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Introduction

The well-known finite-dimensional complement theorems in shape theory (see e.g. [16], [17], [20]) assert that compacta in $\mathbb{R}^n$ (or $S^n$), satisfying suitable dimension and embedding conditions, have the same shape if and only if their complements are homeomorphic. This classifies the shapes of compacta by a geometric property of the complements, but at the same time fails to exhibit a classifying homotopy property. To compensate this deficiency, we developed in [13] a theory of finite-dimensional categorical complement theorems. The reader is assumed to be familiar with notation and results of [13], and should recall in particular that $(\mathcal{URR}_m, (\text{Ad}_m), \mathcal{W}_mC)$ are data of a categorical complement theorem for $\text{Sh}(\mathcal{CM}_m)$, the Borsuk-Mardesić shape category of compacta $X$ with fundamental dimension $FdX \leq m$. This result removes all geometric niceness conditions on the ambient spaces (such conditions are indispensable to establish 'ordinary' complement theorems) and specifies what embeddings are acceptable in order that the complements keep sufficient homotopical information for the classification of shapes. An immediate consequence is the following 'homotopical complement theorem': Compacta $X_i$ with $FdX_i \leq m$, being $m$-admissibly embedded into $m$-connected ANR's $M_i$ with a complete uniform structure, have the same shape if and only if their complements $M_i - X_i$ have the same weak complete $m$-homotopy type.

The present paper continues the program of [13] in the framework of strong shape theory (concerning 'strong shape' see [1], [3], [4], [5], [7], [10]). Our main result is the following.

THEOREM A. $(\mathcal{URR}_{m+1}, (\text{Ad}_{m+2}), \mathcal{H}_{m+1}C)$, where $\mathcal{H}_{m+1}C$ is the complete $(m + 1)$-homotopy category (cf. section 2), are data of a categorical complement theorem for the strong shape category $\text{SSh}(\mathcal{CM}_{m+1})$ of compacta $X$ with $FdX \leq m$. In particular: Compacta $X_i$ with $FdX_i \leq m$, being $(m + 2)$-admissibly embedded into $(m + 1)$-connected ANR's $M_i$ with a complete uniform structure, have the same
shape if and only if their complements have the same complete \((m + 1)\)-homotopy type.

We see that the passage from shape to strong shape costs one additional dimension of control on the complements — \(\text{wH}_m C\) has to be replaced by \(\text{H}_{m+1} C\) (this ‘additional dimension of control’ reflects exactly the conceptual difference between shape and strong shape). The essential advantage of Theorem A is that the complementary category \(\text{H}_{m+1} C\) is a ‘homotopy category’ and not a ‘weak homotopy category’ as in the case of ordinary shape. The price for such a more natural complementary category is a more restrictive embedding condition — \((\text{Ad}_m)\) has to be replaced by \((\text{Ad}_{m+2})\). Results improving upon dimensions in Theorem A, however, are available for compacta of stable shape (for details see section 4).

Fixing the sphere \(S^n\) as an ambient space, Theorem A implies that the assignment \(X \mapsto S^n - X\) induces a covariant category isomorphism from a full subcategory of \(\text{SSh}(\text{DM})\) to a suitable ‘homotopy category of proper maps’. On the other hand, the Strong Shape S-Duality Theorem of Q. Haxhibeqiri and S. Nowak [6] asserts that \(X \mapsto S^n - X\) induces a contravariant category isomorphism from the stable strong shape category to the stable homotopy category. We may therefore expect that there exists a ‘principle of reversing the direction of maps between open subsets of spheres’. This general idea is made precise by the following duality theorem.

**THEOREM B.** Let \(d(n) = \max\{k | 2k + 2 \leq n\}\) and \(c(n) = \max\{k | 4k \leq n\}\). Let \(T_n\) be the full subcategory of the homotopy category \(\text{HTop}\) whose objects are all complements \(S^n - X\) of \(c(n)\)-shape-connected compacta \(X \subset S^n\) having \(FdX \leq d(n) - 1\) and satisfying the inessential loops condition \(\text{ILC}\) [20], and let \(\text{H}_{d(n)} P\) be the proper \(d(n)\)-homotopy category (cf. section 2). There exists a contravariant full embedding

\[
\Delta: T_n \rightarrow \text{H}_{d(n)} P
\]

such that \(\Delta(U) = U\) for each object \(U\).

Roughly speaking, Theorem B says that each map \(f: U \rightarrow V\) between suitable open subsets \(U, V\) of \(S^n\) can be ‘canonically reversed’ to obtain a proper map \(f^*: V \rightarrow U\), and vice versa. In certain special cases we can say even more about this reversing process: \(f: U \rightarrow V\) and \(f^*: V \rightarrow U\) can be chosen in such manner that they are ‘inverse to each other’ in a very weak sense; for details see Theorem 5.5.

1. Filtered spaces

Let \(\tilde{X} = (X; X_a)\) be a filtered space (consisting of an underlying space \(X\) and a sequence of closed subspaces \(X_n \subset X\) such that \(X_0 = X\) and \(X_{n+1} \subset \text{int}\ X_n\) for
each \( n \). We call \( \tilde{X} \) functionally filtered if there is a map \( h: X \to [0, \infty) \) with \( X_n = h^{-1}([n, \infty)) \) for each \( n \). The symbol \( F \) will denote the category of functionally filtered spaces and filtered maps. Here are some properties of functionally filtered spaces \( \tilde{X} \).

1.1. **Proposition.** (a) \( \tilde{X} \times I = (X \times I; X_n \times I) \) is a functionally filtered space, and each \( i_t: X \to X \times I, i_t(x) = (x, t) \), is a filtered map \( i_t: X \times I \to \tilde{X} \).

(b) For each \( A \subseteq X \), \( \tilde{X} \cap A = (A; X_n \cap A) \) is a functionally filtered space, and the inclusion \( i: A \to X \) is a filtered map \( i: \tilde{X} \cap A \to \tilde{X} \).

A filtered subspace \( \tilde{A} \subseteq \tilde{X} \) (i.e. \( \tilde{A} = \tilde{X} \cap A \) for some \( A \subseteq X \)) is called cofinal if \( A \) contains some \( X_n \). In this case also the inclusion \( i: \tilde{A} \to \tilde{X} \) is said to be cofinal. Let \( \Sigma_F \) denote the class of all cofinal inclusions in \( F \), and let \( F_\infty \) denote the quotient category \( F \Sigma_F \). It is easy to show that \( F_\infty \) admits a calculus of right fractions (cf. [5] §6.2).

Next, let \( C \) be the category of \( \sigma \)-complete uniform spaces and complete maps (introduced in [13]). Recall that a uniform space \( X \) is \( \sigma \)-complete iff it has a filtration function \( h_X \), i.e. a map \( h_X: X \to [0, \infty) \) with

\[
(C1) \text{ For each complete } S \subseteq X, \text{ the closure } \text{cl}(h_X(S)) \text{ is compact.}
\]

\[
(C2) \text{ For each } n, h_X^{-1}([0, n]) \text{ is complete.}
\]

By a filtration functor on \( C \) we mean a functor \( V: C \to F \) such that

\[
(V1) \text{ For each } X \in \text{Ob } C, V(X) \text{ is a filtered model of } X \text{ (i.e. } V(X) = (X; h_X^{-1}([n, \infty))) \text{ for some filtration function } h_X).\]

\[
(V2) \text{ For each } f \in C(X, Y), V(f) = f.
\]

From [13] 2.3(2) we infer

1.2 **Proposition.** (a) There exist filtration functors on \( C \); in fact, each choice of filtration functions \( h_X, X \in \text{Ob } C \), determines a unique filtration functor on \( C \).

(b) Any two filtration functors on \( C \) are naturally isomorphic.

(c) Each filtration functor on \( C \) is a full embedding.

An easy consequence of the definition of the quotient categories \( C_\infty \) (cf. [13]) and \( F_\infty \) is

1.3 **Proposition.** Each filtration functor \( V: C \to F \) induces a unique functor \( V_\infty: C_\infty \to F_\infty \) with \( U_FV = V_\infty U_C \) (where \( U_C: C \to C_\infty \) and \( U_F: F \to F_\infty \) are the canonical quotient functors); \( V_\infty \) is a full embedding.

2. **Homotopy relations**

The concept of homotopy was defined in [5] for the categories \( P \) (= proper category = category of \( \sigma \)-compact spaces and proper maps) and \( P_\infty \) (= proper
category at $\infty = P \backslash \text{cofinal inclusions in } P)$, and in [12] for the category $C$. More generally, in each of the categories $D = P, P_{\infty}, C, C_{\infty}, F, F_{\infty}$ the cylinder functor $' \times I'$ can be used to define the relation of homotopy: $\alpha_0, \alpha_1 \in D(A, B)$ are called homotopic in $D$, $\alpha_0 \cong \alpha_1$, if there exists a homotopy $H \in D(A \times I, B)$ such that $H_{ik} = \alpha_k, k = 0, 1$ (where $i_k : A \rightarrow A \times I$ are the obvious morphisms in $D$).

We shall also need the following ‘relative’ version of homotopy: Given any class of objects $\mathfrak{M} \subset \text{Ob } D$, $\alpha_0, \alpha_1 \in D(A, B)$ are called $\mathfrak{M}$-homotopic in $D$, $\alpha_0 \cong_{\mathfrak{M}} \alpha_1$, if $\alpha_0 \varphi \cong \alpha_1 \varphi$ for each $\varphi \in D(A', A)$ with $A' \in \mathfrak{M}$. It is fairly obvious that ‘$\mathfrak{M}$-homotopy in $D$’ is an equivalence relation for morphisms in $D$ which is compatible with composition. The $\mathfrak{M}$-homotopy category $H_{\mathfrak{M}}D$ is obtained from $D$ by identifying $\mathfrak{M}$-homotopic morphisms. For $\mathfrak{M} = \text{Ob } D$ we get the homotopy category $HD = H_{\text{Ob } D}D$ (note that $\alpha_0 \cong \text{Ob } D \alpha_1$ iff $\alpha_0 \cong \alpha_1$).

2.1 EXAMPLE. Let $\Psi(m)$ be the class of all $\sigma$-compact polyhedra $P$ with $\dim P \leq m$ (cf. [13] 1.4). Then we obtain the proper $m$-homotopy category, $H_mP = H_{\Psi(m)}P$, the proper $m$-homotopy category at $\infty$, $H_mP_{\infty} = H_{\Psi(m)}P_{\infty}$, the complete $m$-homotopy category, $H_mC = H_{\Psi(m)}C$, and the complete $m$-homotopy category at $\infty$, $H_mC_{\infty} = H_{\Psi(m)}C_{\infty}$.

2.2 REMARK. Recall from [13] that we regard $P, P_{\infty}$ as full subcategories $P \subset C, P_{\infty} \subset C_{\infty}$. This induces the following inclusions as full subcategories: $HP \subset HC, HP_{\infty} \subset HC_{\infty}, H_mC \subset H_mC, H_mC_{\infty} \subset H_mC_{\infty}$.

From 1.3 we infer

2.3 PROPOSITION. Each filtration functor $V : C \rightarrow F$ induces a unique $HV_\infty : HC_{\infty} \rightarrow HF_{\infty}; HV_\infty$ is a full embedding.

One also readily checks that $U_C : C \rightarrow C_{\infty}$ induces a quotient functor $H_mC : H_mC \rightarrow H_mC_{\infty}$.

3. The end functor

Let $H(\text{pro-Top})$ denote the homotopy category of $\text{pro-Top}$ with respect to its closed model structure as defined in [5] §2.3. Recall that the natural functor $\pi : H(\text{pro-Top}) \rightarrow \text{pro-HTop}$ has the following property (see [5] 5.2.9):

(3.1) Let $X, Y$ be inverse systems in $\text{pro-Top}$ which are isomorphic in $H(\text{pro-Top})$ to towers. Then $X$ and $Y$ are isomorphic in $H(\text{pro-Top})$ iff $\pi X$ and $\pi Y$ are isomorphic in $\text{pro-HTop}$.

We now study the end functor $\varepsilon : C_{\infty} \rightarrow \text{pro-Top}$ constructed in [13].

3.2 THEOREM. $\varepsilon$ induces a full embedding $\varepsilon : HC_{\infty} \rightarrow H(\text{pro-Top})$. 

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Proof. The proof follows the lines of [5] §3.7 and §6.3. Let us first observe that \( \varepsilon: C_\infty \to \text{pro-Top} \) carries homotopies in \( C_\infty \) to left homotopies in \( \text{pro-Top} \) (cf. [15]), and therefore actually induces a functor \( \varepsilon: HC_\infty \to H(\text{pro-Top}) \).

We now construct an end functor \( \varepsilon': F \to \text{tow-Top} \subset \text{pro-Top} \) (\( \text{tow-Top} \subset \text{pro-Top} \) is the full subcategory of towers). For each object \( \vec{X} = (X; X_n) \) of \( F \), let \( \varepsilon'(\vec{X}) \) be the tower \( \{X_n\} \) in \( \text{pro-Top} \) (bonded by inclusions). For each morphism \( f: \vec{X} = (X; X_n) \to \vec{Y} = (Y; Y_n) \) in \( F \), define \( f^*: N \to N \) by \( f^*(n) = \min\{r \mid f(X_r) \subset Y_n\} \). Let \( f_n: X_{f^*(n)} \to Y_n \) be the restriction of \( f \). Then \( (f_n, f^*) \) is a map of inverse systems which represents a unique morphism \( \varepsilon'(f) \in \text{tow-Top}(\varepsilon'(\vec{X}), \varepsilon'(\vec{Y})) \).

If \( j: \vec{A} \to \vec{X} \) is a cofinal inclusion in \( F \), then \( \varepsilon'(j) \) is an isomorphism in \( \text{tow-Top} \), and therefore \( \varepsilon': F \to \text{tow-Top} \) induces a unique functor \( \varepsilon': F_\infty \to \text{tow-Top} \) such that \( \varepsilon_UF = \varepsilon' \).

Since \( \varepsilon \) carries homotopies in \( F_\infty \) to left homotopies in \( \text{tow-Top} \), it induces a functor \( He': HF_\infty \to H(\text{two-Top}) \subset H(\text{pro-Top}) \).

Let us fix an arbitrary filtration functor \( V: C \to F \). It is easy to verify

(3.3) The two functors \( \varepsilon, \varepsilon' \circ V: C \to \text{pro-Top} \) are naturally isomorphic.

From this one readily infers

(3.4) The two functors \( \varepsilon, He' \circ HV_\infty : HC_\infty \to H(\text{pro-Top}) \) are naturally isomorphic.

The telescope construction discussed in [5] §3.7 is readily seen to furnish a functor \( \text{Tel}: H(\text{tow-Top}) \to HF_\infty \). Using the techniques of [5] §3.7 and §6.3 one can verify

(3.5) There are natural isomorphisms \( \text{Tel} \circ He' \cong 1 \) and \( He' \circ \text{Tel} \cong 1 \). In particular, \( He' \) is an equivalence of categories and a fortiori a full embedding.

This completes the proof, since \( HV_\infty \) is a full embedding.

3.6. REMARK. The above proof shows that \( HF_\infty \) is a 'category of geometric models' for \( H(\text{tow-Top}) \). Note that it also follows from [5] 3.7.20 that \( H(\text{tow-Top}) \) embeds as a full subcategory into \( HF_\infty \).

Now, let \( \theta_m \) resp. \( \theta^*_m \) be the full subcategories of \( H(\text{pro-Top}) \) whose objects are all inverse systems which are isomorphic in \( H(\text{pro-Top}) \) to some tower \( P = \{P_n, p_n\} \) such that the \( P_n \) are compact resp. arbitrary polyhedra with \( \dim P_n \leq m \).

Let \( [\Psi(m)] \) be the full subcategory of \( HC_\infty \) such that \( \text{Ob}[\Psi(m)] = \Psi(m) \). Then \( H_mC_\infty = HC_\infty/[\Psi(m)] \) (cf. [13] §2). We set

\[ \theta_m = \text{Sat}(\varepsilon([\Psi(m)])) \cap \theta^*_m \]

and obtain (cf. [13] 2.10)
3.7. PROPOSITION. \( \varepsilon: \text{HC} \rightarrow \text{H(pro-Top)} \) induces a full embedding \( \varepsilon_m: \text{H}_m \rightarrow \text{H(pro-Top)}/\theta_m \).

It is a routine exercise (using simplicial approximation, (3.1) and the telescope construction) to verify

3.8. PROPOSITION. \( \theta_m \subset \theta_{m+1} \).

It is unfortunately not true that \( \theta_m \subset \theta_m \). This can easily be seen for \( m = 0 \). The only non-compact 0-dimensional \( \sigma \)-compact polyhedron is a discrete space \( \Delta \) with infinitely many points. Given any tower \( X = \{X_n\} \) consisting of pathwise connected spaces \( X_n \), there is only one morphism \( \varepsilon(\Delta) \rightarrow X \) in \( \text{H(pro-Top)} \); i.e. any two morphisms \( \phi_0, \phi_1: Y \rightarrow X \) (\( Y \) arbitrary!) are \( \varepsilon(\Psi(0)) \)-equal although they need not be \( \varepsilon_0 \)-equal.

The following is a partial substitute for the failure of ‘\( \theta_m \subset \theta_m \).

3.9 LEMMA. Let \( Y \) be an object of \( \text{H(pro-Top)} \) which admits a \( \varepsilon_m \)-equivalence \( \beta: Y \rightarrow Y' \) onto a stable object \( Y' \) of \( \text{H(pro-Top)} \).

(a) Any two \( \varepsilon_m \)-equal morphisms \( \phi_0, \phi_1: X \rightarrow Y \) in \( \text{H(pro-Top)} \) are also \( \varepsilon_m \)-equal; i.e. for each \( X \) the canonical \( \rho^*_{\varepsilon_m}: \text{H(pro-Top)}(X, Y) \rightarrow \text{H(pro-Top)}(X, Y) \) induces a unique \( \rho^*_{\varepsilon_m}: \text{H(pro-Top)}(X, Y) \rightarrow \text{H(pro-Top)}(X, Y) \).

(b) \( \rho^*_{\varepsilon_m} \) is a bijection provided \( X \) admits a \( \varepsilon_m \)-equivalence \( \alpha: X \rightarrow X' \) onto some \( X' \in \text{Ob} \theta_m \).

Proof. (a) Let \( P \in \text{Ob} \theta_m \) and \( \psi: P \rightarrow X \). We have to show \( \phi_0 \psi = \phi_1 \psi \). Since \( \beta: Y \rightarrow Y' \) is a \( \varepsilon_m \)-equivalence and \( P \in \text{Ob} \theta_m \), it suffices to show \( \beta \phi_0 \psi = \beta \phi_1 \psi \). Since \( Y' \) is stable in \( \text{H(pro-Top)} \), the canonical map \( \pi: \text{H(pro-Top)}(P, Y) \rightarrow \text{pro-HTop}(\pi P, \pi Y') \) is a bijection. It therefore suffices to show \( \pi(\beta)\pi(\phi_0)\pi(\psi) = \pi(\beta)\pi(\phi_1)\pi(\psi) \). We may assume that \( P = \{P_n, p_n\} \) with compact polyhedra \( P_n \) of dimension \( \leq m \). We set \( P^* = \bigcup P_n \times \{r\} \) and obtain (as in the proof of [13] 2.11) a ‘canonical’ morphism \( f: \varepsilon(P^*) \rightarrow P \) in \( \text{pro-Top} \). Let \( \gamma: \varepsilon(P^*) \rightarrow P \) be the image of \( f \) in \( \text{H(pro-Top)} \). Since \( \phi_0, \phi_1 \) are \( \varepsilon_m \)-equal, \( \phi_0 \psi \gamma = \phi_1 \psi \gamma \). Hence \( \pi(\phi_0)\pi(\psi)\pi(\gamma) = \pi(\phi_1)\pi(\psi)\pi(\gamma) \), which implies \( \pi(\phi_0)\pi(\psi) = \pi(\phi_1)\pi(\psi) \) (see again [13] 2.11). This completes the proof of (a).

(b) Since \( \theta_m \subset \theta_m \), \( \rho^*_{\varepsilon_m}: \text{H(pro-Top)} \rightarrow \text{H(pro-Top)}/\theta_m \) induces a full functor \( F_m: \text{H(pro-Top)}/\theta_m \rightarrow \text{H(pro-Top)}/\theta_m \); since \( \theta_m \subset \theta_m \), \( \rho^*_{\varepsilon_m}: \text{H(pro-Top)} \rightarrow \text{H(pro-Top)}/\theta_m \) induces a full functor \( F_m: \text{H(pro-Top)}/\theta_m \rightarrow \text{H(pro-Top)}/\theta_m \). Consider the following commutative diagram (cf. [13] 2.4, 2.7):

\[ \begin{array}{ccc}
\text{H(pro-Top)}/\theta_m(X, Y) & \xrightarrow{\Delta} & \text{H(pro-Top)}/\theta_m(X', Y) \\
F_m & \simeq & F_m \\
\rho^*_{\varepsilon_m} & \downarrow & \downarrow \\
\text{H(pro-Top)}/\theta_m(X, Y) & \xrightarrow{\Delta} & \text{H(pro-Top)}/\theta_m(X', Y)
\end{array} \]
We infer that the $F_m$ are bijections; hence $G_m$ is an injection. Since $G_m$ is a full functor, (b) is proved.

3.10 THEOREM. Let $A$ be a $\sigma$-complete metrizable space and $B$ be a $\sigma$-complete $m$-connected ANR.

(a) $H_m U_C: H_m C(A, B) \rightarrow H_m C_\infty(A, B)$ is a bijection provided $\varepsilon(A)$ admits a $\theta_m$-equivalence $\varphi: \varepsilon(A) \rightarrow A$ onto some $A \in \text{Ob } \theta_m$.

(b) $H_m U_C: H_m C(A, B) \rightarrow H_m C_\infty(A, B)$ is a bijection provided $\varepsilon(A)$ admits a $\theta_m^*$-equivalence $\varphi: \varepsilon(A) \rightarrow A$ onto some $A \in \text{Ob } \theta_m^* \varepsilon$ and $\varepsilon(B)$ admits a $\theta_m^*$-equivalence $\psi: \varepsilon(B) \rightarrow B$ onto a stable object $B$ of $\mathbf{H(pro-top)}$.

\textbf{Proof.} (1) Injectivity. A straightforward modification of the proof of [13] 2.14 yields injectivity in both cases.

(2) Surjectivity. We only prove (b), the other case is similar. By 3.7 and 3.9, $\varepsilon$ induces a bijection $\varepsilon_m^*: H_m C_\infty(A, B) \rightarrow H_m C_\infty(A, B)$ such that

\[ \varepsilon([g]) = \varepsilon([f]) \varphi', \quad \text{where } [\ ] \text{ denotes equivalence class in } HC_\infty. \]

We have $\varepsilon_m^*([g]) = \rho_m^*(\varepsilon([g])) = \rho_m^*(\varepsilon([f])) = \varepsilon_m^*([f])$, where $[\ ]_m$ denotes equivalence class in $H_m C_\infty$. This implies $[g]_m = [f]_m$. But now the equation $\pi \varepsilon([g]) = \pi(\varepsilon([f]) \varphi') \pi \varphi$ shows, as in the proof of [13] 2.14, that $g$ has the form $g = U_C(G)$ with a complete $G: A \rightarrow B$.

4. The categorical complement theorems

The following result is basic for our purposes (see [13] for notation).

4.1 THEOREM. Let $M$ be an ANR and $X \subset M$ be an $(m + 1)$-admissible compactum. Then $i: U^*(X) \rightarrow U(X)$ is a $\theta_m^*$-equivalence in $\mathbf{H(pro-top)}$.

The proof of 4.1 will be prepared by two technical Lemmas.

4.2. LEMMA. Let $\tilde{P} = (P; P_n)$ be a filtered CW-complex (i.e. $P$ is a CW-complex and each $P_n$ is a subcomplex of $P$). For each subcomplex $Q \subset P$, the inclusion $i: \tilde{P} \cap Q \rightarrow \tilde{P}$ is a filtered cofibration, i.e. each filtered map $F: \tilde{P} \times I \cap (P \times \{0\} \cup Q \times I) \rightarrow \tilde{X}$ has a filtered extension $F': \tilde{P} \times I \rightarrow \tilde{X}$.

The proof is a slight modification of the classical homotopy extension theorem for CW-complexes and is left to the reader.

4.3. LEMMA. Let $M$ be an ANR and $X \subset M$ be an $m$-admissible compactum.

(a) There is a cofinal subtower $N(X) = \{N_r\}$ of $U(X)$ such that $\pi_k(N_{r+1}, N_{r+1} - X) \rightarrow \pi_k(N_r, N_r - X)$ is trivial for all $k = 0, \ldots, m + 1$ and all $r$. 


(b) Let \( \tilde{P} = (P; P_n) \) be a filtered CW-complex, \( Q \subset P \) be a subcomplex with \( \dim(P - Q) \leq m + 1 \), and let \( f: \tilde{P} \to \text{Tel} N(X) \) be a filtered map such that \( f(P) \subset \text{Tel}_{m+2} N(X) \) and \( f(Q) \subset \text{Tel} N^*(X) \), where \( N^*(X) = \{ N_r - X \} \). There is a filtered map \( g: \tilde{P} \to \text{Tel} N(X) \) such that \( g(P) \subset \text{Tel} N^*(X) \) and \( f \simeq g \) via a filtered homotopy rel. \( Q \).

**Proof.** (a) is trivial.

(b) We construct by induction filtered maps \( f^i: \tilde{P} \to \text{Tel} N(X) \) such that \( f^i(P) \subset \text{Tel}_{m+1-i} N(X) \), \( f^i(P^i \cup Q) \subset \text{Tel} N^*(X) \) and \( f \simeq f^i \) via a filtered homotopy rel \( Q \) (then \( g = f^{m+1} \) is the required map). The induction starts with \( f^{(-1)} = f \). Given \( f^i \), \( i \leq m \), we have to construct \( f^{i+1} \). Choose integers \( 0 = n_0 < n_1 < n_2 \cdots \) such that \( f^i(P_{n_r}) \subset \text{Tel}_{m+1-i+r} N(X) \). Consider an open \((i+1)-\)cell \( \sigma \) of \( P - Q \) with a characteristic map \( \varphi: D_{i+1} \to \text{cl}(\sigma) \subset P \). Let \( \partial \sigma = \phi(S^i) \subset P^i \) and \( r(\sigma) = \max \{ r | \sigma \subset P_{n_r} \} \). A homotopy \( h_\sigma: \text{cl}(\sigma) \times I \to \text{Tel} N(X) \) is defined by \( h_\sigma(x, t) = (f_1(x), (1-t)f_2(x) + t(m-i + r(\sigma))) \), where \( f_1(x) = (f(x), t(1-t)m + t(m-i + r(\sigma))) \), \( f_2(x) \in N^*(X) \). Define \( \lambda: D^1 \times I \to D^1 \times I \) by \( \lambda(x, t) = (2x/(2-t), t) \) for \( ||x|| \leq (2-t)/2 \) and \( \lambda(x, t) = (x/||x||, 2 - 2||x||) \) for \( ||x|| \geq (2-t)/2 \). Let \( H = h_\sigma(\phi \times 1): D^1 \times I \to \text{Tel} N(X) \). Since \( H(\phi \times 1)^{-1} \) is single-valued, there is a homotopy \( H_\sigma: \text{cl}(\sigma) \times I \to \text{Tel} N(X) \) with \( H = H_\sigma(\phi \times 1) \). Let \( D' = \{ x \in D^1 | ||x|| \leq \frac{1}{2} \} \) and \( D = \phi(D') \subset \sigma \). By construction, \( H_\sigma \) is stationary on \( \partial \sigma \) and \( H_\sigma(x, 1) \in \text{Tel}^* N(X) \) for \( x \notin \tilde{D} \). Consider \( g = H_{\sigma,1}|D; \) this is a map into \( N^*_{m+1-i+r(\sigma)} \times \{ m-i+r(\sigma) \} \subset N^*_{m-i+r(\sigma)} \times \{ m-i+r(\sigma) \} \subset \text{Tel} N(X) \). Since \( g(\partial D) \subset (N^*_{m+1-i+r(\sigma)} - X) \times \{ m-i+r(\sigma) \} \), \( g \) is homotopic rel \( \partial D \) (in \( N^*_{m-i+r(\sigma)} \times \{ m-i+r(\sigma) \} \)) to a map \( g': D \to \text{Tel} N(X) \) with \( g'(D) \subset (N^*_{m-i+r(\sigma)} - X) \times \{ m-i+r(\sigma) \} \). Combining this homotopy with \( H_\sigma \), we obtain a homotopy \( G_\sigma: \text{cl}(\sigma) \times I \to \text{Tel} N(X) \) which is stationary on \( \partial \sigma \) and satisfies \( G_{\sigma,0} = f^i|Q \), \( G_{\sigma,1}(\text{cl}(\sigma)) \subset \text{Tel} N^*(X) \) and \( G_\sigma(\text{cl}(\sigma)) \subset \text{Tel}_{m-i+r(\sigma)} N(X) \). Consider the homotopy \( G: (\tilde{P} \cap (P^{i+1} \cup Q)) \times I \subset \text{Tel}_{m-i} N(X) \) defined by \( G_t|Q = f^i|Q \) and \( G_t|\text{cl}(\sigma) \times I = G_\sigma \) for \( \sigma \in P^{i+1} - Q \). Clearly, \( G \) is filtered. By 4.2, \( G \) extends to a filtered homotopy \( H^{i+1}: \tilde{P} \times I \to \text{Tel} N(X) \) such that \( H_0^{i+1} = f^i \) and \( H^{i+1}(\tilde{P} \times I) \subset \text{Tel}_{m-i} N(X) \). By construction, \( f^{i+1} = H_1^{i+1} \) has the desired properties.

We are now ready to prove 4.1:

Choose a cofinal subtower \( N(X) \subset U(X) \) as in 4.3(a). It suffices to show that \( i: N^*(X) \to N(X) \) is a \( \theta^* \)-equivalence. Let \( P = \{ P_n, p_n \} \) be a tower of polyhedra \( P_n \), \( \dim P_n \leq m \), and piecewise-linear bondings \( p_n \). We have to show that \( i \) induces a bijection \( \mathbf{H}(_{\text{pro-Top}})(P, N^*(X)) \to \mathbf{H}(_{\text{pro-Top}})(P, N(X)) \). Since \( \text{Tel}: \mathbf{H}(_{\text{top-Top}}) \to \mathbf{H}_\infty \) is a full embedding, this is equivalent to showing that \( \mathbf{H}_\infty(\text{Tel} P, N^*(X)) \to \mathbf{H}_\infty(\text{Tel} P, N(X)) \) is a bijection. Since \( \text{Tel} P \) has the structure of a filtered CW-complex of dimension \( \leq m + 1 \), this actually follows from 4.3(b) (note that we can always choose representatives for morphisms resp. homotopies in \( \mathbf{F}_\infty \) which satisfy all assumptions in 4.3(b)).
Putting together the pieces collected so far, we obtain

4.4 THEOREM. \((\Omega \mathcal{R}, (\text{Ad}_{m+2}), H_{m+1}C_{\infty})\) are data of a categorical complement theorem for \(\mathbb{SSh}(C_{\infty})\).

Proof. Let \(V(X)\) be the Vietoris system associated to a compactum \(X\) (cf. [5] 8.2.7). If \(X \subseteq M\), \(M\) an ANR, there exists an isomorphism \(U(X) \rightarrow V(X)\) in \(H(\text{pro-Top})\) (note that \(U(X)\) and \(V(X)\) admit cofinal subtowers and apply 3.1). Since \(\varepsilon(M - X)\) can be identified with a cofinal subsystem of \(U^*(X)\), we infer from 4.1 the following.

(4.5) If \(X \subseteq M\) is \((m + 2)\)-admissible, then there exists a \(\theta_{m+1}\)-equivalence \(\vartheta_{(M,X)}: \varepsilon(M - X) \rightarrow V(X)\) in \(H(\text{pro-Top})\).

Moreover, we clearly have

(4.6) If \(FdX \leq m\), then \(V(X) \in \text{Ob} \theta_m \subset \text{Ob} \theta_{m+1}\).

We can write \(\mathbb{SSh}(X, Y) = H(\text{pro-Top})(V(X), V(Y))\), i.e. for \(X \in C_{\infty}\) (cf. [13] 2.4):

(4.7) \(\mathbb{SSh}(X, Y) = H(\text{pro-Top})/\theta_{m+1}(V(X), V(Y))\).

The proof of 4.4 is now very similar to that of [13] 4.2: Define \(T: H_{m+1}C_{\infty}(C_{\infty}, \Omega \mathcal{R}, (\text{Ad}_{m+2})) \rightarrow \mathbb{SSh}(C_{\infty})\) by \(T(\alpha) = [\varepsilon_{m+1}(x)[\varepsilon_{m+1}(x)]^{-1} \in \mathbb{SSh}(X, Y)\) for \(x \in H_{m+1}C_{\infty}(M - X, N - Y)\). Here \(\lfloor \cdot \rfloor\) denotes equivalence class in \(H(\text{pro-Top})/\theta_{m+1}\). Clearly, \(T\) is an equivalence of categories. The following diagram may be useful to illustrate the definition of \(T\):

\[
\begin{array}{ccc}
H_{m+1}C_{\infty}(M - X, N - Y) & \xrightarrow{T} & \mathbb{SSh}(X, Y) \\
\approx & \downarrow \varepsilon_{m+1} & \\
H(\text{pro-Top})/\theta_{m+1}(\varepsilon(M - X), \varepsilon(N - Y)) & \cong & H(\text{pro-Top})/\theta_{m+1}(V(X), V(Y))
\end{array}
\]

THEOREM A from the Introduction follows now immediately from 4.4 and 3.10(a).

4.8 REMARK. There is a modification of Theorem A with a slightly weaker embedding condition. Let \(\mathcal{I}(m)\) be the class of telescopes \(\text{Tel}(\{P_n, p_n\})\), where \(\{P_n, p_n\}\) is a tower of compact polyhedra \(P_n\) with \(\text{dim} P_n \leq m - 1\) and piecewise-linear bondings \(p_n\). We now obtain the telescope \(m\)-homotopy categories \(H_{\mathcal{I}(m)}C_{\infty}\) and \(H_{\mathcal{I}(m)}C\) (cf. §2). It is fairly obvious that \(\varepsilon: H_{\mathcal{I}(m)}C_{\infty} \rightarrow H(\text{pro-Top})\) induces a full embedding \(\varepsilon_{\mathcal{I}(m)}: H_{\mathcal{I}(m)}C_{\infty} \rightarrow H(\text{pro-Top})/\theta_m\). Adapting the proof of 4.4, we see that \((\Omega \mathcal{R}, (\text{Ad}_{m+1}), H_{\mathcal{I}(m+1)}C_{\infty})\) are data of a categorical complement theorem.
for $\text{SSH}(\mathcal{C}_m)$. Using a suitable version of 3.10, we infer moreover that $(\mathfrak{d}\mathfrak{R}_{m+1}, (\mathfrak{A}_m+1), H_{(m+1)}C)$ are data of a categorical complement theorem for $\text{SSH}(\mathcal{C}_m)$.

Theorem A yields various corollaries (cf. §4 of [13]), for example

4.9 COROLLARY. Let $M$ be an $(m + 1)$-connected ANR with a complete uniform structure. Then the strong shape category of $(m + 2)$-admissible compacta $X \subset M$ with $FdX \leq m$ is isomorphic to the complete $(m + 1)$-homotopy category of their complements $M - X$.

4.10 COROLLARY. Let $M$ be an $r$-connected piecewise-linear manifold with a complete uniform structure such that $r \geq 0$ and $\dim M \geq 2$. Let $k \in \{0, \ldots, \min(r, d(M))\}$, where $d(M) = \max \{s \in \mathbb{N} | 2s + 2 \leq \dim M\}$, and let $m \in \{k, \ldots, \min(r, \dim M - 2 - k)\}$. Then the strong shape category of ILC compacta $X$ in the interior of $M$ with $FdX \leq k - 1$ is isomorphic to the complete $m$-homotopy category of their complements $M - X$. If $r \geq d(M)$, one can always choose $k = m = d(M)$.

4.11 REMARK. If $M$ is compact, one can replace the complete $m$-homotopy in the above two results by the proper $m$-homotopy category.

The rest of this section is devoted to compacta of stable shape, i.e. of the shape of a (not necessarily compact) polyhedron.

4.12 THEOREM. Let $\mathcal{C}_m$ be the class of compacta $X \in \mathcal{C}_m$ which have stable shape. The following are data of a categorical complement theorem for $\text{SSH}(\mathcal{C}_m)$:

(a) $(\Psi R, (\mathfrak{A}_m+1), H_mC_{\infty})$

(b) $(\Psi R_m, (\mathfrak{A}_m+1), H_mC)$

4.13 REMARK. Recall that the canonical functor $\text{SSH}(\mathcal{C}_m) \to \text{Sh}(\mathcal{C}_m)$ is a category isomorphism.

Proof of 4.12. (a) By 3.9, we can define an equivalence of categories $T: H_mC_{\infty}(\mathcal{C}_m, \Psi R, (\mathfrak{A}_m+1)) \to \text{SSH}(\mathcal{C}_m)$ by $T(\alpha) = [\varphi_{(N,Y)}]_m(\alpha)[\varphi_{(M,X)}]^{-1}$ for $\alpha \in H_mC_{\infty}(M - X, N - Y)$. Here, $[ \ ]$ denotes equivalence class in $H(\text{pro-Top})/\theta_m$.

(b) This follows from (a).

Let us now say that a compactum $X$ in an ANR $M$ satisfies the embedding condition $(\mathfrak{A}_m)$, if $X \subset M$ is $m$-admissible and $\pi\varepsilon(M - X)$ is stable in $\text{pro-HTop}$.

4.14 EXAMPLE. Let $X$ be an ILC compactum in the interior of a piecewise-linear manifold $M$. If $X$ is a subpolyhedron of $M$, or if $\dim M \geq 5$ and $X$ has the shape of a compact polyhedron $P$ with $\dim P \leq \dim M - 3$, then $X \subset M$ satisfies $(\mathfrak{A}_{\dim M - 2 - FdX})$; see [16] Theorem 5.6. Notice that $X \subset M$ (in general) does not satisfy $(\mathfrak{A}_{\dim M - 1 - FdX})$. 
4.15 LEMMA. Let $X \subset M$ satisfy $(\text{Ad}_{\text{stm}}^m)$. If $X \in \text{CWR}_m$, then $X \in \text{CWR}_m^m$.

Proof. $\pi\varepsilon(M - X) \to U(X)$ is an $\Omega_\varepsilon$-equivalence in $\text{pro-HTop}$ (cf. [13]). Hence, $U(X)$ is dominated in $\text{pro-HTop}$ by the stable object $\pi\varepsilon(M - X)$ and a fortiori by a single space $Z$. Choose morphisms $u: U(X) \to Z$ and $d: Z \to U(X)$ such that $du = 1$. Obviously, $u$ factors through an ANR (the open neighbourhoods of $X$ form a cofinal subsystem of $U(X)$). Hence, $X$ is shape dominated by an ANR, and therefore $X$ has stable shape.

4.16 THEOREM. The following are data of a categorical complement theorem for $\text{SSh}(\text{CWR}_m)$:

(a) $(\text{URR}_m, (\text{Ad}_{\text{stm}}^m), H_mC_\infty)$
(b) $(\text{URR}_m, (\text{Ad}_{\text{stm}}^m), H_mC)$

Proof. We have shown in [13] that $(\text{URR}_m, (\text{Ad}_{\text{stm}}^m), wH_mC_\infty)$ and $(\text{URR}_m, (\text{Ad}_{\text{stm}}^m), wH_mC)$ are data of a categorical complement theorem for $\text{Sh}(\text{CWR}_m)$. It therefore suffices to show: If $X \subset M, Y \subset N$ and $\pi\varepsilon(N - Y)$ is stable, then

(a) $H_mC_\infty(M - X, N - Y) \to wH_mC_\infty(M - X, N - Y)$ is a bijection.
(b) $H_mC(M - X, N - Y) \to wH_mC(M - X, N - Y)$ is a bijection for $m$-connected $N - Y$.

It is clear that both maps are surjective since homotopy implies weak homotopy. Let $f_0, f_1 \in C_\infty(M - X, N - Y)$ represent the same morphism in $wH_mC_\infty$, and let $g \in C_\infty(P, M - X)$, where $P$ is a $\sigma$-compact polyhedron of dimension $\leq m$. Then $f_0g$ and $f_1g$ represent the same morphism in $wHC_\infty$. Hence $\pi\varepsilon([f_0g]) = \pi\varepsilon([f_1g])$, where $[\ ]$ denotes equivalence class in $HC_\infty$ (cf. [13] 2.2). Since $\pi\varepsilon(N - Y)$ is stable, $\varepsilon([f_0g]) = \varepsilon([f_1g])$. This means $[f_0g] = [f_1g]$. Therefore, $f_0, f_1$ represent the same morphism in $H_mC_\infty$, which proves (a). Now let $f_0, f_1 \in C(M - X, N - Y)$ represent the same morphism in $wH_mC$, and let $g \in C(P, M - X)$, where $P$ is as above. By (a), $[UC(f_0g)] = [UC(f_1g)]$. Then the argument of [13] 2.14 shows that $f_0g, f_1g$ are homotopic in $C$. Thus, $f_0, f_1$ represent the same morphism in $H_mC$, which proves (b).

5. The Duality Theorem

In this section we prove Theorem B of the Introduction. In addition to our categorical complement theorems we shall need the following two ingredients.

5.1 THEOREM (Strong Shape S-duality Theorem of Q. Haxhibeqiri and S. Nowak [6]). Let $\text{Stab-SSh}_n$ be the full subcategory of the stable strong shape category $\text{Stab-SSh}$ having as objects all compacta $X \subset S^n$, and let $\text{Stab-HTop}_n$ be the full subcategory of the stable homotopy category $\text{Stab-HTop}$ having as objects all complements $S^n - X$ of compacta $X \subset S^n$. Then there exist a contravariant category isomorphism $D_n: \text{Stab-SSh}_n \to \text{Stab-HTop}_n$ with $D_n(X) = S^n - X$ for all objects $X$. 
5.2 THEOREM (Strong Shape Suspension Theorem). Let $X$ be a compactum of fundamental dimension $FdX = m$, and let $Y$ be an $r$-shape connected compactum. Then the suspension $\Sigma: SSH(X, Y) \to SSH(\Sigma X, \Sigma Y)$ is a surjection provided $m \leq 2r$ and a bijection provided $m \leq 2r - 1$.

5.3 REMARK. Let $CM$ be the category of compacta (= compact metrizable spaces) and continuous maps, and let $S$ be the class of strong shape equivalences in $CM$. Then the strong shape category $SSH$ is the quotient category $CM/S$ (see [2], [3]). We let $q: CM \to SSH$ denote the quotient functor. The suspension functor $\Sigma$ on $CM$ has the property $\Sigma(S) \subseteq S$; hence there is a unique functor $\Sigma = \Sigma_{SSH}: SSH \to SSH$ with $\Sigma_{SSH}q = q\Sigma$, called the strong shape suspension. Theorem 5.2 is the strong shape analogue of the classical suspension theorem in homotopy theory (see e.g. [18], [19]). We remark that the corresponding result in the Borsuk-Mardesic shape category $Sh$ says that $\Sigma: Sh(X, Y) \to Sh(\Sigma X, \Sigma Y)$ is a surjection provided $m \leq 2r + 1$ and a bijection provided $m \leq 2r$ (this was essentially established by S. Nowak in [14]). A proof of Theorem 5.2 can be based on Yu. T. Lisica's description of $SSH$ via the coherent homotopy category of towers (cf. [8]), applying the classical suspension theorem to maps and homotopies. Details are given in the Appendix.

The proof of Theorem B is now straightforward: Let $X, Y \subset S^n$ be ILC compacta such that $FdX, FdY \leq d(n) - 1$ and $X, Y$ are $c(n)$-shape-connected. Using Theorem 5.2, Theorem 5.1 and the classical Suspension Theorem (noticing that $S^n - Y$ is $(n - FdY - 2)$-connected; cf. [13] 3.9), we obtain a bijection

$$D: SSH(Y, X) \cong Stab-SSH(Y, X) \cong Stab-HTop(S^n - X, S^n - Y) \cong HTop(S^n - X, S^n - Y).$$

Finally, the category isomorphism theorem 4.10 yields a bijection $R: SSH(Y, X) \cong H_{d(n)}P(S^n - Y, S^n - X)$. Now set $\Delta = RD^{-1}$; this yields the desired functor $\Delta: T_n \to H_{d(n)}P$.

In the polyhedral case, there is a modification of Theorem B with a slightly improved connectivity condition.

5.4 THEOREM. Let $c(n) = \max\{k | 4k + 2 \leq n\}$, and let $T'_n$ be the full subcategory of $HTop$ having as objects all complements $S^n - X$ of $c(n)$-connected piecewise-linear embedded compact polyhedra $X \subset S^n$ with $\dim X \leq d(n) - 1$. Then there is a contravariant full embedding

$$\Delta': T'_n \to H_{d(n)}P$$

such that $\Delta'(S^n - X) = S^n - X$ for each object $S^n - X$.

Proof. There are bijections $D': HTop(Y, X) \cong HTop(S^n - X, S^n - Y)$ and $R': HTop(Y, X) \cong H_{d(n)}P(S^n - Y, S^n - X)$.
Let us say that a map $f: A \to B$ is homotopically associated in degree $m$ to a proper map $g: A \to B$, $f \sim_m g$, if the following holds: For each closed $A' \subset A$ such that $f|A'$ is proper, $f|A' \cong_m g|A'$. Here $\cong_m$ denotes proper $m$-homotopy.

The functor $\Delta'$ has the following remarkable property.

**5.5 THEOREM.** For each $\alpha \in \mathcal{H}^n(S^n - X, S^n - Y)$, there exist representatives $f: S^n - X \to S^n - Y$ of $\alpha$ and $f^*: S^n - Y \to S^n - X$ of $\Delta'(\alpha) \in \mathcal{H}_d(n)P(S^n - Y, S^n - X)$ such that

1. $ff^*$ is a proper map with $ff^* \cong_{d(n)} 1$;
2. $f^*f \sim_{d(n)} 1$.

**5.6 REMARK.** Since $f^*$ is proper, $f^*(S^n - Y)$ is closed in $S^n - X$. It follows then from (1) that $f|f^*(S^n - Y)$ must be proper, which sheds some more light on (2). We may regard $f$ and $f^*$ as 'inverse to each other' in a very weak sense; however, we emphasize that this strongly depends on the 'correct' choice of representatives $f \in \alpha, f^* \in \Delta'(\alpha)$.

**Proof of 5.5.** Let us first observe that we can extend the functor $\Delta'$ to the full subcategory $\mathcal{T}'$ of $\mathcal{H}$Top having as objects all complements $S^n - W$ of $c'(n)$-connected $PL$ embedded compact polyhedra $W \subset S^n$ with $FdW \leq d(n) - 1$ and $\dim W \leq n - 3$. We now consider $\beta \in HTop(Y, X)$ and construct adequate representatives $f$ of $D'(\beta)$ and $f^*$ of $R'(\beta)$; see the proof of 5.4. Let $\beta = [g]$ with a PL map $g: Y \to X$, and let $Z$ be the PL mapping cylinder of $g$. Clearly, the problem is unchanged if we move $X$ and $Y$ to other places by PL homeomorphisms of $S^n$. In particular, by the dimension hypothesis $\dim X, \dim Y \leq d(n) - 1$, we may assume that $X, Y \subset Z \subset S^{n-1} \subset S^n$, and that there is a collapsing map $r: Z \to X$ with $g = ri$ (where $i: Y \to Z$ is the inclusion). Note that $Z$ is $c'(n)$-connected with $FdZ \leq d(n) - 1$ and $\dim Z \leq n - 3$.

**Step 1.** Clearly $D'([i])$ is represented by the inclusion $i^#: S^n - Z \to S^n - Y$. Define $\lambda: S^n \to S^n_+ = \{(x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} \geq 0\}$ by $\lambda(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, |x_{n+1}|)$. Let $\pi: S^n_+ \to D^n = \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1}| \Sigma x_i^2 \leq 1\}$ be the homeomorphism $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, 0)$. Choose $\omega: D^n \to I$ such that $\omega^{-1}(1) = Y$ and define $\chi_Y: D^n \to D^n$ by $\chi_Y(x) = \omega(x)x$. Finally, define $i^*: S^n - Y \to S^n - Z$ by $i^*(x) = \pi^{-1}\chi_Y \pi \lambda(x)$. This is a proper map which can be pieced together with $i: Y \to Z$ to a continuous $F: S^n \to S^n$. But now [13] 4.9 and the proof of 4.16 above imply that $R([i])$ is represented by $i^*$. By construction, $i^#i^*$ is a proper map which can be pieced together with $1_Y$ to the above map $F$. Hence $R([i])$ is represented by $i^#i^*$, i.e. $i^#i^* \cong_{d(n)} 1_{S^n - Y}$. Finally, let $\lambda_x: S^n - X \to S^n - X, \lambda_x(x) = \lambda(x)$. This is a proper map with $\lambda_x \cong_{d(n)} 1$ (repeat the arguments above). It is easy to see that $i^#i^* \sim_{d(n)} 1$. Thus $\Delta'$ has the following remarkable property.

**Step 2.** We have shown in [11] that there exists a map $H: D^{n+1} \to D^{n+1}$ such
that $H$ carries $D^{n+1} - Z$ homeomorphically onto $D^{n+1} - X$, $H$ restricts to a
retraction $r': Z \to X$, and $H = \text{id}$ outside of a regular neighbourhood $N'$ of $Z$ in
$D^{n+1}$.

Let $h: S^n - Z \to S^n - X$ be the homeomorphism induced by $H$. Then $R'(\{r\})$ is
represented by $h$ (note $[r] = [r']$ and argue as in Step 1). By construction, $h$ is
the identity outside a regular neighbourhood $N$ of $Z$ in $S^n$. Hence, $h$ is
homotopic to the inclusion $j: S^n - Z \to S^n - X$ (recall that $N - Z \approx \partial N \times
(0, 1] \approx N - X$). We conclude that $D'([r])$ is represented by $h^{-1}: S^n - X \to
S^n - Z$: Let $i: X \to Z$ be the inclusion, and let $\{ \}$ denote stable homotopy
classes. Then $\{[r]\} = \{[i']\}^{-1}$, thus $\{D'([r])\} = \{D'([i'])\}^{-1} = \{[j]\}^{-1} = \{[h]\}^{-1} = \{[h^{-1}]\}$.

**Step 3.** Set $f = i \# h^{-1}$ and $f^* = h^*$. Then $f^* \simeq_{d(n)} i^* h^{-1}$. Moreover, $f^* f = h^* i^* h^{-1}$ is clearly homotopically associated in degree $d(n)$ to $h^{-1}$. By construction, $[f] = [i^*][h^{-1}] = D'(\{i\})D'(\{r\}) = D'\{g\}$ and $[f^*]_{d(n)} = [h^*]_{d(n)}[i^*]_{d(n)} = R'(\{r\})R'(\{i\}) = R'(\{g\})$.

**Appendix. Proof of the Strong Shape Suspension Theorem**

We begin with some notation. Let $\partial I = \{0, 1\}$ be the boundary of $I = [0, 1]$. For
each homotopy $H: A \times I \to B$, we define $H_t: A \to B$ by $H_t(x) = H(x, t)$. If $f_0$,
$f_1: A \to B$ are maps and $A' \subset A$, the notation $H: f_0 \simeq f_1 \text{ rel } A'$ indicates that
$H: X \times I \to Y$ is a homotopy such that $H|_{A' \times I}$ is stationary and $H_k = f_k$, $k = 0, 1$
(if $A' = \emptyset$, we write $H: f_0 \simeq f_1$). For homotopies $H^0, H^1: A \times I \to B$ we prefer to
write $\theta: H^0 \simeq H^1$ instead of $\theta: H^0 \simeq H^1 \text{ rel } A \times \partial I$ ($\theta$ is then a homotopy of
homotopies).

For each pointed space $(A, a)$, let $p = p_{(A, a)}: A \times I \to A \times I/\{a\} \times I$ denote the
projection map, written as $p(x, t) = [x, t]$, and let $(A, a) \otimes I$ denote the pointed
space $(A \times I/\{a\} \times I, \ast)$ with $\ast = p(\{a\} \times I)$. By $(A, a) \otimes \partial I$ we mean the pointed
subspace $(p(A \times \partial I), \ast)$. For each $t \in I$, let $i_t: (A, a) \to (A, a) \otimes I$ be the pointed
map $i_t(x) = [x, t]$. If $f_0, f_1: (A, a) \to (B, b)$ are pointed maps, we also write $H: f_0 \simeq f_1$ instead of $H: f_0 \simeq f_1 \text{ rel } \{a\}$; $H$ is then a pointed homotopy. Moreover,
for two such pointed homotopies $H^0, H^1$ we also write $\theta: H^0 \simeq_{d(n)} H^1$ instead of
$\theta: H^0 \simeq H^1 \text{ rel } A \times \partial I \cup \{a\} \times I$. Given any pointed homotopy $H$, we let
$\tilde{H}: (A, a) \otimes I \to (B, b)$ denote the unique pointed map satisfying $\tilde{H}p = H$. We let
$q = q_{(A, a)}: A \times I \to A \times I/\{a\} \times I \cup A \times \partial I$ denote the projection map, written as
$q(x, t) = \langle x, t \rangle$, and let $S(A, a) = (A \times I/\{a\} \times I \cup A \times \partial I, \ast)$ denote the reduced suspension of
$(A, a)$, where $\ast = q(\{a\} \times I \cup A \times \partial I)$.

It is easy to see that $\mu = \mu_{(A, a)}: S(A, a) \otimes I \to S((A, a) \otimes I)$, $\mu(\langle x, s \rangle, \langle t \rangle) = \langle [x, s], [t] \rangle$, defines a natural homeomorphism. Then for each pointed map $G: S(A, a) \otimes I \to (B, b)$, the map
$G^* = G \circ \mu: S(A, a) \times I \to B$ is a pointed homotopy from $GS(i_0)$ to $GS(i_1)$.
Recall that the reduced suspension $S$ and the loop space functor $Q$ are adjoint: In fact, the exponential law yields a natural bijection $03B2: \text{Topo}(S(A, a), (B, b)) \to \text{Topo}_0((A, a), \Omega(B, b))$ between sets of pointed maps. For $(B, b) = S(A, a)$ we obtain the map $\rho = \rho_{(A, a)} = 03B1: (A, a) \to \Omega S(A, a)$ which is a $(2r + 1)$-equivalence provided $(A, a)$ is an $r$-connected pointed CW-complex; see [18] or [19].

A.1 PROPOSITION. Let $(B, b)$ be an $r$-connected pointed CW-complex, $(A, a)$ be a pointed CW-complex, and $f_0, f_1: (A, a) \to (B, b)$ be pointed maps.

1. Let $\dim A \leq 2r$ and $H: S(f_0) \simeq S(f_1)$. Then there exists $G: f_0 \simeq_f f_1$ such that $S(G) \# \simeq_f H$.
2. Let $\dim A \leq 2r - 1$ and $G^0, G^1: f_0 \simeq_f f_1$. If $S(G^0)\# \simeq_f S(G^1)\#$, then $G^0 \simeq_f G^1$.

Proof. (1) Define $f: (A, a) \otimes \partial I \to (B, b)$ by $f([x, k]) = f_k(x)$. One verifies $\rho_{(B, b)}f = \beta(\tilde{H}^1(A, a))|_{(A, a) \otimes \partial I}$. Since $\rho_{(B, b)}$ is a $(2r + 1)$-equivalence, there is a map $g: (A, a) \otimes I \to B$ such that $g|_{(A, a) \otimes \partial I} = f$ and $\rho_{(B, b)}g \simeq \beta(\tilde{H}^1(A, a))|_{(A, a) \otimes \partial I}$ (cf. [18] p. 404). Let $\theta: (A, a) \otimes I \otimes I \to \Omega S(B, b)$ be such a homotopy; in particular, $\theta$ is a pointed homotopy. One readily verifies $(\beta^{-1}(\theta))\#: \beta^{-1}(\rho_{(B, b)}g) \simeq \beta^{-1}(\tilde{H}^1(A, a, a))|_{(A, a) \otimes \partial I}$. Since $\beta^{-1}(\rho_{(B, b)}g) = S(g)$, we have found $\Gamma: S(g) \simeq \tilde{H}^1(A, a, a)\otimes \partial I$. Set $G = gp: A \times I \to B$ and $\Gamma^+ = (\mu_{(A, a), p(A, a) \otimes 1I}: (A, a) \times I \times I \to S(B, b))$. Then $G: f_0 \simeq_f f_1$ and $\Gamma^+: S(G)\# \simeq_f S(G^1)\#$.

(2) $S(G^0)\#, k = 0, 1$, are pointed homotopies from $S(f_0)$ to $S(f_1)$. Let $\theta: S(A, a) \times I \times I \to B$, $\theta: S(G^0)\# \simeq_f S(G^1)\#$. There is a unique map $\delta: (S(A, a) \otimes I) \times I \to B$ satisfying $\delta|((A, a) \otimes \partial I) \cup (A, a) \otimes \partial I \otimes I = \theta$ and one easily verifies that $\delta: S(G^0) \simeq S(G^1) \simeq S((A, a) \otimes \partial I)$. In particular, $\delta$ is a pointed homotopy. Define $Z = (A, a) \otimes I \otimes \partial I \cup (A, a) \otimes \partial I \otimes I$ and define $f: Z \to B$ by $f([[x, t'], k]) = G(x, t)$, $f([[x, k], t]) = f_k(x)$. Straightforward computations show that $\rho_{(B, b)} f = \beta(\tilde{H}^1(A, a, a) \otimes 1I)|_Z$. Since $\rho_{(B, b)}$ is a $(2r + 1)$-equivalence, there exists a map $g: (A, a) \otimes I \otimes I \to B$ such that $g|_Z = f$. Set $\psi = g|_{(A, a) \otimes 1I}$.

We shall now apply A.1 to the unpunctured setting. For each space $A$, let $\Sigma A$ denote its unreduced suspension and $q' = q'_A: A \times I \to \Sigma A$ the canonical quotient map, usually written as $q'(x, s) = \langle x, s \rangle'$. We adopt the convention $\Sigma \varnothing = S'0$. Given a homotopy $H: A \times I \to B$, we define $\Sigma H: (\Sigma A) \times I \to \Sigma B$ by $\Sigma H(\langle x, s \rangle', t) = \langle H(x, t), s' \rangle$. Note that $H: f_0 \simeq f_1$ implies $\Sigma H: \Sigma f_0 \simeq \Sigma f_1$. Moreover, if we have pointed maps $f_0, f_1: (A, a) \to (B, b)$ and a pointed homotopy $H: f_0 \simeq f_1$ then $\pi_{(B, b), \Sigma} H = S(H)\#(\pi_{(A, a)} \times 1I)$, where $\pi = \pi_{(C, c)}: \Sigma C \to S(C, c)$ denotes the obvious quotient map, $C = A, B$. 

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A.2 THEOREM. Let \( B \) be an \( r \)-connected CW-complex, \( A \) be a CW-complex, and \( f_0, f_1: A \to B \).

(1) Let \( \dim A \leq 2r \) and \( H: \Sigma f_0 \simeq \Sigma f_1 \). Then there exists \( G: f_0 \simeq f_1 \) such that \( \Sigma G \simeq H \).

(2) Let \( \dim A \leq 2r - 1 \) and \( G^0, G^1: f_0 \simeq f_1 \). If \( \Sigma G^0 \simeq \Sigma G^1 \), then \( G^0 \simeq G^1 \).

Proof. The case \( A = \emptyset \) is trivial. For \( A \neq \emptyset \), let us fix basepoints \( a \in A \) and \( b \in B \).

(1) Step 1. Assume that \( f_0, f_1: (A, a) \to (B, b) \) are pointed maps and \( H(\langle a, s \rangle, t) = \langle b, s \rangle \) for all \( s, t \).

Then \( H \) induces a pointed homotopy \( H': S(A, a) \times I \to S(B, b) \). By A.1, there is \( G: f_0 \simeq f_1 \) with \( S(G) \simeq S(H) \). We shall show \( \Sigma G \simeq H \). Let \( \theta: S(A, a) \times I \times I \to S(B, b) \), \( \theta: S(G) \simeq S(H) \). Define \( Z = S(A, a) \times I \times I \), \( Z' = S(B, b) \) and \( F: Z \to \Sigma B \) by \( F(\xi, \xi, t) = H(\xi, t) \) and \( F(\xi, k, t) = (\Sigma f_k)(\xi) \). One checks \( \bar{\pi}_{(b, k)} F = \bar{\pi}_{(a, k)} H \). Since \( \pi_{(b, k)} \) is a homotopy equivalence, there is \( \psi: A \times I \times I \to B \) with \( \psi|_{Z} = F \). It is obvious that \( \psi: \Sigma G \simeq H \).

Step 2. Assume that \( f_0, f_1: (A, a) \to (B, b) \) are pointed maps.

Then, since the inclusion \( \Sigma A \times \partial I \cup \Sigma A \times I \to \Sigma A \times I \) is a cofibration, and \( \Sigma B \) is 1-connected, one can use a standard homotopy extension argument to find a homotopy \( H': \Sigma f_0 \simeq \Sigma f_1 \) such that \( H' \simeq H \) and \( H'(<a, s>, t) = <b, s> \). Now apply Step 1.

Step 3. General case.

Since \( a \in A \) is nondegenerate and \( B \) is path-connected, \( f_0, f_1 \) are homotopic to pointed maps \( f'_0, f'_1: (A, a) \to (B, b) \). This reduces the problem to Step 2: Let \( h: f_0 \simeq f_1 \) and set \( H' = (\Sigma h^1) \circ H \circ (\Sigma h^0)^{-1} \), where \( \circ \) denotes juxtaposition of homotopies (starting from the right side) and \( \cdot^{-1} \) denotes the inverse homotopy. Then \( H': \Sigma f'_0 \simeq \Sigma f'_1 \), and we find \( G': f'_0 \simeq f'_1 \) with \( \Sigma G' \simeq H' \). The desired homotopy \( G \) is defined by \( G = (h^1)^{-1} \circ G' \circ h^0 \).

(2) This is proved by similar arguments and left to the reader (the case \( r = 0 \) is trivial; for \( r \geq 1 \) the first step is to assume that \( f_0, f_1: (A, a) \to (B, b) \) are pointed maps, \( G^0, G^1: f_0 \simeq f_1 \) and \( \theta: \Sigma G^0 \simeq \Sigma G^1 \) with \( \theta(\langle a, s \rangle, t, t) = \langle b, s \rangle \) for all \( s, t \), \( t \).)

We now come to the proof of the Strong Shape Suspension Theorem. We first need explicit descriptions of \( S\Sigma \) and \( \Sigma S\Sigma \). \( \Sigma \) extends in an obvious way to a functor \( \Sigma: \text{pro-Top} \to \text{pro-Top} \); if \( f \) is a level homotopy equivalence in \( \text{pro-Top} \), then so is \( \Sigma f \). Hence, \( \Sigma \) induces a functor \( \Sigma: \text{H(pro-Top)} \to \text{H(pro-Top)} \). The Vietoris functor \( V: \text{CM} \to \text{pro-Top} \) (cf. [5] §8, [10] III §9) induces a full embedding \( V: \text{S\Sigma} \to \text{H(pro-Top)} \) with the property \( V \Sigma = \Sigma V \) (we identify \( V(\Sigma X) \) and \( \Sigma V(X) \); cf. [5]). Now, a category \( K \) is defined as follows. The objects
are triples \((X, X, p)\), where \(X\) is a compactum, \(X\) is an inverse system of spaces and \(p: V(X) \to X\) is an isomorphism in \(\text{H(pro-Top)}\). The morphisms are defined by \(K(X, X, p, X', X', p') = \text{H(pro-Top)}(X, X')\). It is obvious that the suspension functor on \(\text{H(pro-Top)}\) induces a suspension functor \(\Sigma: K \to K\) (for the objects \(\Sigma(X, X, p) = (\Sigma X, \Sigma X, \Sigma p)\)). A functor \(\phi: K \to \text{SSh}\) is defined as follows. For the objects \(\phi(X, X, p) = X\); for the morphisms \(f \in K((X, X, p), (X', X', p'))\), \(\phi(f) = V^{-1}((p')^{-1}(fp))\). Obviously \(\phi\) is an equivalence of categories which satisfies \(\Sigma \phi = \phi \Sigma\).

For \(-1 \leq r, m \leq \infty\) let \(T(r, m)\) be the class of towers \(X = \{X_n\}\) of compact \(r\)-connected polyhedra \(X_n\) with \(\dim X_n \leq m\).

Given a compactum \(X\) with \(FdX = m\) and an \(r\)-shape-connected compactum \(Y\), there exist \(X \in T(-1, m)\) and \(Y \in T(r, \infty)\) admitting isomorphisms \(p: V(X) \to X\) and \(q: V(Y) \to Y\) in \(\text{H(pro-Top)}\). By the above considerations, the Strong Shape Suspension Theorem follows from

\[(A.3)\] Let \(X \in T(-1, m)\) and \(Y \in T(r, \infty)\). Then \(\Sigma: \text{H(tow-Top)}(X, Y) \to \text{H(tow-Top)}(\Sigma X, \Sigma Y)\) is a surjection if \(m \leq 2r\) and a bijection if \(m \leq 2r - 1\).

To prove \((A.3)\), we shall employ Lisica’s description [8] of \(\text{H(tow-Top)}\) via the coherent homotopy category of towers, \(\text{Coh}\), which is defined as follows. Objects are all towers \(X = \{X_n\}\) of spaces. A pre-morphism \(f: X \to Y\) consists of a strictly increasing index function \(\varphi: \mathbb{N} \to \mathbb{N}\), maps \(f_n: X_{\varphi(n)} \to Y_n\) and homotopies \(h_n: X_{\varphi(n+1)} \times I \to Y_n, h_n: f_{n+1} \simeq f_n\) bond. Pre-morphisms \(f = \{\varphi, f_n, h_n\}\) and \(f' = \{\varphi', f'_n, h'_n\}\) are homotopic if there exists a pre-morphism \(H = \{\chi, F_n, H_n\}: X \times I \to Y\) such that \(\chi \geq \varphi, \varphi', F_n: f_n \simeq f'_n\) bond and \(H_n(x, 0, t) = h_n(bond(x), t), H_n(x, 1, t) = h'_n(bond(x), t)\). A morphism of \(\text{Coh}\) is then a homotopy class of pre-morphisms; composition comes from the obvious composition of pre-morphisms (see [8] for details). There is an obvious suspension functor \(\Sigma: \text{Coh} \to \text{Coh}, \Sigma X = \{\Sigma X_n\}, \Sigma f = \{\varphi, \Sigma f_n, \Sigma h_n\}\) for a pre-morphism \(f\). Moreover, the canonical functor \(\lambda: \text{tow-Top} \to \text{Coh}\) given by \(\lambda(X) = X\) and \(\lambda(\{(\varphi, f_n)\}) = \{(\varphi, f_n, \text{stationary homotopy})\}\), is readily seen to induce a category isomorphism \(\lambda: \text{H(tow-Top)} \to \text{Coh}\) such that \(\lambda \Sigma = \Sigma \lambda\). Note also that the strong shape theories based on \(\text{H(tow-Top)}\) (cf. [5]) and on \(\text{Coh}\) (cf. [8]) are already known to be equivalent by [9], Theorems 2 and 3. We have now seen that \((A.3)\) is equivalent to

\[(A.4)\] Let \(X \in T(-1, m)\) and \(Y \in T(r, \infty)\). Then \(\Sigma: \text{Coh}(X, Y) \to \text{Coh}(\Sigma X, \Sigma Y)\) is a surjection if \(m \leq 2r\) and a bijection if \(m \leq 2r - 1\).

In the proof of \((A.4)\) we need the following easily established fact.

\[(A.5)\] Given a pre-morphism \(f = \{\varphi, f_n, h_n\}: X \to Y\) and homotopies \(\Phi_n: X_{\varphi(n)} \times I \to Y_n, \Phi_n: f_n \simeq f'_n\). Then the pre-morphism \(f^* = \{\varphi, f'_n, \Phi_n(bond \times 1) \circ h_n \circ bond \Phi_n^{-1}\}\) is homotopic to \(f\).

Proof of \((A.4)\).
(1) Let $m \leq 2r$. Consider any pre-morphism $f = \{\varphi, f_n, h_n\}: \Sigma X \to \Sigma Y$. Each
$f_n: \Sigma X_{\varphi(n)} \to \Sigma Y$ desuspends, i.e. there is $g_n: X_{\varphi(n)} \to Y$ with $f_n \simeq \Sigma g_n$. By (A.5) we may assume that already $f_n = \Sigma g_n$. But then $h_n: (\Sigma \text{bond})(\Sigma g_{n+1}) \simeq (\Sigma g_n)(\Sigma \text{bond})$. By A.2, there is $h'_n: \text{bond} g_{n+1} \simeq g_n \text{bond}$ with $\Sigma h'_n \simeq h_n$. It is now obvious that $g = \{\varphi, g_n, h'_n\}: X \to Y$ is a pre-morphism such that $\Sigma g$ is homotopic to $f$.

(2) Let $m \leq 2r - 1$. Consider pre-morphisms $f = \{\varphi, f_n, h_n\}, f' = \{\varphi', f'_n, h'_n\}: X \to Y$ such that $\Sigma f$ and $\Sigma f'$ are homotopic. This homotopy is realized by a pre-morphism $H = \{\chi, F_n, H_n\}: \Sigma X \times I \to \Sigma Y$. We may assume $\chi = \varphi = \varphi'$. By A.2, the homotopies $F_n: \Sigma f_n \simeq \Sigma f'_n$ desuspend, i.e. there are $\phi_n: f_n \simeq f'_n$ with $\Sigma \phi_n \simeq F_n$. By (A.5) the pre-morphism $f^* = \{\varphi, f'_n, h^*_n = \phi_n(\text{bond} \times 1) \circ h_n \circ \text{bond} \circ 1\}$ is homotopic to $f$; hence $\Sigma f^*$ is homotopic to $\Sigma f$ and therefore homotopic to $\Sigma f'$. The homotopy between $\Sigma f^*$ and $\Sigma f'$ is now realized by a premorphism $H^* = \{\varphi, F_n^*, H_n^*\}$ where each $F_n^*: f_n \simeq f'_n$ is a stationary homotopy. But then the $H_n^*$ may be regarded as homotopies of homotopies $\Sigma h^*_n \simeq \Sigma h'_n$. By A.2, we infer $h^*_n \simeq h'_n$. But this implies that $f^*$ and $f'$ are homotopic, i.e. that $f$ and $f'$ represent the same morphism of $\text{Coh}$.

References


