centre(\mathscr{A}_q)$. The actions of the comultiplication and counit on D are given by $\Phi(D) = D \otimes D$ and $e(D) = 1$.

The algebra $\mathrm{Pol}(U_q(n))$ is defined as the extension of $\mathscr{A}_q$ with the central element D^{-1} subject to the relation $DD^{-1} = I = D^{-1}D$. The comultiplication and counit extend uniquely to $\mathrm{Pol}(U_q(n))$ if we put $\Phi(D^{-1}) = D^{-1} \otimes D^{-1}$ and $e(D^{-1}) = 1$. Now if we define $\kappa(t_{ij}) = (-q)^{i-j} D^{ji} D^{-1}$ and $\kappa(D^{-1}) = D$, then we can extend $\kappa\colon \mathrm{Pol}(U_q(n)) \to \mathrm{Pol}(U_q(n))$ as a unital linear antimultiplicative mapping, so that $\mathrm{Pol}(U_q(n))$ becomes a Hopf algebra with comultiplication Φ, counit e and antipode, or *coinverse* κ. Finally, we introduce a $*$-operation on $\mathrm{Pol}(U_q(n))$ by putting

$$t_{ij}^* = \kappa(t_{ji}) = (-q)^{j-i} D^{ij} D^{-1}$$

and $(D^{-1})^* = D$. This gives $\mathrm{Pol}(U_q(n))$ a Hopf $*$-algebra structure. Note that $DD^* = I = D^*D$, so D is a unitary element.

To end this section we will state some commutation relations in $\mathrm{Pol}(U_q(n))$ concerning t_{ij} and t_{rs}^*. The first relation is simply a restatement of (2.10) and (2.11).

$$\sum_{k=1}^{n} t_{ik} t_{jk}^* = \delta_{ij} I = \sum_{k=1}^{n} t_{ki}^* t_{kj}. \tag{2.12}$$

The next relation can be proved by multiplying (2.3) from the left by t_{ip}^* and from the right by t_{sn}^* and then sum over i, n using (2.12). This yields

$$t_{rs}^* t_{ij} = t_{ij} t_{rs}^*, \qquad r \neq i,\ s \neq j, \tag{2.13}$$

$$t_{rs}^* t_{is} = q^{-1} t_{is} t_{rs}^* + (q^{-1} - q) \sum_{l=1}^{s-1} t_{il} t_{rl}^*, \qquad r \neq i, \tag{2.14}$$

$$t_{rs}^* t_{rj} = q t_{rj} t_{rs}^* + (q^2 - 1) \sum_{l=r+1}^{n} t_{ls}^* t_{lj}, \qquad s \neq j, \tag{2.15}$$

$$t_{rs}^* t_{rs} = t_{rs} t_{rs}^* + (q^2 - 1) \sum_{l=r+1}^{n} t_{ls}^* t_{ls} + (1 - q^2) \sum_{l=1}^{s-1} t_{rl} t_{rl}^*. \tag{2.16}$$

3. The Poincaré-Birkhoff-Witt theorem and the Verma module construction

In this section we construct a basis for $\mathscr{A}_q$ and $\mathrm{Pol}(U_q(n))$ by proving a Poincaré-Birkhoff-Witt theorem for $\mathscr{A}_q$. This result will then be applied to construct representations of $\mathscr{A}_q$ and $\mathrm{Pol}(U_q(n))$ by means of the Verma module construction.

THEOREM 3.1. *For any total ordering $<$ on the elements t_{ij} there is a basis for $\mathscr{A}_q$ consisting of*

$$\{t_{i_1,j_1}^{r_1}\cdots t_{i_m,j_m}^{r_m} \mid m = n^2,\ r_i \in \mathbf{Z}_+,\ t_{i_1j_1} < t_{i_2j_2} < \cdots < t_{i_mj_m}\}.$$

The proof of theorem 3.1 falls into two pieces. First we prove the theorem for a special case using the diamond lemma (Bergman [1, theorem 1.2]) in lemma 3.2. Next the general case is a direct consequence of lemma 3.2 and lemma 3.3.

Let us first introduce some notation. If a pair (t_{ab}, t_{cd}) of matrix elements satisfies the second commutation relation of (2.2), then we call the pair *bad*. For a product of matrix elements $x = t_{i_1,j_1}\cdots t_{i_p,j_p}$ we define the badness $b(x)$ of x by

$$b(x) = \#\{1 \leqslant r < s \leqslant p \mid (t_{i_r,j_r}, t_{i_s,j_s}) \text{ is bad}\}.$$

We also introduce a special ordering on the matrix elements by $t_{ij} <_0 t_{kl}$ if $i < k$ or if $i = k$ and $j > l$.

LEMMA 3.2. *For the ordering $<_0$ theorem* 3.1 *holds true.*

Proof. We will use freely the notions of Bergman [1, section 1]. Put $X = \{t_{ij} \mid 1 \leqslant i, j \leqslant n\}$, then we want to introduce a reduction system S for the free associative algebra $\mathbf{C}\langle X\rangle$. We extend the ordering $<_0$ to monomials of $\mathbf{C}\langle X\rangle$ by first ordering by the degree and for monomials with the same degree by the lexicographical ordering. In order to write this reduction system in a simple manner we introduce the operator L defined by

$$\begin{cases} L(t_{il}t_{ij}) = qt_{ij}t_{il} & l < j;\\ L(t_{ij}t_{kl}) = t_{kl}t_{ij} & i > k \text{ and } j < l;\\ L(t_{kj}t_{ij}) = q^{-1}t_{ij}t_{kl} & k > i;\\ L(t_{kl}t_{ij}) = t_{ij}t_{kl} + (q^{-1} - q)t_{il}t_{kj} & i < k \text{ and } j < l;\\ L(t_{ij}t_{kl}) = t_{ij}t_{kl} & i < k \text{ or } i = k \text{ and } j \geqslant l. \end{cases} \tag{3.1}$$

The reduction system S is now given by $\{t_{ij}t_{kl}, L(t_{ij}t_{kl})\}$ for $t_{kl} <_0 t_{ij}$. The ordering $<_0$ is compatible with the reduction system S and there are only overlap ambiguities. Now every element of $\mathbf{C}\langle X\rangle$ is reduction finite, since the number of monomials smaller than a given monomial is finite. Lemma 3.2 is implied by the diamond lemma if we prove that all ambiguities are resolvable.

Consider $x = t_{rs}t_{kl}t_{ij}$ with $t_{ij} <_0 t_{kl} <_0 t_{rs}$, then we have to show that compositions of reductions, r and r', exist, so that $rL_{(12)}(x) = r'L_{(23)}(x)$, where $L_{(12)}(x) = L(t_{rs}t_{kl})t_{ij}$ and $L_{(23)}(x) = t_{rs}L(t_{kl}t_{ij})$. To prove this we have to consider some special cases. Let $b(x)$ be the badness of x, then $b(x) \in \{0, 1, 2, 3\}$.

In case $b(x) = 0$ we have $L_{(12)}L_{(23)}L_{(12)}x = L_{(23)}L_{(12)}L_{(23)}x$. This also happens if $b(x) = 1$ or if $b(x) = 3$, which can be checked easily by using (3.1). So in these cases the ambiguities are resolvable. Finally we assume $b(x) = 2$, then there are four possibilities, which we list by a typical example.

$$x = \begin{cases} t_{23}t_{11}t_{12} \\ t_{22}t_{23}t_{11} \\ t_{32}t_{22}t_{11} \\ t_{32}t_{23}t_{11} \end{cases}$$

In the last two possibilities we have again $L_{(12)}L_{(23)}L_{(12)}x = L_{(23)}L_{(12)}L_{(23)}x$ and for the first two possibilities it is easily shown that

$$L_{(12)}L_{(23)}L_{(12)}(t_{23}t_{11}t_{12}) = L_{(12)}L_{(23)}L_{(12)}L_{(23)}(t_{23}t_{11}t_{12})$$

and

$$L_{(23)}L_{(12)}L_{(23)}L_{(12)}(t_{22}t_{23}t_{11}) = L_{(23)}L_{(12)}L_{(23)}(t_{22}t_{23}t_{11}).$$

This proves that all ambiguities are resolvable, which proves the lemma. □

REMARK. There is a relation between the operator L defined in (3.1) and the Yang-Baxter operator R describing the commutation relations defined in (2.4), (2.5). Denote by R^{-1} the inverse of R and put

$$\tilde{R}(t_{ir}t_{js}) = \sum_{k,l,m,p=1}^{n} (R^{-1})_{ij}^{kl} t_{km} t_{lp} R_{mp}^{rs}.$$

Then $L(t_{ir}t_{js})$ is defined as $\tilde{R}(t_{ir}t_{js})$ in case $t_{ir} >_0 t_{js}$ and as the identity operator if $t_{ir} <_0 t_{js}$.

Next we introduce $\mathscr{A}_q^k$, the span of all elements of products of precisely k matrix elements. Since $\mathscr{A}_q$ is defined by quadratic relations, this is well defined. By the previous lemma the set

$$\{t_{i_1,j_1} \cdots t_{i_k,j_k} \mid t_{i_l,j_l} \leqslant_0 t_{i_m,j_m}, \ l < m\}$$

yields a basis for $\mathscr{A}_q^k$. Now theorem 3.1 follows immediately from the following lemma.

LEMMA 3.3. *For an arbitrary total ordering $<$ on the set of matrix elements we have*

$$\mathscr{A}_q^k = \operatorname{span}\{t_{i_1,j_1} \cdots t_{i_k,j_k} \mid t_{i_l,j_l} \leqslant t_{i_m,j_m}, \ l < m\}. \tag{3.2}$$

Proof. For $k = 0, 1$ this is obvious. Now pick an arbitrary product of $k \geqslant 2$ matrix elements. So put $x = t_{r_1,s_1} \cdots t_{r_k,s_k}$. We will prove by induction on the badness of x, that x can be written as a linear combination of elements in (3.2). If $b(x) = 0$ this is obvious, because we can interchange all matrix elements of x at the cost of some constant.

Now we suppose that $b(x) \geqslant 1$ and we define the index of x by $\operatorname{ind}(x) = \#\{(l, m) \mid 1 \leqslant l < m \leqslant k, \ t_{r_l,s_l} > t_{r_m,s_m}\}$. If $\operatorname{ind}(x) = 0$, then x would already have the required form. So suppose that $\operatorname{ind}(x) \geqslant 1$ and pick $a \in \{1, \ldots, k-1\}$ so that $t_{r_a,s_a} > t_{r_{a+1},s_{a+1}}$. Put

$$x' = t_{r_1,s_1} \cdots t_{r_{a-1},s_{a-1}} t_{r_{a+1},s_{a+1}} t_{r_a,s_a} t_{r_{a+2},s_{a+2}} \cdots t_{r_k,s_k},$$

then $\operatorname{ind}(x') < \operatorname{ind}(x)$.

There are two cases to be considered. If $(t_{r_a,s_a}, t_{r_{a+1},s_{a+1}})$ is not bad, then we have $x = cx'$ for some constant $c \in \{1, q, q^{-1}\}$ and the induction on $\operatorname{ind}(x)$ does the job. The difficult case arises when $(t_{r_a,s_a}, t_{r_{a+1},s_{a+1}})$ is bad. In this case we put $[t_{r_a,s_a}, t_{r_{a+1},s_{a+1}}] = \pm(q - q^{-1})t_{uv}t_{wp}$ and

$$\tilde{x} = t_{r_1,s_1} \cdots t_{r_{a-1},s_{a-1}} t_{uv} t_{wp} t_{r_{a+2},s_{a+2}} \cdots t_{r_k,s_k}.$$

Then we have $x = x' \pm (q - q^{-1})\tilde{x}$, with $\operatorname{ind}(x') < \operatorname{ind}(x)$. So the first term can be dealt with by induction on $\operatorname{ind}(x)$. By a simple counting argument it follows that $b(\tilde{x}) < b(x)$, hence the induction on the badness provides for the other term. $\square$

Note that theorem 3.1 yields a basis for every ordering on the t_{ij}'s. We now specify the ordering more. Let $\mathscr{N}^+$, respectively $\mathscr{N}^-$, be the subalgebra generated by the elements $t_{ij}, j < n + 1 - i$, respectively $j > n + 1 - i$. $\mathscr{H}$ is the abelian subalgebra generated by the elements $t_{i,n+1-i}$, $i = 1, \ldots, n$. So $\mathscr{H}$ is generated by the elements on the antidiagonal of T, whereas $\mathscr{N}^+$ (respectively $\mathscr{N}^-$) is generated by the elements above (respectively below) the antidiagonal of T. Then we have

$$\mathscr{A}_q = \mathscr{N}^- \otimes_{\mathbf{C}} \mathscr{H} \otimes_{\mathbf{C}} \mathscr{N}^+.$$

THEOREM 3.4. *The elements $n^- h n^+ (D^{-1})^l$, $l \in \mathbf{Z}_+$ constitute a basis for* $\operatorname{Pol}(U_q(n))$, *whenever n^-, n^+ run through a set of basis vectors of $\mathscr{N}^-$, $\mathscr{N}^+$ and*

$$h = t_{n1}^{p_1} t_{n-1,2}^{p_2} \cdots t_{1n}^{p_n}, \qquad p_i \in \mathbf{Z}_+$$

subject to the condition $\min(p_1, \ldots, p_n, l) = 0$.

Proof. Let $\mathscr{A}_q^{\text{ext}}$ be the algebra $\mathscr{A}_q$ extended with the central element D^{-1} and let $\mathscr{L}$ be the two sided ideal generated by $DD^{-1} - I$, then $\mathrm{Pol}(U_q(n)) = \mathscr{A}_q^{\text{ext}}/\mathscr{L}$. Obviously, $n^- h n^+ (D^{-1})^l$, $l \in \mathbf{Z}_+$ is a basis for $\mathscr{A}_q^{\text{ext}}$, whenever n^-, n^+, h run through a basis of $\mathscr{N}^-$, $\mathscr{N}^+$ and $\mathscr{H}$. Let K denote the linear span of the elements described in the theorem. We will prove that

$$\mathscr{A}_q^{\text{ext}} = K \oplus \mathscr{L},$$

which will prove the theorem.

For an element $h = t_{n1}^{p_1} \cdots t_{1n}^{p_n} \in \mathscr{A}_q^{\text{ext}}$, we define $p(h) = \min p_i$. If $p(h) \geqslant 1$, then we put $h' = t_{n1}^{p_1 - 1} \cdots t_{1n}^{p_n - 1}$. Now we rewrite (2.8) into

$$t_{n1} \cdots t_{1n} = (-q)^{l(\rho_0)} D - (-q)^{l(\rho_0)} \sum_{\substack{\sigma \in S_n \\ \sigma \neq \rho_0}} (-q)^{-l(\sigma)} t_{n,\sigma(n)} \cdots t_{1,\sigma(1)}, \tag{3.3}$$

where $\rho_0 \in S_n$ is defined by $\rho_0 \colon i \mapsto n + 1 - i$, so $l(\rho_0) = \frac{1}{2} n(n-1)$. Note that we can reorder the factors in the summand of (3.3) so that all factors from $\mathscr{N}^-$ precede all factors from $\mathscr{N}^+$. Then (3.3) implies the following identity in $\mathscr{A}_q^{\text{ext}}$, with h as above,

$$n^- h n^+ (D^{-1})^l = (-q)^{l(\rho_0)} n^- h' n^+ D (D^{-1})^l + \sum_i c_i n_i^- h_i n_i^+ (D^{-1})^l \tag{3.4}$$

with $c_i \in \mathbf{C}$ and $p(h_i) \leqslant p(h) - 1$, for all i, because the number of matrix elements in an element $h \in \mathscr{H}$ is not increased if we pull $t_{k,\sigma(k)} \in \mathscr{N}^-$ from the right through h.

For an arbitrary basis element

$$a = n^- h n^+ (D^{-1})^l \in \mathscr{A}_q^{\text{ext}}$$

we define $k(a) = \min(p(h), l)$. First we prove by induction on $p(h) + l$ that $\mathscr{A}_q^{\text{ext}} = K + \mathscr{L}$. From (3.4) it follows that

$$n^- h n^+ (D^{-1})^l = (-q)^{l(\rho_0)} n^- h' n^+ (D^{-1})^{l-1} + \sum_i c_i n_i^- h_i n_i^+ (D^{-1})^l \bmod \mathscr{L}.$$

This gives the induction step.

To prove directness of the sum we construct an endomorphism $L\colon \mathscr{A}_q^{\text{ext}} \to \mathscr{A}_q^{\text{ext}}$ with the properties $L|_K = \mathrm{id}$ and $L|_{\mathscr{L}} = 0$. First, we define an endomorphism L_0 of $\mathscr{A}_q^{\text{ext}}$ by defining it on basis elements. If $k(a) \geqslant 1$ then we put

$$L_0(a) = (-q)^{l(\rho_0)} \left(n^- h' \left(I - \sum_{\substack{\sigma \in S_n \\ \sigma \neq \rho_0}} (-q)^{-l(\sigma)} t_{n,\sigma(n)} \cdots t_{1,\sigma(1)} D^{-1} \right) n^+ (D^{-1})^{l-1} \right)$$

and $L_0(a) = a$ if $k(a) = 0$. Now L_0 does not increase the degree of the monomial and it decreases the k. So we can define $L = (L_0)^\infty$, which is a finite product of L_0's on every monomial. Then $L|_K = id$ and to see that $L|_{\mathscr{L}} = 0$, we note that if $x \in \mathscr{L}$, then x is of the form $(I - L_0)y$ for some $y \in \mathscr{A}_q^{\text{ext}}$. Hence, for sufficient large r we have

$$L(x) = (L_0^r - L_0^{r+1})y = L_0^r y - L_0^r y = 0. \qquad \square$$

Now we will construct representations of $\mathscr{A}_q$ and $\mathrm{Pol}(U_q(n))$ using a Verma module construction. We start with an arbitrary permutation $\rho \in S_n$ and we put

$$N_\rho^- = \{t_{\rho(i),j} | 1 \leqslant i < j \leqslant n\},$$
$$H_\rho = \{t_{\rho(i),i} | 1 \leqslant i \leqslant n\},$$
$$N_\rho^+ = \{t_{\rho(i),j} | 1 \leqslant j < i \leqslant n\}.$$

Let us choose such an ordering on the matrix elements t_{ij} so that all elements of N_ρ^- are smaller than the elements of H_ρ, which are smaller than the elements of N_ρ^+. By $\mathscr{N}_\rho^-$ (respectively $\mathscr{H}_\rho$, $\mathscr{N}_\rho^+$) we denote the linear subspace of $\mathscr{A}_q(n)$ generated by the basis elements (cf theorem 3.1) which are products of elements of N_ρ^- (respectively H_ρ, N_ρ^+). Then

$$\mathscr{A}_q(n) = \mathscr{N}_\rho^- \otimes_{\mathbf{C}} \mathscr{H}_\rho \otimes_{\mathbf{C}} \mathscr{N}_\rho^+, \tag{3.5}$$

in the sense that $n_\rho^- h_\rho n_\rho^+$ yields a basis for $\mathscr{A}_q(n)$, whenever n_ρ^-, h_ρ and n_ρ^+ run through a basis of $\mathscr{N}_\rho^-$, $\mathscr{H}_\rho$ and $\mathscr{N}_\rho^+$. Note that $\mathscr{N}_\rho^-$, $\mathscr{H}_\rho$ and $\mathscr{N}_\rho^+$ are not subalgebras of $\mathscr{A}_q(n)$, unless $\rho = \rho_0 : i \mapsto n + 1 - i$.

Let $\mathscr{L}_\rho$ be the left ideal generated by the matrix elements of N_ρ^+, then $\mathscr{L}_\rho$ is the linear span of $n_\rho^- h_\rho n_\rho^+$ with $n_\rho^+ \neq I$. Indeed, the span of these elements is obviously included in $\mathscr{L}_\rho$ and to prove that every element of $\mathscr{L}_\rho$ is a linear combination of these elements one uses induction on the badness and on the index as in lemma 3.3. This has been worked out more properly in the proof of lemma 4.11.

The elements $t_{\rho(i),i}$ commute modulo $\mathscr{L}_\rho$. To see this we suppose $i < j$, then we have $t_{\rho(i),i} t_{\rho(j),j} = t_{\rho(j),j} t_{\rho(i),i}$ in case $\rho(i) > \rho(j)$ and in case $\rho(i) < \rho(j)$ we have

$$t_{\rho(i),i} t_{\rho(j),j} - t_{\rho(j),j} t_{\rho(i),i} = (q - q^{-1}) t_{\rho(i),j} t_{\rho(j),i}.$$

Since $t_{\rho(j),i} \in N_\rho^+$ it follows that $[t_{\rho(i),i}, t_{\rho(j),j}] = 0 \bmod \mathscr{L}_\rho$.

For an arbitrary $\Gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{C}^n$ we define a left ideal $\mathscr{I}_\rho(\Gamma)$ generated by the matrix elements of N_ρ^+ and by the elements $t_{\rho(i),i} - \gamma_i I$. Then $\mathscr{L}_\rho \subset \mathscr{I}_\rho(\Gamma)$ and it follows by use of the basis (3.5) that $I \notin \mathscr{I}_\rho(\Gamma)$. So we can define

$$V^\rho(\Gamma) = \mathscr{A}_q/\mathscr{I}_\rho(\Gamma), \quad v^+ = I \bmod \mathscr{I}_\rho(\Gamma).$$

and this is left $\mathscr{A}_q$-module by left multiplication. It follows from (3.5) that $V^\rho(\Gamma) \cong \mathscr{N}_\rho^- \cdot v^+$ as vector spaces.

Now we introduce the highest weight modules for an arbitrary element $\rho \in S_n$. A left $\mathscr{A}_q$-module M is a *ρ-highest weight module* if the module is generated by a ρ-highest weight vector $m \in M$. Here a *ρ-highest weight vector $m \in M$ of weight* $\Gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies $t_{\rho(i),i} \cdot m = \gamma_i m$ and $t_{kl} \cdot m = 0$ for $t_{kl} \in N_\rho^+$.

Obviously, $V^\rho(\Gamma)$ is a ρ-highest weight module for $\mathscr{A}_q$ with ρ-highest weight vector v^+ of weight Γ.

PROPOSITION 3.5. (i) *If M is any ρ-highest weight module with ρ-highest weight vector m of weight Γ, then there exists a unique surjective $\mathscr{A}_q$-equivariant linear map φ: $V^\rho(\Gamma) \to M$, so that $\varphi(v^+) = m$.*

(ii) *For $i = n, \ldots, \rho(1) + 1$ we have* $t_{i1} \cdot v = 0$ for all $v \in V^\rho(\Gamma)$.

(iii) *If $\Gamma \in (\mathbf{C}^*)^n$, then $V^\rho(\Gamma)$ has a unique proper maximal submodule.*

(iv) *For every $\rho \in S_n$ and $\Gamma \in (\mathbf{C}^*)^n$ a unique irreducible ρ-highest weight module $L^\rho(\Gamma)$ exists.*

Proof. The first statement is similar to proposition 4.35 of Knapp [8]. So define the linear map φ by

$$\varphi(n_\rho^- \cdot v^+) = n_\rho^- \cdot m,$$

then φ is well defined. Obviously φ is equivariant and since $M \cong \mathscr{N}_\rho^- \cdot m$ as vector spaces, the map φ is surjective. Uniqueness now easily follows.

Since $V^\rho(\Gamma) \mapsto \mathscr{N}_\rho^- \cdot v^+$ as vector spaces we can consider the subspace of $V^\rho(\Gamma)$ generated by elements of the form

$$\prod_{j \leqslant k < l \leqslant n} t_{\rho(k),l}^{p_{kl}} \cdot v^+, \qquad p_{kl} \in \mathbf{Z}_+.$$

For $l > k \geqslant j$, $\rho(i) \geqslant \rho(j)$ and $i \geqslant j$ we have the following commutation relations,

$$t_{\rho(i),j} t_{\rho(k),l} = \begin{cases} t_{\rho(k),l} t_{\rho(i),j}, & \text{if } \rho(k) < \rho(i); \\ q t_{\rho(k),l} t_{\rho(i),j}, & \text{if } \rho(k) = \rho(i); \\ t_{\rho(k),l} t_{\rho(i),j} + (q - q^{-1}) t_{\rho(i),l} t_{\rho(k),j}, & \text{if } \rho(k) > \rho(i). \end{cases} \tag{3.6}$$

Now it follows that for $\rho(i) > \rho(j)$, $i > j$ we have

$$t_{\rho(i),j} \prod_{j \leqslant k < l \leqslant n} t_{\rho(k),l}^{p_{kl}} \cdot v^+ = 0. \tag{3.7}$$

Indeed, if $\rho(i)$ is maximal with respect to $\rho(i) > \rho(j)$, $i > j$, then the last commutation relation of (3.6) does not occur and in this case (3.7) follows from $t_{\rho(i),j} \cdot v^+ = 0$. To prove (3.7) for general $\rho(i)$ we assume that it holds for all $t_{\rho(k),j}$ with $\rho(k) > \rho(j)$, $k > j$, so that $\rho(k) > \rho(i)$ and we use induction on $\sum p_{kl}$. Now (3.6) furnishes the induction step. Only the last case may give any trouble, but this can be dealt with by induction on the $\rho(i)$, since $\rho(k) > \rho(i) > \rho(j)$ and $k \geqslant j$. Taking $j = 1$ in (3.7) proves (ii).

Combining (3.6) and (3.7) yields

$$t_{\rho(j),j} \prod_{j \leqslant k < l \leqslant n} t_{\rho(k),l}^{p_{kl}} \cdot v^+ = \gamma_j \prod_{j < l \leqslant n} q^{p_{jl}} \prod_{j \leqslant k < l \leqslant n} t_{\rho(k),l}^{p_{kl}} \cdot v^+.$$

So let us introduce subspaces of $V^\rho(\Gamma)$ by

$$\begin{aligned} A_k^i &= \{v \in V^\rho(\Gamma) | t_{\rho(j),j} \cdot v = \gamma_j v, j = 1, \ldots, i-1; t_{\rho(i),i} \cdot v = q^k \gamma_i v\} \\ &= \operatorname{span} \left\{ \prod_{i \leqslant r < l \leqslant n} t_{\rho(r),l}^{p_{rl}} \cdot v^+ \,\middle|\, \sum_{i < l \leqslant n} p_{il} = k \right\} \end{aligned}$$

for $k \in \mathbf{Z}_+$ and $i = 1, \ldots, n-1$. If we put $A_0^0 = V^\rho(\Gamma)$, then we have

$$\sum_{k \in \mathbf{Z}_+} A_k^i = A_0^{i-1}, \quad i = 1, \ldots, n-1. \tag{3.8}$$

and

$$V^\rho(\Gamma) = \sum_{i=1}^{n-1} \sum_{k \geqslant 1} A_k^i \oplus A_0^{n-1}, \quad A_0^{n-1} = \langle v^+ \rangle. \tag{3.9}$$

Let K be any submodule of $V^\rho(\Gamma)$, then we have

$$K \cap \sum_{k \in \mathbf{Z}_+} A_k^i = \sum_{k \in \mathbf{Z}_+} (K \cap A_k^i)$$

for all i. To see this we check the $\subset$-inclusion. Pick $u \in K$ and write $u = u_1 + \cdots + u_s$, with $u_j \in A_{k_j}^i$ for different k_j's. Now all the $q^{k_j}\gamma_i$'s are different, so we can construct polynomials p_r in one variable, so that $p_r(q^{k_j}\gamma_i) = \delta_{r,j}$. Then $p_r(t_{\rho(i),i}) \cdot u = u_r$, and hence, since K is a submodule, $u_r \in K$.

By (3.8) and (3.9) we see that every submodule K splits as

$$K = \sum_{i=1}^{n-1} \sum_{k \geqslant 1} (K \cap A_k^i) \oplus (K \cap A_0^{n-1}). \tag{3.10}$$

So every proper submodule is contained in $\sum_{i=1}^{n-1}\sum_{k\geq 1} A_k^i$, since otherwise $v^+ \in K$ and $K = V^\rho(\Gamma)$. Hence the sum of all proper submodules is also contained in this subspace, which gives the unique maximal proper submodule.

Finally, to prove (iv) we note that existence follows from (iii) and the uniqueness is proved as theorem 20.3.A of Humphreys [6]. □

Since D is a central element D acts as a scalar on $L^\rho(\Gamma)$. It is easy to compute the action of D on the highest weight vector v^+ if we use (2.8). We have

$$D \cdot v^+ = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} t_{\sigma(n),n} \cdots t_{\sigma(1),1} \cdot v^+.$$

All terms cancel except for $\sigma = \rho$, so we have proved the following theorem.

THEOREM 3.6. *If* $\Gamma \in (\mathbf{C}^*)^n$, *then* $L^\rho(\Gamma)$ *is an irreducible* $\mathrm{Pol}(U_q(n))$*-module if we put*

$$D^{-1} \cdot v = (\gamma_1 \cdots \gamma_n)^{-1}(-q)^{l(\rho)} v, \quad \forall\, v \in L^\rho(\Gamma).$$

COROLLARY 3.7. *Suppose* H *is a* ρ*-highest weight module of weight* $\Gamma \in (\mathbf{C}^*)^n$ *for* $\mathrm{Pol}(U_q(n))$, *so that the corresponding representation is a* $*$*-representation, then* H *is irreducible.*

Proof. Any submodule K of H splits as in (3.10). If K is proper, then $K \cap A_0^{n-1} = \{0\}$. The highest weight vector is an element of the invariant subspace $K^\perp$, which must be equal to H. □

4. Representation theory of Pol($U_q(n)$)

In this section we consider primary and irreducible $*$-representations of $\mathrm{Pol}(U_q(n))$. We formulate some necessary conditions for a representation to be a primary or irreducible $*$-representation of $\mathrm{Pol}(U_q(n))$. By using induction on n we are able to classify all ireducible $*$-representations of $\mathrm{Pol}(U_q(n))$. We will also show that every primary representation contains an irreducible $*$-subrepresentation.

Let us first introduce some notions from Mackey [10, Chapter 1]. For two $*$-representations π_1 and π_2 of $\mathrm{Pol}(U_q(n))$ we denote by $R(\pi_1, \pi_2)$ the space of all intertwining operators from the representation space of π_1 to the representation space of π_2. Furthermore, $R(\pi) = R(\pi, \pi)$. A $*$-representation of $\mathrm{Pol}(U_q(n))$ is *primary* if the centre of $R(\pi)$ consists of λId, $\lambda \in \mathbf{C}$. This is equivalent to the following condition: if $\pi = \pi_1 \oplus \pi_2$ is a non-trivial decomposition, then a non-zero element in $R(\pi_1, \pi_2)$ exists. Note that an irreducible $*$-representation is primary. A primary representation is of *type* I if it contains an irreducible *-subrepresentation. Note that we assume primary representations to be *-representations.

PROPOSITION 4.1. *If π is a primary representation of* $\mathrm{Pol}(U_q(n))$, *then there exists r, $0 \leqslant r < n$, so that*

$$\pi(t_{n1}) = \cdots = \pi(t_{n-r+1,1}) = 0; \quad \ker(\pi(t_{n-r,1})) = \{0\}.$$

Proof. Let π be a primary representation of $\mathrm{Pol}(U_q(n))$, then it follows directly from the commutation relations that $\ker(\pi(t_{n1}))$ is an invariant subspace and thus

$$H = \ker(\pi(t_{n1})) \oplus (\ker(\pi(t_{n1})))^{\perp}$$

as a decomposition of the representation space H of π in invariant subspaces. Consequently, $\pi = \pi_1 \oplus \pi_2$ and, if the decomposition is non-trivial, there exists a non-zero $T \in R(\pi_1, \pi_2)$, so that $T\pi_1(x) = \pi_2(x)T$ for all $x \in \mathrm{Pol}(U_q(n))$.

Take $x = t_{n1}$ to obtain $\mathrm{range}(T) \subset \ker(\pi_2(t_{n1}))$. Since $\ker(\pi_2(t_{n1})) = \{0\}$ we obtain $T = 0$, a contradiction. Hence $\pi = \pi_1$ or $\pi = \pi_2$. In the second case we can take $r = 0$ and we are finished. In the first case the commutation relations and $\pi(t_{n1}) = 0$ imply that $\ker(\pi(t_{n-1,1}))$ is invariant and we can play the game once again. Continuing in this way we prove the proposition.

Finally we remark that $r < n$, since otherwise $\pi(D) = 0$, contradicting the unitarity of $\pi(D)$. □

PROPOSITION 4.2. *Let π be a primary representation of* $\mathrm{Pol}(U_q(n))$ *with r defined as in proposition* 4.1, *then $\pi(t_{n-r,1})$ is normal and, for some $\lambda \in \mathbf{C}^*$,*

$$\lambda \in \sigma(\pi(t_{n-r,1})) \subseteq \{\lambda q^k\}_{k \in \mathbf{Z}_+} \cup \{0\},$$

where $\sigma(\pi(t_{n-r,1}))$ denotes the spectrum of $\pi(t_{n-r,1})$.

Before we take up the proof of this proposition we will demonstrate a lemma which will be needed in the proof.

LEMMA 4.3. (cf. Rudin [15, 12.23]). *Let $N, B \in B(H)$ and suppose N is normal with spectral decomposition E. Assume there exists a $c \in \mathbf{C}^*$ so that*

$$BN = cNB; \qquad BN^* = \bar{c}N^*B.$$

Then $BE(\Omega) = E(c^{-1}\Omega)B$, for all Borel sets $\Omega \subset \mathbf{C}$.

Proof. Let p be a polynomial in two variables and let $v, w \in H$. Then

$$\begin{aligned}\int_{\mathbf{C}} p(z, \bar{z}) dE_{v,B^*w}(z) &= (p(N, N^*)v, B^*w) \\ &= (p(cN, \bar{c}N^*)Bv, w) \\ &= \int_{\mathbf{C}} p(cz, \bar{c}\bar{z}) dE_{Bv,w}(z) \\ &= \int_{\mathbf{C}} p(z, \bar{z}) dE_{Bv,w}(c^{-1}z).\end{aligned}$$

This yields $dE_{v,B^*w}(z) = dE_{Bv,w}(c^{-1}z)$. So for any Borel set $\Omega \subset \mathbf{C}$ we find

$$\begin{aligned}(BE(\Omega)v, w) &= \int_{\Omega} dE_{v,B^*w}(z) = \int_{\Omega} dE_{Bv,w}(c^{-1}z) \\ &= \int_{c^{-1}\Omega} dE_{Bv,w}(z) = (E(c^{-1}\Omega)Bv, w).\end{aligned}$$ □

Proof of proposition 4.2. The equality (2.16) yields

$$t^*_{n-r,1} t_{n-r,1} = t_{n-r,1} t^*_{n-r,1} + (q^2 - 1) \sum_{l=n-r+1}^{n} t^*_{l1} t_{l1}. \tag{4.1}$$

If we apply π to (4.1), then proposition 4.1 yields the normality of $\pi(t_{n-r,1})$.

Now suppose $\lambda_1, \lambda_2 \in \sigma(\pi(t_{n-r,1}))$ and $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. We intend to prove the existence of a $k \in \mathbf{Z}$ so that $\lambda_1 = q^k \lambda_2$. This result combined with the compactness of the spectrum, implies the proposition.

Suppose the contrary, then there exist closed sets Ω_1, Ω_2 of $\mathbf{C}$ so that $\lambda_i \in \operatorname{int}(\Omega_i)$, $q\Omega_i = \Omega_i$ and $\Omega_1 \cap \Omega_2 = \{0\}$. So we find for the spectral decomposition E of $\pi(t_{n-r,1})$ that $0 \neq E(\Omega_i) \neq id$. We claim that

$$E(\Omega_i)\pi(a) = \pi(a)E(\Omega_i), \quad \forall a \in \operatorname{Pol}(U_q(n)). \tag{4.2}$$

It is sufficient to prove (4.2) for all matrix elements t_{ij} and D^{-1}. If $a = t_{i,1}$, $(i \geqslant n - r)$, $a = D^{-1}$ or $a = t_{ij}$ $(i \neq n - r, j \neq 1)$ this is obvious, since $[\pi(t_{n-r,1}), \pi(a)] = 0$ and in these cases (4.2) even holds for arbitrary Borel sets. In case $a = t_{ij}$ with $i = n - r, j \geqslant 2$ or $i < n - r, j = 1$ it follows from (2.1), (2.14) or from (2.1), (2.15), proposition 4.1 that the conditions of lemma 4.3 hold with $c = q^{\mp 1}$. Hence lemma 4.3 implies (4.2) for these choices of a, which completes the proof of (4.2).

Now (4.2) implies that we have proper closed π-invariant subspaces $H_{\Omega_i} = E(\Omega_i)H$. Since $\ker(\pi(t_{n-r,1})) = 0$ we have $E(\{0\}) = 0$, so $H_{\Omega_1} \perp H_{\Omega_2}$. Since π is primary there exist non-zero π-invariant subspaces H_i of H_{Ω_i}, so that there exists an intertwining isometry $T: H_1 \to H_2$. In particular T is an intertwining isometry for $\pi(t_{n-r,1})|_{H_1}$ and $\pi(t_{n-r,1})|_{H_2}$, so these two operators must have the same spectrum. Since restriction of an operator to some closed subspace decreases the spectrum, the two operators $\pi(t_{n-r,1})|_{H_i}$ must have the same spectrum $\{0\} = \Omega_1 \cap \Omega_2$. This contradicts $\ker(\pi(t_{n-r,1})) = \{0\}$. □

Put $K = \ker(\pi(t_{n-r,1}) - \lambda I)$, then K is a closed subspace of H.

PROPOSITION 4.4. *For $i \neq n-r$ and $j \neq 1$ we have $\pi(D)$, $\pi(D^{-1})$, $\pi(t_{ij})$, $\pi(t_{ij})^*$: $K \to K$. Furthermore, we have $\pi(t_{p1})K = \{0\}$ for $p \neq n-r$ and $\pi(t_{n-r,i})^*K = \{0\}$ for $i \neq 1$.*

Proof. The first part follows from $[\pi(t_{n-r,1}), \pi(t_{ij})] = 0$, because of (2.1), (2.2) and proposition 4.1, and from $[\pi(t_{n-r,1}), \pi(t_{ij})^*] = 0$, because of (2.13), for $i \neq n-r$ and $j \neq 1$ and from $D, D^{-1} \in centre(\mathrm{Pol}(U_q(n)))$. The second assertion for $p > n-r$ is proposition 4.1. To prove the rest of the statement we use (2.1), (2.15) and proposition 4.1 to see that for $v \in K$ we have

$$\pi(t_{n-r,1})\pi(t_{p1})v = \lambda q^{-1}\pi(t_{p1})v$$

and

$$\pi(t_{n-r,1})\pi(t_{n-r,i})^*v = \lambda q^{-1}\pi(t_{n-r,i})^*v.$$

Since $\lambda q^{-1} \notin \sigma(\pi(t_{n-r,1}))$ the proposition follows. □

COROLLARY 4.5. $|\lambda| = 1$.

Proof. Apply π to $I = \sum_{k=1}^{n} t_{k1}^* t_{k1}$ (cf. (2.12)) and restrict to a vector $v \in K$. Then proposition 4.4 yields the result. □

Since $\pi(t_{n-r,1})$ is normal, we have an orthogonal decomposition

$$H = \bigoplus_{k \in \mathbf{Z}_+} H_k, \quad H_k = \ker(\pi(t_{n-r,1}) - \lambda q^k I), \tag{4.3}$$

with $H_0 = K$. Note that

$$\pi(t_{n-r,i}): H_k \to H_{k+1}, \ i = 2, \ldots, n;$$

$$\pi(t_{ij}): H_k \to H_k, \ i \neq n-r \text{ and } j \neq 1;$$

$$\pi(t_{p1}): H_k \to H_{k-1}, \ 1 \leqslant p < n-r. \tag{4.4}$$

For this general primary representation π of $\mathrm{Pol}(U_q(n))$ we are interested in the decomposition of a closed invariant subspace with respect to the orthogonal decomposition (4.3). The results are contained in the following lemma.

LEMMA 4.6. *Let π be a primary representation of* $\mathrm{Pol}(U_q(n))$ *in* $H = \bigoplus_{k \in \mathbf{Z}_+} H_k$.

(i) *For every subrepresentation π_1 of π we have*

$$\sigma(\pi_1(x)) = \sigma(\pi(x)), \qquad \forall x \in \mathrm{Pol}(U_q(n)).$$

(ii) *For every non-zero closed π-invariant subspace $V \subset H$ we have $V \cap H_k \neq \{0\}$ for all k with $H_k \neq \{0\}$.*

(iii) $H = \overline{\pi(\mathrm{Pol}(U_q(n)))H_0}$.

Proof. To prove (i) we assume the existence of a $*$-subrepresentation π_i of π so that an element $x \in \mathrm{Pol}(U_q(n))$ exists with $F = \sigma(\pi_1(x)) \neq \sigma(\pi(x))$. Then F is a proper closed subset of $\sigma(\pi(x))$. Without loss of generality we may assume that every $*$-subrepresentation $\tilde{\pi}$ of π with $\sigma(\tilde{\pi}(x)) \subset F$ is a $*$-subrepresentation of π_1. Put $\pi = \pi_1 \oplus \pi_2$ corresponding to $H = H^1 \oplus H^2$, then $H^2 \neq \{0\}$. Since π is primary, equivalent $*$-subrepresentations π'_1, π'_2 of π_1, π_2 exist. But this contradicts $\sigma(\pi'_1(x)) \subset F$ and $\sigma(\pi'_2(x)) \not\subset F$.

To see (ii) we apply (i) with $\pi_1 = \pi|_V$ and $x = t_{n-r,1}$, with r as in proposition 4.1. Finally, (iii) follows from (ii) with $k = 0$. Indeed, if $W = \overline{\pi(\mathrm{Pol}(U_q(n)))H_0}$ has a non-zero complement in H, say V, then $V \cap H_0 \neq \{0\}$, contradicting $H = W \oplus V$. □

For $1 \leqslant j \leqslant n-1$ we define

$$s_{ij} = \begin{cases} t_{i,j+1}, & 1 \leqslant i \leqslant n-r-1; \\ t_{i+1,j+1}, & n-r \leqslant i \leqslant n-1. \end{cases} \tag{4.5}$$

So $S = (s_{ij})_{i,j=1,\dots,n-1}$ is obtained from $T = (t_{ij})_{i,j=1,\dots,n}$ by deleting the first column and the $(n-r)$th row. Then the s_{ij} also satisfy the commutation relations (2.1)–(2.2). Thus (4.5) defines an embedding of $\mathscr{A}_q(n-1)$ in $\mathscr{A}_q(n)$. Let D_S and D_S^{-1} denote the determinant and its inverse for S and let $\mathrm{Pol}(U_q(n-1))$ be generated by the s_{ij}, $1 \leqslant i, j \leqslant n-1$, and D_S^{-1} subject to the customary commutation relations.

We define the following operators in $B(K)$, cf. (4.4):

$$\tau(D_S^{-1}) = \lambda \pi(D^{-1})|_K,$$

$$\tau(s_{ij}) = -\frac{1}{q}\pi(t_{i,j+1})|_K,\ 1 \leqslant j \leqslant n-1,\ 1 \leqslant i \leqslant n-r-1,$$

$$\tau(s_{ij}) = \pi(t_{i+1,j+1})|_K,\ 1 \leqslant j \leqslant n-1,\ n-r \leqslant i \leqslant n-1. \tag{4.6}$$

THEOREM 4.7. *If we extend τ to* $\mathrm{Pol}(U_q(n-1))$ *as a homomorphism, then τ becomes a $*$-representation of* $\mathrm{Pol}(U_q(n-1))$ *in the Hilbert space K.*

Furthermore, τ is primary and τ is irreducible if and only if π is irreducible.

Proof. The proof is rather long and falls into three pieces. First we show that τ is a representation of $\mathrm{Pol}(U_q(n-1))$. Then we prove that it is actually a $*$-representation of $\mathrm{Pol}(U_q(n-1))$. Finally we will prove the last statement of the theorem.

One can directly check that τ as defined in (4.6) yields a representation of $\mathscr{A}_q(n-1)$, since the scaling factor $-1/q$ appears in every term of the relations (2.1)–(2.2) the same number of times. Keeping in mind that D_S coincides with the quantum minor $D^{n-r,1}$ it is easily seen that

$$\pi(D^{n-r,1})|_K = (-q)^{n-r-1}\tau(D_S). \tag{4.7}$$

By (2.10), proposition 4.4 and (4.7) we have

$$\begin{aligned}\pi(D)|_K &= \sum_{k=1}^{n} (-q)^{1-k}\pi(D^{k1})\pi(t_{k1})|_K\\ &=\lambda(-q)^{1-(n-r)}\pi(D^{n-r,1})|_K = \lambda\tau(D_S),\end{aligned}$$

which shows that τ preserves $D_S D_S^{-1} = I = D_S^{-1}D_S$ and hence τ is a representation of $\mathrm{Pol}(U_q(n-1))$.

To prove that τ is a $*$-representation of $\mathrm{Pol}(U_q(n-1))$ we have to consider several cases. We will need

$$\pi((D^{n-r,1})^{i,j+1})|_K = (-q)^{n-r-2}\tau(D_S^{ij}), \quad 1 \leqslant i \leqslant n-r-1, \tag{4.8}$$

$$\pi((D^{n-r,1})^{i+1,j+1})|_K = (-q)^{n-r-1}\tau(D_S^{ij}), \quad n-r \leqslant i \leqslant n-1, \tag{4.9}$$

which can be proved as (4.7). First we consider the case $1 \leqslant i \leqslant n-r-1$, then by the definition of $*$: $\mathrm{Pol}(U_q(n)) \to \mathrm{Pol}(U_q(n))$, the development of the quantum minor $D^{i,j+1}$, proposition 4.4 and (4.8)

$$\begin{aligned}\tau(s_{ij})^* &= -\frac{1}{q}\,\pi(t_{i,j+1})^*|_K\\ &= -\frac{1}{q}\,(-q)^{1+j-i}\pi(D^{-1})\pi(D^{i,j+1})|_K\\ &=(-q)^{j-i}\pi(D^{-1})\sum_{\substack{k=1\\k\neq i}}^{n} (-q)^{\alpha(k)}\pi((D^{i,j+1})^{k1})\pi(t_{k1})|_K\\ &=\lambda(-q)^{j-i}(-q)^{1-(n-r-1)}\pi(D^{-1})\pi((D^{n-r,1})^{i,j+1})|_K\\ &=(-q)^{j-i}\tau(D_S^{-1})\tau(D_S^{ij})\\ &=\tau(s_{ij}^*).\end{aligned}$$

Here

$$\alpha(k) = \begin{cases} 1-k, & k = 1, \ldots, i-1; \\ 1-(k-1), & k = i+1, \ldots, n-1. \end{cases}$$

In a similar way one proves $\tau(s_{ij})^* = \tau(s_{ij}^*)$ for $n-r \leqslant i \leqslant n-1$ using (4.9). Also, by (2.10), (4.6) and (4.7),

$$\begin{aligned} \tau(D_S^{-1})^* &= \bar{\lambda}\pi(D)|_K \\ &= |\lambda|^2(-q)^{1-(n-r)}\pi(D^{n-r,1})|_K \\ &= |\lambda|^2\tau(D_S), \end{aligned}$$

which proves $\tau(D_S^{-1})^* = \tau((D_S^{-1})^*)$ by corollary 4.5.

In order to prove the last statement of the theorem we investigate the relation between the decomposition of K into closed τ-invariant subspaces and the decomposition of H into closed π-invariant subspaces. So assume that $\pi = \pi_1 \oplus \pi_2$ corresponds to $H = V \oplus W$ and let $T \in R(\pi_1, \pi_2)$, then for $v \in V_k = V \cap H_k$ we have

$$\lambda q^k Tv = T\pi_1(t_{n-r,1})v = \pi_2(t_{n-r,1})Tv \Rightarrow T(V_k) \subset W_k = W \cap H_k.$$

In particular $T|_{V_0}\colon V_0 \to W_0$ and by the construction of τ we have $T|_{V_0} \in R(\tau_1, \tau_2)$, where $\tau = \tau_1 \oplus \tau_2$ corresponds to the decomposition $K = V_0 \oplus W_0$. We claim that

$$T = 0 \Leftrightarrow T|_{V_0} = 0. \tag{4.10}$$

It is sufficient to prove for $v \in V_k$ we have $Tv = 0$, when $T|_{V_0} = 0$. Lemma 4.6(ii), (iii) imply that we can write v as

$$v = \sum_P C_P \pi_1(P) v_0^P \in V_0,$$

with $P = t_{n-r,2}^{p_2} \cdots t_{n-r,n}^{p_n}$, $\sum_{i=2}^n p_i = k$. This way of writing is in general not unique. Then we have

$$\begin{aligned} Tv &= T\left(\sum_P C_P \pi_1(P) v_0^P\right) \\ &= \sum_P C_P \pi_2(P) T v_0^P = 0, \end{aligned}$$

which proves our claim.

Hence there is a $1-1$ correspondence between the decomposition of $H = V \oplus W$ into closed π-invariant subspaces and the decomposition of $K = V_0 \oplus W_0$ into closed τ-invariant subspaces. This proves that τ is irreducible if and only if π is irreducible and (4.10) implies that τ is primary. □

Theorem 4.7 indicates that it is worthwhile to study the irreducible $*$-representations of $\mathrm{Pol}(U_q(1))$, which can be identified with the abelian algebra $\mathbf{C}[t, t^{-1}]$. Hence all irreducible $*$-representations are of the form $t \mapsto e^{i\varphi}$. As an example we classify the irreducible $*$-representations of $\mathrm{Pol}(U_q(2))$. They are described in the following theorem. (See Vaksman and Soibelman [19, theorem 3.2].)

THEOREM 4.8. (i) *Let $\mathscr{B}$ be the unital algebra generated by t_{11}, t_{21}, t_{12} and let π be an irreducible $*$-representation of* $\mathrm{Pol}(U_q(2))$ *in the Hilbert space H, then a unique $\pi(\mathscr{B})$-invariant line $\langle v \rangle \subset H$ exists.*

(ii) *The representations $\pi^1_{\theta,\varphi}$ and $\pi^\infty_{\theta,\varphi}$, θ, $\varphi \in [0, 2\pi)$ defined below constitute all irreducible, mutually inequivalent $*$-representations of* $\mathrm{Pol}(U_q(2))$.

(a) *$\pi^1_{\theta,\varphi}$ is a one dimensional representation defined by*

$$\pi^1_{\theta,\varphi}(t_{11}) = e^{i\theta};\ \pi^1_{\theta,\varphi}(t_{22}) = e^{i\varphi};\ \pi^1_{\theta,\varphi}(D^{-1}) = e^{-i(\theta+\varphi)};$$

$$\pi^1_{\theta,\varphi}(t_{12}) = 0 = \pi^1_{\theta,\varphi}(t_{21}).$$

(b) *$\pi^\infty_{\theta,\varphi}$ is an infinite dimensional representation in a Hilbert space H with orthonormal basis $\{e_k : k \in \mathbf{Z}_+\}$ defined by*

$$\pi^\infty_{\theta,\varphi}(t_{12})e_j = -q^{j+1}e^{i\varphi}e_j;\ \pi^\infty_{\theta,\varphi}(t_{21})e_j = q^j e^{i\theta}e_j;\ \pi^\infty_{\theta,\varphi}(D^{-1})e_j = e^{-i(\varphi+\theta)}e_j;$$

$$\pi^\infty_{\theta,\varphi}(t_{22})e_j = \sqrt{1 - q^{2(j+1)}}\,e_{j+1};$$

$$\pi^\infty_{\theta,\varphi}(t_{11})e_j = \begin{cases} e^{i(\theta+\varphi)}\sqrt{1-q^{2j}}\,e_{j-1}, & j \in \mathbf{N}; \\ 0, & j = 0. \end{cases}$$

Proof. For every irreducible $*$-representation π of $\mathrm{Pol}(U_q(2))$, we can construct the irreducible $*$-representation τ of the abelian algebra $\mathrm{Pol}(U_q(1)) \cong \mathbf{C}[t, t^{-1}]$. Hence τ is one dimensional and by proposition 4.4 the representation space of τ is $\pi(\mathscr{B})$-invariant.

To prove (ii) we first note that $\pi^1_{\theta,\varphi}$ and $\pi^\infty_{\theta,\varphi}$ are mutually inequivalent irreducible $*$-representations of $\mathrm{Pol}(U_q(2))$. We consider an irreducible $*$-representation π of $\mathrm{Pol}(U_q(2))$. Hence the results of this section are applicable. If $r = 1 - r$ as in Proposition 4.1 —, then it easily follows that $\pi \cong \pi^1_{\theta,\varphi}$ for some θ, $\varphi \in [0, 2\pi)$.

If $r = 0$, then application of the results of this section yields

$$\pi(t_{11})e_0 = 0, \qquad \pi(t_{21})e_0 = e^{i\theta}e_0, \qquad \pi(t_{12})e_0 = -qe^{i\varphi}e_0,$$

where e_0 denotes the normalised v of (i). This gives $\pi(D)e_0 = e^{i(\varphi+\theta)}e_0$ by (2.8). Now $K = H_0 = \mathrm{span}(e_0)$ and $H_k = \mathrm{span}(\pi(t_{22}^k)e_0)$, so $\pi(t_{22}^k)e_0$ yields an orthogonal basis for H. Furthermore, $t_{22}^* t_{22} = D^{-1}t_{11}t_{22} = I + qt_{12}t_{21}D^{-1}$ implies

$$(\pi(t_{22}^*)^k\pi(t_{22}^k)e_0, e_0) = (\pi(t_{22}^*)^{k-1}\pi(t_{22}^{k-1})e_0, e_0) \\ + q^{2k-1}(\pi(t_{22}^*)^{k-1}\pi(t_{22}^{k-1})\pi(t_{12}t_{21}D^{-1})e_0, e_0),$$

so that $\|\pi(t_{22}^k)e_0\|^2 = (1-q^2)\cdots(1-q^{2k})$. The action of $\pi(t_{kl})$ for $k, l = 1, 2$ on the normalised basis vector e_k can now be calculated and it corresponds to $\pi_{\theta,\varphi}^{\infty}(t_{kl})e_k$ as described in the theorem. Hence $\pi \cong \pi_{\theta,\varphi}^{\infty}$. □

The next theorem describes under which conditions on Γ the irreducible $\mathrm{Pol}(U_q(n))$-modules $L^\rho(\Gamma)$ yield irreducible $*$-representations. To formulate this theorem we need the following definition.

Define $l(\rho, i) = \#\{j \mid 1 \leqslant j < i, \rho(j) > \rho(i)\}$, then $l(\rho) = \sum_{i=1}^n l(\rho, i)$.

THEOREM 4.9. *For $\Gamma = (\gamma_1, \ldots, \gamma_n)$ with $|\gamma_i| = q^{l(\rho,i)}$ the modules $L^\rho(\Gamma)$ yield irreducible $*$-representations of* $\mathrm{Pol}(U_q(n))$.

We only have to show the existence of an inner product on $L^\rho(\Gamma)$ for these conditions on Γ, under which $L^\rho(\Gamma)$ becomes a $*$-representation. To prove this we start with a linear functional ω on $\mathscr{A}_q$, which we extend to $\mathrm{Pol}(U_q(n))$. We show that this linear functional yields a sesquilinear form on $V^\rho(\Gamma)$, which will finally yield the inner product on $L^\rho(\Gamma)$.

Next we state that these representations yield all possible mutually inequivalent irreducible $*$-representations of $\mathrm{Pol}(U_q(n))$. We will denote such a representation by $\pi^\rho(\Gamma)$.

THEOREM 4.10. *$\pi^\rho(\Gamma)$ for $\rho \in S_n$ and Γ as in theorem 4.9 yield all mutually inequivalent irreducible $*$-representations of* $\mathrm{Pol}(U_q(n))$.

The rest of this section is devoted to the proof of these theorems. As a byproduct we find a suitable basis for the representation space $L^\rho(\Gamma)$.

Let us introduce the linear map $\omega\colon \mathscr{A}_q \to \mathbf{C}$ by

$$\omega(n_\rho^- t_{\rho(1),1}^{p_1} \cdots t_{\rho(n),n}^{p_n} n_\rho^+) = \begin{cases} 0, & \text{if } n_\rho^- \neq I \neq n_\rho^+; \\ \gamma_1^{p_1} \cdots \gamma_n^{p_n}, & \text{otherwise.} \end{cases} \tag{4.11}$$

This is well-defined because of (3.5), where the n_ρ^- and n_ρ^+ are to be understood as ordered monomials. If we let $\mathscr{R}_\rho$ be the right ideal in $\mathscr{A}_q$ generated by the matrix elements of N_ρ^-, then we see that ω is defined on $\mathscr{R}_\rho\backslash\mathscr{A}_q/\mathscr{L}_\rho$, since $\mathscr{R}_\rho$ is the span of $n_\rho^- h_\rho n_\rho^+$ with $n_\rho^- \neq I$. This can be proved in a similar way as the corresponding statement for $\mathscr{L}_\rho$.

LEMMA 4.11. *For* $a \in \mathscr{A}_q(n)$ *we have*

(i) $\omega(at_{kl}) = 0$, *for* $t_{kl} \in N_\rho^+$,
(ii) $\omega(at_{\rho(i),i}) = \gamma_i \omega(a)$,
(iii) $\omega(t_{kl}a) = 0$, *for* $t_{kl} \in N_\rho^-$,
(iv) $\omega(t_{\rho(i),i}a) = \gamma_i \omega(a)$.

Proof. The proof uses the notions of badness and index as introduced in section 3. To prove (i) we let a be any, not necessarily ordered monomial and we put $x = at_{kl}$. We will show that $\omega(x) = 0$ by induction with respect to the badness of x, $b(x)$, and for fixed $b(x)$ by induction with respect to the index of x, $\mathrm{ind}(x)$.

If $b(x) = 0$ or if $\mathrm{ind}(x) = 0$, then clearly $\omega(x) = 0$. To make the induction step we have to consider two cases. First assume that $\mathrm{ind}(a) > 0$, then we have $at_{kl} = c_1 a_1 t_{kl} + c_2 a_2 t_{kl}$, where $b(a_i) < b(a)$ or $b(a_i) = b(a)$ and $\mathrm{ind}(a_i) < \mathrm{ind}(a)$. Hence $\omega(x) = 0$ in this case. Otherwise, if $\mathrm{ind}(a) = 0$, but $\mathrm{ind}(x) > 0$, then we must have $x = a_3 t_{\rho(i),j} t_{\rho(r),l}$ with $i > j$, $\rho^{-1}(k) = r > l$ and $t_{\rho(i),j} > t_{\rho(r),l}$. If $b(t_{\rho(i),j} t_{\rho(r),l}) = 0$ then we have $\omega(x) = c_3 \omega(a_3 t_{\rho(r),l} t_{\rho(i),j}) = 0$ by induction on the index. If $b(t_{\rho(i),j} t_{\rho(r),l}) = 1$ we have

$$\omega(x) = \omega(a_3 t_{\rho(r),l} t_{\rho(i),j}) \pm (q - q^{-1}) \omega(a_3 t_{\rho(i),l} t_{\rho(r),j}).$$

The first term yields zero by the induction on the index. To deal with the second term we note that $t_{\rho(i),l}$ and $t_{\rho(r),j}$ commute and at least one of them is an element of N_ρ^+, hence induction with respect to the badness shows that the second term yields zero as well.

For (ii) we let a be as in (i) and we use induction with respect to the length, badness and index of $at_{\rho(i),i}$. If the length is 1 or if $b(at_{\rho(i),i}) = 0$ or if $\mathrm{ind}(at_{\rho(i),i}) = 0$, then (ii) is true. Suppose next that $\mathrm{ind}(at_{\rho(i),i}) > 0$, then we have two possibilities. If $\mathrm{ind}(a) > 0$, then it can be dealt with in a similar way as in the proof of (i). If $\mathrm{ind}(at_{\rho(i),i}) > 0$, $\mathrm{ind}(a) = 0$, then we must have $at_{\rho(i),i} = a_1 t_{\rho(k),l} t_{\rho(i),i}$ with $k \geqslant l$ and $t_{\rho(k),l} > t_{\rho(i),i}$. If $k = l$, then $t_{\rho(k),k}$ and $t_{\rho(i),i}$ commute modulo $\mathscr{L}_\rho$ and hence $\omega(at_{\rho(i),i}) = \omega(a_1 t_{\rho(i),i} t_{\rho(k),k})$. By induction on the index and on the length this equals $\gamma_k \gamma_i \omega(a_1)$ and by induction on the length we have $\omega(a) = \gamma_k \omega(a_1)$.

If $k > l$ and $b(t_{\rho(k),l} t_{\rho(i),i}) = 0$, then $\omega(a) = 0$. But in this case part (i) implies that $\omega(at_{\rho(i),i}) = c\omega(a_1 t_{\rho(i),i} t_{\rho(k),l}) = 0$. Finally if $k > l$ and $b(t_{\rho(k),l} t_{\rho(i),i}) = 1$, then $\omega(a) = 0$ and

$$\omega(at_{\rho(i),i}) = \omega(a_1 t_{\rho(i),i} t_{\rho(k),l}) \pm (q - q^{-1}) \omega(a_1 t_{\rho(k),i} t_{\rho(i),l}).$$

Part (i) shows that the first term yields zero, but also that the second term vanishes, since $t_{\rho(k),i}$ and $t_{\rho(i),l}$ commute and at least one of them is an element of N_ρ^+.

The proof of (iii) and (iv) is analogous to the proof of (i) and (ii). □

Now for a basis element a as in (4.11) we calculate aD. By (2.8) we have

$$aD = \sum_{\sigma \in S_n} (-q)^{-l(\sigma)} n_\rho^- t_{\rho(1),1}^{p_1} \cdots t_{\rho(n),n}^{p_n} t_{\sigma(n),n} \cdots t_{\sigma(1),1} n_\rho^+$$

and from lemma 4.11 it follows that $\omega(aD) = 0$ if $n_\rho^- \neq I \neq n_\rho^+$. In case $n_\rho^- = I = n_\rho^+$ all terms vanish except for $\sigma = \rho$ by lemma 4.11. Hence for all a we find

$$\omega(aD) = (-q)^{-l(\rho)}(\gamma_1 \cdots \gamma_n)\omega(a).$$

This shows that if we define

$$\omega(aD^{-1}) = (-q)^{l(\rho)}(\gamma_1 \cdots \gamma_n)^{-1}\omega(a), \tag{4.12}$$

the map ω: $\mathrm{Pol}(U_q(n)) \to \mathbf{C}$ is well-defined. Note that we can extend lemma 4.11 to $a \in \mathrm{Pol}(U_q(n))$.

From now on we assume the condition of theorem 4.9 fulfilled, i.e. that $|\gamma_i| = q^{l(\rho,i)}$.

LEMMA 4.12. *For* $a \in \mathrm{Pol}(U_q(n))$ *we have*

(i) $\omega(t_{kl}^* a) = 0$, *for* $t_{kl} \in N_\rho^+$,
(ii) $\omega(t_{\rho(i),i}^* a) = \bar{\gamma}_i \omega(a)$.

Proof. To prove (ii) we consider $t_{\rho(i),i}^* = (-q)^{i-\rho(i)} D^{\rho(i),i} D^{-1}$ modulo the right ideal $\mathscr{R}_\rho$. We find

$$t_{\rho(i),i}^* = (-q)^{i-\rho(i)}(-q)^{-l(\rho')}(t_{\rho(n),n} \cdots \hat{t}_{\rho(i),i} \cdots t_{\rho(1),1})D^{-1}, \quad \mathrm{mod}\ \mathscr{R}_\rho, \tag{4.13}$$

where $\rho' \in S_{n-1}$ is obtained from ρ by restriction to $\{1, \ldots, \hat{i}, \ldots, n\}$. Hence,

$$\omega(t_{\rho(i),i}^* a) = (-q)^{i-\rho(i)}(-q)^{-l(\rho')}(-q)^{l(\rho)}\gamma_i^{-1}\omega(a).$$

Now $\bar{\gamma}_i = \gamma_i^{-1} q^{2l(\rho,i)}$ and since

$$l(\rho') = l(\rho) - l(\rho, i) - \#\{k \mid i < k,\ \rho(i) > \rho(k)\}$$

and

$$l(\rho, i) - \#\{k \mid i < k,\ \rho(i) > \rho(k)\} = i - \rho(i)$$

(ii) follows.

Finally, to see (i) we develop the quantum minor D^{kl} consecutively along the $\rho(1)$th row until the $\rho(l)$th row. Application of ω after each of the developments along the row $\rho(1), \ldots, \rho(l-1)$ leaves only one term, which can be dealt with by lemma 4.11. The development along the $\rho(l)$th row yields zero. □

Lemmas 4.11 and 4.12 imply that

$$(\, , \,): V^{\rho}(\Gamma) \times V^{\rho}(\Gamma) \to \mathbf{C}$$

$$(a \cdot v^{+}, b \cdot v^{+}) = \omega(b^{*}a) \tag{4.14}$$

is a well defined sesquilinear form on $V^{\rho}(\Gamma)$. Note that the only possible eigenvalues for the action of $t_{\rho(1),1}$ on $V^{\rho}(\Gamma)$ are $\gamma_1 q^k$ for $k \in \mathbf{Z}_+$. This follows directly from the proof of proposition 3.5(iii).

It follows from (4.14) and proposition 3.5(ii) that the occurrence of t_{i1} or t_{i1}^{*}, $\rho(1) < i \leqslant n$, in a product a of matrix elements gives $\omega(a) = 0$.

PROPOSITION 4.13. *Suppose* $n_S^{+}, m_S^{+} \in \mathcal{N}_{\rho}^{+} \cap \mathrm{Pol}(U_q(n-1))$ *with the identification* (4.5) *with* $r = \rho(1)$. *Then we have with* $s = \rho(1) - 1$

$$\omega(m_S^{+} t_{11}^{q_1} \cdots t_{s1}^{q_s} (t_{s1}^{*})^{p_s} \cdots (t_{11}^{*})^{p_1} (n_S^{+})^{*}) = \begin{cases} 0, & \text{if } (p_1, \ldots, p_s) \neq (q_1, \ldots, q_s); \\ C\omega(m_S^{+}(n_S^{+})^{*}), & \text{with } C > 0 \text{ if } (p_1, \ldots, p_s) = (q_1, \ldots, q_s), \end{cases}$$

and

$$m_S^{+} t_{11}^{q_1} \cdots t_{s1}^{q_s} (t_{s1}^{*})^{p_s} \cdots (t_{11}^{*})^{p_1} (n_S^{+})^{*} \cdot v^{+} = \begin{cases} 0, & \text{if } (p_1, \ldots, p_s) < (q_1, \ldots, q_s); \\ C m_S^{+}(n_S^{+})^{*} \cdot v^{+}, & \text{with } C > 0 \text{ if } (p_1, \ldots, p_s) = (q_1, \ldots, q_s). \end{cases}$$

The ordering is defined by $(p_1, \ldots, p_s) < (q_1, \ldots, q_s)$ *if* $\sum_{i=1}^{s} p_i < \sum_{i=1}^{s} q_i$ *or if* $\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i$ *and* $(p_1, \ldots, p_s) < (q_1, \ldots, q_s)$ *in the lexicographical ordering.*

Proof. We start with some identities in $V^{\rho}(\Gamma)$.

$$t_{i1}(n_S^{+})^{*} \cdot v^{+} = \begin{cases} 0, & \text{if } i \neq \rho(1); \\ \gamma_1 (n_S^{+})^{*} \cdot v^{+}, & \text{if } i = \rho(1), \end{cases} \tag{4.15}$$

$$t_{\rho(1),1}^{*} t_{\rho(1),1} (n_S^{+})^{*} \cdot v^{+} = (n_S^{+})^{*} \cdot v^{+}. \tag{4.16}$$

First we prove (4.15). By (2.13) $t_{\rho(1),1}$ commutes with $(n_S^{+})^{*}$, which gives the result for $i = \rho(1)$. If $i < \rho(1)$, then $t_{\rho(1),1} t_{i1} (n_S^{+})^{*} \cdot v^{+} = q^{-1} \gamma_1 t_{i1} (n_S^{+})^{*} \cdot v^{+}$, which proves (4.15) in this case, because of the remark following (4.14). If $i > \rho(1)$, then it follows from proposition 3.5(ii).

For (4.16) we need to show that $t^*_{\rho(1),1}(n_S^+)^*\cdot v^+ = \bar\gamma_1(n_S^+)^*\cdot v^+$, which can be proved by induction on the degree of $(n_S^+)^*$. If the degree of $(n_S^+)^*$ is zero, we use (4.13) with $\mathscr{R}_\rho$ replaced by $\mathscr{L}_\rho$, which is easily seen to be true. This implies that $t^*_{\rho(1),1}\cdot v^+ = \bar\gamma_1\cdot v^+$. To make the induction step we note that either $t^*_{\rho(1),1}$ and t^*_{kl} commute or

$$t^*_{kl}t^*_{\rho(1),1} - t^*_{\rho(1),1}t^*_{kl} = (q-q^{-1})t^*_{k1}t^*_{\rho(1),l}.$$

The first case is obvious and in the second case we develop the quantum minor $D^{\rho(1),l}$ along the first column and apply (4.15). This proves (4.15) and (4.16).

We first prove the second assertion. Put $P = n_S^+ t_{11}^{p_1}\cdots t_{b,1}^{p_b}$, $(p_b \geqslant 1)$, $Q = m_S^+ t_{11}^{q_1}\cdots t_{a,1}^{q_a}$, $(q_a\geqslant 1)$, and $Q = Q't_{a1}$. Consider the case $k = \sum_{i=1}^s p_i < \sum_{i=1}^s q_i = l$, then the vector $v = QP^*\cdot v^+$ is an eigenvector of $t_{\rho(1),1}$ with eigenvalue $\gamma_1 q^{k-l}$. According to the remark following (4.14) we have $v = 0$.

Assume now that $k = \sum_{i=1}^s p_i = \sum_{i=1}^s q_i$, then we will use induction on k, the case $k=0$ being trivial. Since (2.14) reduces to $qt^*_{r1}t_{i1} = t_{i1}t^*_{r1}$ for $s=1$, we have

$$QP^*\cdot v^+ = q^{p_b+\cdots+p_{a+1}}Q'(t^*_{b1})^{p_b}\cdots(t^*_{a+1,1})^{p_{a+1}}t_{a1}(t^*_{a1})^{p_a}\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+.$$

In particular, if $b<a$ we find $QP^*\cdot v^+ = 0$ by (4.15). Now we assume $a\leqslant b$ and we use (2.16) with $s=1$ to interchange t_{a1} and t^*_{a1}. This yields

$$\begin{aligned} QP^*\cdot v^+ = q^{p_b+\cdots+p_{a+1}}\{&Q'(t^*_{b1})^{p_b}\cdots(t^*_{a+1,1})^{p_{a+1}}t^*_{a1}t_{a1}(t^*_{a1})^{p_a-1}\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+ \\ &+(1-q^2)\sum_{i=a+1}^n Q'(t^*_{b1})^{p_b}\cdots(t^*_{a+1,1})^{p_{a+1}}t^*_{i1}t_{i1}(t^*_{a1})^{p_a-1}\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+\}.\end{aligned}$$

By (2.14), (2.1) we see that we can pull $t^*_{i1}t_{i1}$ to the right in the sum at the cost of a factor $q^{2(p_a-1+p_{a-1}+\cdots+p_1)}$. Now (4.15) and (4.16) give

$$\begin{aligned} QP^*\cdot v^+ = q^{p_b+\cdots+p_{a+1}}\{&Q'(t^*_{b1})^{p_b}\cdots(t^*_{a+1,1})^{p_{a+1}}t^*_{a1}t_{a1}(t^*_{a1})^{p_a-1}\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+ \\ &+(1-q^2)q^{2(p_a-1+p_{a-1}+\cdots+p_1)}Q'(t^*_{b1})^{p_b} \\ &\cdots(t^*_{a+1,1})^{p_{a+1}}(t^*_{a1})^{p_a-1}\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+\}.\end{aligned}$$

Repeating this process on the first part of the right hand side yields

$$\begin{aligned} QP^*\cdot v^+ = q^{p_b+\cdots+p_{a+1}}\{&Q'(t^*_{b1})^{p_b}\cdots(t^*_{a+1,1})^{p_{a+1}}(t^*_{a1})^{p_a}t_{a1}(t^*_{a-1,1})^{p_{a-1}} \\ &\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+ \\ &+(1-q^{2p_a})q^{2(p_{a-1}+\cdots+p_1)}Q'(t^*_{b1})^{p_b}\cdots(t^*_{a+1,1})^{p_{a+1}}(t^*_{a1})^{p_a-1}\cdots(t^*_{11})^{p_1}(n_S^+)^*\cdot v^+\}.\end{aligned}$$

Now we can pull t_{a1} to the right by (2.14) at the cost of some power of q and

(4.15) shows that the first term yields zero. The induction hypothesis can now be applied to the second term and it yields zero if $P < Q$ and $Cm_S^+(n_S^+)^* \cdot v^+$ if $P = Q$.

If $k = \sum_{i=1}^s p_i > \sum_{i=1}^s q_i = l$, then $PQ^* \cdot v^+ = 0$. So (4.14) with $a = I$ and $b = PQ^*$ implies $\omega(QP^*) = 0$. If $k < l$, then it follows from part two that $\omega(QP^*) = 0$. Next we assume $k = l$. Taking inner products with v^+ in the equalities above then yields

$$\omega(QP^*) = q^{p_b + \cdots + p_{a+1}}(1 - q^{2p_a})q^{2(p_{a-1} + \cdots + p_1)}$$
$$\times \omega(Q'(t_{b1}^*)^{p_b} \cdots (t_{a+1,1}^*)^{p_{a+1}}(t_{a1}^*)^{p_a - 1} \cdots (t_{11}^*)^{p_1}(n_S^+)^*)$$

and induction on k finishes the job. □

Next we will establish a link between the linear functional ω on $\mathrm{Pol}(U_q(n))$ and a similarly defined linear functional ω' on $\mathrm{Pol}(U_q(n-1))$ We use (4.5) with $n - r$ replaced by $\rho(1)$ to obtain an embedding $\mathscr{A}_q(n-1) \subset \mathscr{A}_q(n)$. If we define $\rho'(i-1) = \rho(i)$ if $\rho(i) < \rho(1)$ and $\rho'(i-1) = \rho(i) - 1$ if $\rho(i) > \rho(1)$, then $\rho' \in S_{n-1}$ and $t_{\rho(i),i} = s_{\rho'(i-1),i-1}$ for $i = 2, \ldots, n$. Note that $l(\rho) = l(\rho') + \#\{i \mid \rho(i) < \rho(1)\}$.

Define

$$\gamma'_{i-1} = \begin{cases} \gamma_i, & \text{if } \rho(i) > \rho(1); \\ -q^{-1}\gamma_i, & \text{if } \rho(i) < \rho(1), \end{cases} \tag{4.17}$$

then $|\gamma'_i| = q^{l(\rho',i)}$. Using the above identification we consider

$$\omega'\colon \mathrm{Pol}(U_q(n-1)) \to \mathbf{C}$$

defined by

$$\begin{cases} \omega'(n_S^- t_{\rho(2),2}^{q_2} \cdots t_{\rho(n),n}^{q_n} n_S^+) = \begin{cases} 0, & \text{if } n_S^- \neq I \neq n_S^+, \\ (\gamma'_1)^{q_2} \cdots (\gamma'_{n-1})^{q_n}, & \text{if } n_S^- = I = n_S^+. \end{cases} \\ \omega'(aD_S^{-1}) = (-q)^{l(\rho')}(\gamma'_1 \cdots \gamma'_{n-1})^{-1}\omega'(a). \end{cases} \tag{4.18}$$

From (4.18) and (4.11) with (4.17) it follows that for $i_k \neq \rho(1)$, $j_k \neq 1$ we have

$$\omega\left(\prod_{k=1}^N t_{i_k,j_k}\right) = (-q)^{\#\{k \mid i_k < \rho(1)\}}\omega'\left(\prod_{k=1}^N t_{i_k,j_k}\right). \tag{4.19}$$

We can even generalise (4.19) to the whole of $\mathrm{Pol}(U_q(n-1))$ as follows.

PROPOSITION 4.14. *For products of matrix elements $x, y \in \mathscr{A}_q(n-1)$ we have*

$$\omega(xy^*) = (-q)^{\mu(x)+\mu(y)}\omega'(xy^{*_S}),$$

where $*_S$ denotes the $*$-operation in $\mathrm{Pol}(U_q(n-1))$ *and $\mu(x)$ denotes the number of matrix elements t_{ij} in x with $i > \rho(1)$.*

Proof. Pick $t_{ij} \in \mathscr{A}_q(n-1)$ and develop the quantum minor D^{ij} along the first column. All terms cancel except for $\rho(1)$, which yields a factor γ_1. To see this we note that for $i_k \neq \rho(1)$, $j_k \neq 1$ we have

$$\omega(xt_{i1}t^*_{i_1,j_1}\cdots t^*_{i_N,j_N}) = \begin{cases}\gamma_1\omega(xt^*_{i_1,j_1}\cdots t^*_{i_N j_N}), & \text{if } i = \rho(1),\\ 0, & \text{if } i \neq \rho(1).\end{cases}$$

Since $t_{\rho(1),1}$ commutes with all possible elements the case $i = \rho(1)$ is a direct consequence of lemma 4.11. To prove the case $i \neq \rho(1)$ we use induction on N, the case $N=0$ being contained in lemma 4.11. If $i_1 \neq i$, then $[t_{i1}, t^*_{i_i,j_1}]=0$ by (2.13) and this gives the induction step in this case. Finally, if $i = i_1$, then by (2.15)

$$\begin{aligned}\omega(xt_{i1}t^*_{i,j_1}\cdots t^*_{i_N j_N}) &= q^{-1}\omega(xt^*_{i,j_1}t_{i1}t^*_{i_2,j_2}\cdots t^*_{i_N j_N})\\ &\quad+(q^{-1}-q)\sum_{l=i+1}^{n}\omega(xt^*_{l,j_1}t_{l1}t^*_{i_2\,j_2}\cdots t^*_{i_N j_N}).\end{aligned}$$

If $i > \rho(1)$ this is zero by the induction hypothesis. If $i < \rho(1)$, then we use (2.12) with $j = 1$ and $i = j_1$. Then $\sum_{l=i+1}^{n} t^*_{l,j_1}t_{l1} = -\sum_{l=1}^{i} t^*_{l,j_1}t_{l1}$, and we obtain the same conclusion.

Hence,

$$\omega\left(x\prod_{k=1}^{N} t^*_{i_k,j_k}\right) = (\gamma_2\cdots\gamma_n)^{-N}(-q)^{Nl(\rho)}\omega\left(x\prod_{k=1}^{N}(-q)^{j_k-i_k}(-q)^{\alpha_{i_k}}(D^{\rho(1),1})^{i_k,j_k}\right),$$

where $\alpha_i = 1-\rho(1)$ if $\rho(1) < i$ and $\alpha_i = 2-\rho(1)$ if $\rho(1) > i$. Under the mentioned identification we have $D_S = D^{\rho(1),1}$ and

$$t^{*_S}_{ij} = (-q)^{j-i}c_i D_S^{-1}(D^{\rho(1),1})^{ij},$$

where $c_i = 1$ if $\rho(1) < i$ and $c_i = (-q)^{-1}$ if $\rho(1) > i$. Thus

$$\begin{aligned}&\omega'\left(x\prod_{k=1}^{N} t^{*_S}_{i_k,j_k}\right)\\ &= (\gamma'_1\cdots\gamma'_{n-1})^{-N}(-q)^{Nl(\rho')}\omega'\left(x\prod_{k=1}^{N}(-q)^{j_k-i_k}c_{i_k}(D^{\rho(1),1})^{i_k,j_k}\right).\end{aligned}$$

Now (4.17), (4.18) and (4.19) imply the proposition. □

Proof of theorem 4.9. We will use the following basis for $\mathcal{N}_\rho^+$

$$n_0P^+ = \left(\prod_{j=1}^{n} \prod_{\substack{i=n \\ \rho^{-1}(i)>j}}^{\rho(j)+1} t_{ij}^{q_{ij}}\right)\left(\prod_{j=1}^{n} \prod_{\substack{i=1 \\ \rho^{-1}(i)>j}}^{\rho(j)-1} t_{ij}^{p_{ij}}\right),$$

where we order the first part in an arbitrary fashion and the second part according to $t_{ij} < t_{kl}$ if $j > l$ or if $j = l$ and $i < k$. Then $n_\rho^- h_\rho (D^{-1})^l n_0 P^+$ span $\mathrm{Pol}(U_q(n))$.

Let us assume for the moment that ω satisfies

$$\omega(n_\rho^- h_\rho (D^{-1})^l n_0 P^+ (Q^+)^* (m_0)^* (D)^p g_\rho^* (m_\rho^-)^*) = 0 \tag{4.20}$$

whenever $n_0 \neq I$ or $m_0 \neq I$ or $n_\rho^- \neq I$ or $m_\rho^- \neq I$ or $P^+ \neq Q^+$. Then we can write an arbitrary $a \in \mathrm{Pol}(U_q(n))$ in a possibly non-unique way as

$$a = \sum_{i\in I} (P_i^+)^* (n_0)_i^* D^{l_i} h_i^* (n_i^-)^* + \sum_{j\in J} \sum_{k\in K_j} (P_j^+)^* D^{l_k} h_k^*,$$

where for all $i \in I$ either $(n_0)_i \neq I$ or $n_i^- \neq I$. Then $\omega(a^*a)$ is independent of the form of a, since ω is well defined on $\mathrm{Pol}(U_q(n))$. By (4.20) we have

$$\omega(a^*a) = \sum_{j\in J} \sum_{k,p\in K_j} \omega(h_p (D^{-1})^{l_p} P_j^+ (P_j^+)^* D^{l_k} h_k^*).$$

By propositions 4.13 and 4.14 and (4.11), (4.12), (4.17), (4.18), (4.19) as well as lemmas 4.11 and 4.12 we can find $C_j > 0$, $b_k \in \mathbf{C}$ and $a_k \in \mathscr{A}_q(n-1)$ so that

$$\omega(a^*a) = \sum_{j\in J} C_j \sum_{k,p\in K_j} b_p \overline{b_k} \omega'(a_p a_k^{*s}).$$

This implies that ω is a positive linear functional on $\mathrm{Pol}(U_q(n))$ if ω' is a positive linear functional on $\mathrm{Pol}(U_q(n-1))$ and ω satisfies (4.20). We will show that these conditions are fulfilled by induction with respect to n.

Let us therefore assume that for every $\rho' \in S_{n-1}$ and $\Gamma' \in \mathbf{C}^{n-1}$ as formulated in theorem 4.9 an irreducible $*$-representation $\pi^{\rho'}(\Gamma')$ in $L^{\rho'}(\Gamma')$ exists so that the corresponding positive linear functional

$$\omega' \colon \mathrm{Pol}(U_q(n-1)) \to \mathbf{C}$$

$$\omega'(a) = (\pi^{\rho'}(\Gamma')(a)v^+, v^+)$$

satisfies (4.18) and (4.20) in the $\mathrm{Pol}(U_q(n-1))$ case. This is obvious for $n-1=1$.

So it is sufficient to prove (4.20). This is a direct consequence of the assumption on ω', propositions 4.13 and 4.14, lemmas 4.11 and 4.12 as well as (4.14).

Hence, for arbitrary $\rho \in S_n$ and $\Gamma \in \mathbf{C}^n$ as described in the theorem we have a positive linear functional ω. Now

$$K = \{a \cdot v^+ \in V^\rho(\Gamma) \mid \omega(a^*a) = 0\}$$

is a proper invariant subspace and $V^\rho(\Gamma)/K$ yields a ρ-highest weight $*$-representation of weight Γ of $\mathrm{Pol}(U_q(n))$. According to corollary 3.7 this representation is irreducible and by proposition 3.5 it is isomorphic to $L^\rho(\Gamma)$. □

Proof of theorem 4.10. The representations $\pi^\rho(\Gamma)$ are mutually inequivalent. Now it can be proved that every irreducible $*$-representation of $\mathrm{Pol}(U_q(n))$ is a ρ-highest weight module of weight Γ with $|\gamma_i|q^{l(\rho,i)}$ by induction on n by use of theorems 4.8, 4.7, (4.6) and proposition 4.4 and corollary 4.5. □

COROLLARY 4.15. (i) *With the ordering* $t_{ij} < t_{kl}$ *if* $j > l$ *or if* $j = l$ *and* $i < k$ *an orthogonal basis of* $L^\rho(\Gamma)$ *is given by*

$$\pi^\rho(\Gamma)((P^+)^*)v^+ = \prod_{j=1}^{n} \prod_{\substack{i=1 \\ \rho^{-1}(i) > j}}^{\rho(j)-1} \pi^\rho(\Gamma)((t_{ij}^*)^{p_{ij}})v^+, \qquad p_{ij} \in \mathbf{Z}_+.$$

(ii) *There exists a total ordering on the elements* P^+ *as in* (i) *so that*

$$\pi^\rho(\Gamma)(Q^+(P^+)^*)v^+ = 0, \qquad Q^+ > P^+;$$

$$\pi^\rho(\Gamma)(P^+(P^+)^*)v^+ = Cv^+, \qquad C > 0.$$

Proof. Part (i) is clear and we could prove part (ii) by induction on n and proposition 4.13, if we could show that for $n_S \in \mathscr{A}_q(n-1))$ we have

$$\pi^\rho(\Gamma)(n_S^*)v^+ = C\pi^{\rho'}(\Gamma')(n_S^{*s})v^+, \qquad C \in \mathbf{C}^*.$$

Since $\pi^\rho(\Gamma)$ is an irreducible $*$-representation this follow from theorem 4.7. □

5. The quantum group $C(U_q(n))$

In this section we complete the Hopf $*$-algebra $\mathrm{Pol}(U_q(n))$ into a C^*-algebra $C(U_q(n))$. Then we will recall the definition of a compact matrix quantum group and we will show that the constructed C^*-algebra $C(U_q(n))$ fits into this definition. We end this section by proving that the C^*-algebra $C(U_q(n))$ is of type I.

By $\mathscr{R}$ we denote the collection of all $*$-representations of $\mathrm{Pol}(U_q(n))$. For an element $a \in \mathrm{Pol}(U_q(n))$ we put

$$\|a\| = \sup_{\pi \in \mathscr{R}} \|\pi(a)\|, \tag{5.1}$$

where we use the operator norm in $B(H_\pi)$ on the right hand side and H_π is the representation space of π. Note that $\|a\| < \infty$, since $\|\pi(t_{ij})\| \leqslant 1$ for all $\pi \in \mathscr{R}$ because of (2.12).

We want to complete $\mathrm{Pol}(U_q(n))$ with respect to the norm (5.1) to obtain a C^*-algebra in which $\mathrm{Pol}(U_q(n))$ is dense. So we have to show that all non-zero elements of $\mathrm{Pol}(U_q(n))$ have a non-zero norm. To prove this we restrict our attention to the irreducible $*$-representations $\pi(\Gamma) = \pi^{\rho_0}(\Gamma)$ with ρ_0: $i \mapsto n + 1 - i$. Note that τ as constructed in theorem 4.7 from $\pi(\Gamma)$ possesses the same property.

For the representation space of $\pi(\Gamma)$ one can give two natural bases. First the one described in corollary 4.15, $\pi(\Gamma)((n^+)^*)v^+$, $n^+ \in \mathcal{N}^+_{\rho_0} = \mathcal{N}^+$, but in this special case one proves in a similar way that $\pi(\Gamma)(n^-)v^+, n^- \in \mathcal{N}^-_{\rho_0} = \mathcal{N}^-$ yields an orthogonal basis for $L^\rho(\Gamma)$ as well. This basis cannot be expected to exist for other choices of ρ, since $\# N^-_\rho = \binom{n}{2}$ for all $\rho \in S_n$, whereas

$$\#\{t_{ij} \mid 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant \rho(j) - 1,\ \rho^{-1}(i) > j\} < \binom{n}{2}$$

for all $\rho \neq \rho_0$.

PROPOSITION 5.1. *Let* $a \in \mathrm{Pol}(U_q(n))$ *with* $\|a\| = 0$, *then* $a = 0$.

Proof. Theorem 3.4 implies that we can write an arbitrary element $a \in \mathrm{Pol}(U_q(n))$ as

$$a = \sum_{k \in K} \sum_{i \in I_k} \sum_{j \in J_{i,k}} c_{i,j,k} n_i^- (D^{-1})^{l_j} h_j n_k^+,$$

where are all index sets are finite. We suppose that all n_k^+ are different for different $k \in K$ and that the n_i^-'s are different for $i \in I_k$. Let n_s^+ be the smallest element in the ordering of corollary 4.15 so that $c_{i,j,s} \neq 0$ for some i and j. We will prove that $\|a\| = 0$ implies $c_{i,j,s} = 0$.

Let $\pi(\Gamma)(a) = 0$ act on the basis element $\pi(\Gamma)((n_s^+)^*)v^+$ of the representation space of $\pi(\Gamma)$. By corollary 4.15 we have

$$0 = \sum_{i\in I_s} \sum_{j\in J_{i,s}} c_{i,j,s}(-q)^{(l_j/2)n(n-1)}\Gamma^{\alpha_j - l_j(1,\dots,1)}\pi(\Gamma)(n_i^-)v^+,$$

where α_j is some multiindex. From the remark above it follows that $\pi(\Gamma)(n_i^-)v^+$ also yields a basis for the representation space of $\pi(\Gamma)$. Hence,

$$0 = \sum_{j\in J_{i,s}} c_{i,j,s}(-q)^{(l_j/2)n(n-1)}\Gamma^{\alpha_j - l_j(1,\dots,1)} \qquad \forall\, i\in I_s.$$

Since all the $\Gamma^{\alpha_j - l_j(1,\dots,1)}$ are different by theorem 3.4, we have a polynomial in $\gamma_1, \dots, \gamma_n$, which must be zero for all possible choices of $\gamma_1, \dots, \gamma_n$, as described in theorems 4.9 and 4.10. This implies $c_{i,j,s} = 0$ for all $j \in J_{i,s}$ and all $i \in I_s$, which proves the proposition. □

Let $C(U_q(n))$ be the completion of $\mathrm{Pol}(U_q(n))$ with respect to this norm (5.1), then $C(U_q(n))$ is a C^*-algebra in which $\mathrm{Pol}(U_q(n))$ is dense.

Next we will consider the quantum group associated with $U_q(n)$ in the sense of Woronowicz. We define

$$u = \begin{bmatrix} t_{11} & \cdots & t_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_{n1} & \cdots & t_{nn} & 0 \\ 0 & \cdots & 0 & D^{-1} \end{bmatrix} \in M_{n+1}(\mathrm{Pol}(U_q(n))). \tag{5.2}$$

So we will use an analogue of $U(n) \cong S(U(n) \times U(1)) \subset SU(n+1)$.

Let us now recall the definition of a compact matrix quantum group, cf [21, def. 1.1]. The pair (A, u), consisting of a unital C^*-algebra A and a $N \times N$ matrix $u = (u_{ij})_{i,j=1,\dots,N}$, $u_{ij} \in A$ is a *compact matrix quantum group* if the following three conditions are satisfied:

(i) the $*$-algebra $\mathscr{A}$ generated by the matrix elements u_{ij} is dense in A,
(ii) there exists a unital C^*-homomorphism $\Phi\colon A \to A \otimes_{\min} A$ so that

$$\Phi(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}, \quad \forall i, j \in \{1, \dots, N\}, \tag{5.3}$$

(iii) there exists a linear antimultiplicative mapping $\kappa\colon \mathscr{A} \to \mathscr{A}$ so that

$$\kappa(\kappa(a^*)^*) = a, \quad \forall a \in \mathscr{A}$$

and

$$\sum_{r=1}^{N} \kappa(u_{kr})u_{rl} = \delta_{kl} I = \sum_{r=1}^{N} u_{kr}\kappa(u_{rl}), \quad \forall\, k, l \in \{1, \ldots, N\}.$$

REMARK. The algebraic tensor product $A \otimes A$ is in a natural way a $*$-algebra. We can make it into a C^*-algebra by completing $A \otimes A$ with respect to the injective C^*-cross norm defined by

$$\|x\| = \sup_{\pi_1, \pi_2 \in \mathscr{R}} \|(\pi_1 \otimes \pi_2)(y)\|.$$

The resulting C^*-algebra is the *injective tensor product* $A \otimes_{\min} A$. See Takesaki, [18, Chapter 4.4].

THEOREM 5.2. *$(C(U_q(n)), u)$ with u as in (5.2) is a compact matrix quantum group.*

Proof. Condition (iii) follows directly from the results in sections 1 and 2. Condition (i) is fulfilled because of the above construction of $C(U_q(n))$. From section 2 it also follows that Φ is a unital $*$-homomorphism from $\mathrm{Pol}(U_q(n)) \to \mathrm{Pol}(U_q(n)) \otimes \mathrm{Pol}(U_q(n))$ satisfying (5.3). So we only have to prove that Φ can be extended from $\mathrm{Pol}(U_q(n))$ to $C(U_q(n))$.

Let us prove the continuity of Φ on $\mathrm{Pol}(U_q(n))$. Pick $\pi_1, \pi_2 \in \mathscr{R}$, then $(\pi_1 \otimes \pi_2) \circ \Phi$ is a $*$-representation of $\mathrm{Pol}(U_q(n))$ in the Hilbert space $H_{\pi_1} \hat{\otimes}_2 H_{\pi_2}$ (cf Takesaki [18, Chapter 4.1]). Hence $(\pi_1 \otimes \pi_2) \circ \Phi \in \mathscr{R}$ and

$$\|\Phi(x)\| = \sup_{\pi_1, \pi_2 \in \mathscr{R}} \|(\pi_1 \otimes \pi_2) \circ \Phi(x)\| \leqslant \|x\|. \qquad \square$$

In case A is a type I C^*-algebra there is only one C^*-cross norm on the algebraic tensor product $A \otimes A$. Since $C(U_q(1))$ is abelian it is a type I C^*-algebra. For $C(U_q(2))$ we can prove that it is a type I C^*-algebra as follows. First we note that all irreducible $*$-representations (cf theorem 4.8) contain a Hilbert-Schmidt operator. In case $\pi^1_{\theta,\varphi}$ this is trivial and in the infinite dimensional case we see that $\pi^\infty_{\theta,\varphi}(t_{21})$ is Hilbert-Schmidt. Now Sakai [16, theorem 4.6.4], which states that a C^*-algebra A for which every non-zero irreducible $*$-representation π there exists a non-zero compact operator in $\pi(A)$ is a type I C^*-algebra, implies that $C(U_q(2))$ is a type I C^*-algebra.

This criterion can also be used to show that $C(U_q(3))$ is a type I C^*-algebra. There are $3! = 6$ possibilities for the representations which all have to be checked on the existence of a non-zero compact operator. See Bragiel [2] for the case $C(SU_q(3))$, which is analogous to the case $C(U_q(3))$. However, for $n \geqslant 4$ we need another criterion to prove that $C(U_q(n))$ is a type I C^*-algebra. According to Dixmier [3, théorème 9.1] a separable C^*-algebra is of type I if and only if all primary representations are of type I.

THEOREM 5.3. *$C(U_q(n))$ is a type* I *C^*-algebra.*

Proof. We use induction on n, the cases $n = 1, 2$ being covered. Since $C(U_q(n))$ is separable, we have to show that all primary representations of $C(U_q(n))$ are of type I.

Let π be a primary representation of $C(U_q(n))$, then we can consider π as a primary representation of $\mathrm{Pol}(U_q(n))$ and because of (5.1) all primary representations arise in this way. Construct the primary representation τ of $C(U_q(n-1))$ as in theorem 4.7, then by the induction hypothesis we have $\tau = \tau_1 \oplus \tau_2$, where τ_1 is irreducible. In the corresponding decomposition of $\pi = \pi_1 \oplus \pi_2$ (cf proof of theorem 4.7) π_1 is irreducible. Hence π is of type I. □

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