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Compositio Mathematica, tome 76, n° 1-2 (1990), p. 7-17

<http://www.numdam.org/item?id=CM_1990__76_1-2_7_0>
Monads and cohomology modules of rank 2 vector bundles

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Received 19 August 1988; accepted 20 July 1989

Introduction

Monads are a useful tool to construct and study rank 2 vector bundles on the complex projective space $\mathbb{P}_n$, $n \geq 2$ (compare [O-S-S]). Horrocks' technique of eliminating cohomology [Ho 2] represents a given rank 2 vector bundle $\mathcal{E}$ as the cohomology of a monad

$$(M(\mathcal{E})) \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$$

as follows.

First eliminate the graded $\mathcal{S} = \mathbb{C}[z_0, \ldots, z_n]$-module $H^1(\mathcal{E}(\ast)) = \oplus_{m \in \mathbb{Z}} H^1(\mathcal{P}_n, \mathcal{E}(m))$ by the universal extension

$$0 \to \mathcal{E} \to \mathcal{A} \to 0,$$

where

$$L_0 \to H^1(\mathcal{E}(\ast)) \to 0.$$

is given by a minimal system of generators ($\sim$ stands for sheafification).

If $n = 2$ take this extension as a monad with $\mathcal{A} = 0$.

If $n \geq 3$ eliminate dually $H^{n-1}(\mathcal{E}(\ast))$ by the universal extension

$$0 \to \mathcal{L}_0^* (c_1) \to \mathcal{K} \to \mathcal{E} \to 0$$

(where $c_1 = c_1(\mathcal{E})$ is the first Chern-class). Then notice, that the two extensions

* Partially supported by the DAAD.
can be completed to the display

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & & \downarrow & \\
0 \to \mathcal{L}_0^\vee (c_1) \to \mathcal{K} \to \mathcal{E} \to 0 \\
\downarrow & & \downarrow & \\
0 \to \mathcal{L}_0^\vee (c_1) \overset{\varphi}{\to} \mathcal{B} \to \mathcal{E} \to 0 \\
\downarrow \psi & & \downarrow & \\
\mathcal{L}_0 = \mathcal{L}_0 \\
\downarrow & & \downarrow & \\
0 & 0
\end{array}
\]

of a monad

\[
\mathcal{L}_0^\vee (c_1) \overset{\varphi}{\to} \mathcal{B} \overset{\psi}{\to} \mathcal{L}_0
\]

for \( \mathcal{E} \).

To get a better understanding for \( \mathcal{B}, \varphi \) and \( \psi \) consider first the case \( n = 2, 3 \). Then \( \mathcal{B} \) is a direct sum of line bundles by Horrocks' splitting criterion \([Ho 1]\). Taking cohomology we obtain a free presentation

\[
B \overset{\psi}{\to} L_0 \to H^1 \mathcal{E}(*) \to 0
\]

with \( B = H^0 \mathcal{B}(*) \). The crucial point is that this is minimal \([Ra]\). Moreover, if \( n = 3 \), then \( B \) is self-dual \([Ra]\): \( B^\vee (c_1) \cong B \). We will see below that up to isomorphism \( \varphi \) is the dual map of \( \psi \).

Let us summarize and slightly generalize. Consider an arbitrary graded \( S \)-module \( N \) of finite length with minimal free resolution (m.f.r. for short)

\[
0 \to L_{n+1} \overset{\alpha_{n+1}}{\to} L_n \to \cdots \to L_1 \overset{\alpha_1}{\to} L_0 \to N \to 0.
\]

If \( n = 2 \) then \( N \cong H^1 \mathcal{E}(*) \) for some rank 2 vector bundle \( \mathcal{E} \) on \( \mathbb{P}_2 \) iff \( \text{rk } L_1 = \text{rk } L_0 + 2 \) (compare \([Ra]\)). In this case \( \mathcal{E} \) is uniquely determined as \( \ker \alpha_0 \):

\[
0 \to \mathcal{L}_0^\vee (c_1) \overset{\alpha_0 (c_1)}{\to} \mathcal{L}_1^\vee (c_1) \overset{\alpha_1}{\to} \mathcal{L}_0 \overset{\alpha_0}{\to} \mathcal{L}_0 \to 0.
\]
(This sequence is self-dual by Serre-duality [Ho 1, 5.2], since $\mathcal{F}(c_1) \simeq \mathcal{F}$).

For $n = 3$ there is an analogous result. Answering Problem 10 of Hartshorne’s list [Ha] we prove:

**PROPOSITION 1.** $N$ is the first cohomology module of some rank 2 vector bundle on $\mathbb{P}_3$ iff

1. $\text{rk } L_1 = 2 \text{rk } L_0 + 2$ and
2. there exists an isomorphism $\Phi: L_1(c_1) \cong L_1$ for some $c_1 \in \mathbb{Z}$ such that $\alpha_0 \circ \Phi \circ \alpha_0(c_1) = 0$.

In this case any $\Phi$ satisfying (2) defines a monad

$$
(M_\Phi) \quad \tilde{L}_0(c_1) \xrightarrow{\Phi \circ \alpha_0(c_1)} L_1 \xrightarrow{\alpha_0} \tilde{L}_0
$$

and $\mathcal{F}$ is a 2-bundle on $\mathbb{P}_3$ with $H^1(\mathcal{F}(\bullet)) \simeq N$ (and $c_1 = c_1(\mathcal{F})$) iff $(M(\mathcal{F})) \simeq (M_\Phi)$ for some $\Phi$.

To complete the picture let us mention a result of Hartshorne and Rao (not yet published). If $N \simeq H^1(\mathcal{F}(\bullet))$ as above then $L_0(c_1) \xrightarrow{\rho} L_1$ is part of a minimal system of generators for $\ker \alpha_0$. In other words: There exists a splitting

$$0 \to L_4 \to L_3 \to L_2 \oplus L_0(c_1) \to L_1 \to L_0 \to H^1(\mathcal{F}(\bullet)) \to 0$$

inducing the monad

$$(M(\mathcal{F})) \quad \tilde{L}_0(c_1) \to \tilde{L}_1 \to \tilde{L}_0$$

and the m.f.r.

$$0 \to \tilde{L}_4 \to \tilde{L}_3 \to \tilde{L}_2 \to \mathcal{F} \to 0$$

resp.

For $n \geq 4$ there is essentially only one indecomposable 2-bundle known on $\mathbb{P}_n$: The Horrocks-Mumford-bundle $\mathcal{F}$ on $\mathbb{P}_4$ with Chern-classes $c_1 = -1$, $c_2 = 4$. We prove:

**PROPOSITION 2.** The m.f.r. of $H^2(\mathcal{F}(\bullet))$ decomposes as

$$0 \to H_2 \xrightarrow{\beta_1} H_1 \xrightarrow{(\beta_0, \beta_0)} L_0(c_1) \oplus L_1 \xrightarrow{(0, \alpha_0(c_1))} L_0 \oplus L_1(c_1) \to$$

$$\to H_1(c_1) \to H_2(c_1) \to H^2(\mathcal{F}(\bullet)) \to 0$$
with $B = H^0\mathcal{B}(\ast)$, inducing the monad

$$(M(\mathcal{F})) \quad \tilde{L}_0^\gamma(c_1) \rightarrow \mathcal{B} \rightarrow \tilde{L}_0$$

and the minimal free presentation

$$L_1 \xrightarrow{\alpha_0} L_0 \rightarrow H^1\mathcal{F}(\ast) \rightarrow 0.$$  

The corresponding m.f.r. decomposes as

$$0 \rightarrow L_5 \rightarrow L_4 \rightarrow L_3 \oplus H_2 \xrightarrow{\begin{pmatrix} * & 0 \\ * & \beta_1 \end{pmatrix}} L_2 \oplus H_1 \xrightarrow{\begin{pmatrix} * & \beta_0^\gamma \end{pmatrix}} L_1 \xrightarrow{\alpha_0} L_0 \rightarrow H^1\mathcal{F}(\ast) \rightarrow 0$$

inducing the m.f.r.

$$0 \rightarrow \tilde{L}_5 \rightarrow \tilde{L}_4 \rightarrow \tilde{L}_3 \rightarrow \tilde{L}_2 \rightarrow \mathcal{F} \rightarrow 0.$$  

$(M(\mathcal{F}))$ is the monad given in [H-M]. Using its display we can compute the above m.f.r.’s explicitly. Especially we reobtain the equations of the abelian surfaces in $\mathbb{P}_4$ ([Ma 1], [Ma 2]).

Of course we may deduce from 57 some more bundles by pulling it back under finite morphisms $\pi: \mathbb{P}_4 \rightarrow \mathbb{P}_4$. The above result also holds for the bundles $\pi^*\mathcal{F}$ with $(M(\pi^*\mathcal{F})) = \pi^*(M(\mathcal{F}))$.

There is some evidence (but so far no complete proof), that a splitting as in Proposition 2 occurs for every indecomposable 2-bundle on $\mathbb{P}_4$. This suggests a new construction principle for such bundles by constructing their $H^2$-module first.

**Proof of Proposition 1**

Let $n = 3$ and $N$ be a graded $S$-module of finite length with m.f.r.

$$0 \rightarrow L_4 \rightarrow L_3 \rightarrow L_2 \rightarrow L_1 \xrightarrow{\alpha_0} L_0 \rightarrow N \rightarrow 0.$$

Suppose first that $N \cong H^1\mathcal{E}(\ast)$ for some 2-bundle $\mathcal{E}$ on $\mathbb{P}_3$ (with first Chern-class $c_1$). As seen in the introduction, Horrocks’ construction leads to a monad

$$(M(\mathcal{E})) \quad \tilde{L}_0^\gamma(c_1) \xrightarrow{\phi} L_1 \xrightarrow{\alpha_0} L_0$$
for \( \mathcal{E} \). The dual sequence

\[
L_0(c_1) \xrightarrow{\varphi(c_1)} L_1(c_1) \xrightarrow{\varphi(c_1)} L_0
\]

is a monad for \( \mathcal{E}'(c_1) \simeq \mathcal{E} \). The induced presentation of \( N \) has to be isomorphic to that one given by the m.f.r.:

\[
\begin{array}{cccc}
L_1(c_1) & \xrightarrow{\varphi(c_1)} & L_0 & \xrightarrow{0} N & \xrightarrow{0} 0 \\
\downarrow & & \downarrow & & \downarrow \\
L_1 & \xrightarrow{\varphi(c_1)} & L_0 & \xrightarrow{0} N & \xrightarrow{0} 0.
\end{array}
\]

Dualizing gives (2) since \( \alpha_0 \circ \varphi = 0 \) and thus also a monad \( (M_{\mathcal{E}}) \) for \( \mathcal{E} \), isomorphic to \( (M(\mathcal{E})) \) (replace \( \mathcal{E} \) by \( \mathcal{E}'(c_1) \)).

Conversely if \( N \) satisfies (2), we obtain a monad \( (M_{\mathcal{E}}) \) by sheafification. (Since \( \tilde{N} = 0 \), \( \alpha_0 \) is a bundle epimorphism. Dually \( \alpha_0(c_1) \) is a bundle monomorphism.) Let \( \mathcal{E} \) be the cohomology bundle of \( (M_{\mathcal{E}}) \). Then \( H^1(\mathcal{E})(*) \simeq N \). \( \mathcal{E} \) has rank 2, if \( N \) satisfies (1). \( \square \)

REMARK 1. (i) Let \( N \simeq H^1(\mathcal{E})(*) \) as above with induced splitting

\[
0 \to L_4 \to L_3 \to L_2 \oplus L_1(c_1) \to L_1 \to L_0 \to H^1(\mathcal{E})(*) \to 0
\]

as in the introduction. Recall that \( \mathcal{E} \) is stable iff \( H^0(\mathbb{P}_3, \mathcal{E}(m)) = 0 \) for \( m \leq -c_1/2 \). Thus \( \mathcal{E} \) is stable iff \( L_2 \) has no direct summand \( S(m) \) with \( m \geq c_1/2 \). Notice that this condition only depends on \( N \).

(ii) If \( N \) satisfies (1) and has only one generator, then (2) is obviously equivalent to the symmetry condition \( L_1(c_1) \simeq L_1 \). Thus \([Ra, 3.1]\) is a special case of Proposition 1.

EXAMPLES. (i) The well-known Null correlation bundles are by definition the bundles corresponding to the \( S \)-module \( \mathcal{C} \). Consider the Koszul-presentation

\[
4S \xrightarrow{\alpha_0} S(1) \to \mathcal{C} \to 0, \quad \alpha_0 = (z_0, z_1, z_2, z_3).
\]

The isomorphisms \( 4S \xrightarrow{\alpha_0} 4S \) with \( \alpha_0 \circ \varphi \circ \alpha_0(c_1) = 0 \) are precisely the \( 4 \times 4 \) skew symmetric matrices with nonzero determinant. Two such matrices give isomorphic bundles iff they differ by a scalar (use [O-S-S, II, Corollary 1 to 4.1.3]). The moduli space of Null correlation bundles is thus isomorphic to \( \mathbb{P}_3 \setminus G \), where \( G \) is the Plucker embedded Grassmanian of lines in \( \mathbb{P}_3 \).

Unlike the case \( n = 2 \) the bundle is not uniquely determined by the module.
(ii) The $S$-module

\[ 6S \xrightarrow{\alpha_0} S(1) \oplus S(2) \to N \to 0, \quad \alpha_0 = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & 0 & 0 \\ 0 & 0 & z_0^2 & z_1^2 & z_2^2 & z_3^2 \end{pmatrix} \]

satisfies (1) and the symmetry condition $L_1 \simeq L_1$, i.e. the necessary conditions of [Ra]. But $N$ does not satisfy (2).

**Cohomology modules of the Horrocks-Mumford-bundle $\mathcal{F}$**

We first recall the construction of $\mathcal{F}$ [H-M]. Let

\[ V = \text{Map}(\mathbb{Z}_5, \mathbb{C}) \]

be the vector space of complex valued functions on $\mathbb{Z}_5$. Denote by

\[ H \subset N \subset \text{SL}(5, \mathbb{C}) \]

the Heisenberg group and its normalizer in $\text{SL}(5, \mathbb{C})$ resp.

Let $V_0 = V, V_1, V_2, V_3$ and

\[ W = \text{Hom}_H(V_1, \Lambda^2 V) \]

be defined as in [H-M]. The $V_i$ are irreducible representations of $H$ and $N$ of degree 5. $W$ is an irreducible representation of $N/H$ of degree 2. It is unimodular, so it comes up with an invariant skew symmetric pairing.

Let $\mathbb{P}_4 = \mathbb{P}(V)$ be the projective space of lines in $V$. The Koszul-complex on $\mathbb{P}(V)$ is the exact sequence

\[ (K) \quad \cdots \to \mathcal{O}(-1) \otimes \Lambda^2 V \to \mathcal{O} \otimes \Lambda^3 V \to \cdots, \]

\[ (\Lambda^2 \mathcal{F})(-3) \]

\[ 0 \quad 0 \]

obtained by exterior multiplication with the tautological subbundle

\[ \mathcal{O}(-1) \to V \otimes \mathcal{O}. \]

The exterior product provides $(K)$ with a self-duality (with values in $\mathcal{O}(-1) \otimes \Lambda^5 V \simeq \mathcal{O}(-1)$).
This induces the natural pairings

$$(\Lambda^i \mathcal{F})(-i - 1) \otimes (\Lambda^{4-i} \mathcal{F})(i - 5) \xrightarrow{\alpha_0} (\Lambda^4 \mathcal{F})(-6) \cong \mathcal{O}(-1)$$

and is compatible with the action of $\text{SL}(5, \mathbb{C})$.

It can be extended to $(K) \otimes W$ by tensoring with the invariant form, then being compatible with the action of $N$.

As in the proof of [H-M, Lemma 2.4] it follows, that $(K) \otimes W$ decomposes as given by the splitting into irreducible $N$-modules. Moreover the induced

$$\mathcal{O}(-1) \otimes V_1 \rightarrow (\Lambda^2 \mathcal{F})(-3) \otimes W \rightarrow \mathcal{O} \otimes V_3$$

is the self-dual Horrocks-Mumford-monad, whose cohomology is $\mathcal{F}$ (normalized such that $c_1 \mathcal{F} = -1$).

To proof Proposition 2 consider the display

It first follows that $H^2\mathcal{F}(\ast) = W$ is a vector space, sitting in degree $-2$ (compare [H-M]). Its m.f.r. is the Koszul-complex obtained from $(K) \otimes W$ by taking global
sections. So it decomposes, inducing the presentation

$$S(-1) \otimes V_1 \otimes U \xrightarrow{\alpha_0} S \otimes V_3 \longrightarrow H^1 F(*) \longrightarrow 0$$

and the Horrocks-Mumford-monad. But this is just

$$(M(F)) \quad \mathcal{O}(-1) \otimes V_1 \to \mathcal{B} \to \mathcal{C} \otimes V_3:$$

apply e.g. [O-S-S, II, Corollary 1 to 4.1.3] (notice that $H^0 F = 0$ implies $H^0 B = H^0 B^*(-1) = 0$ by construction of $(M(F))$).

It remains to show that $\alpha_0$ is minimal and that the corresponding m.f.r. of $H^1 F(*)$ decomposes, inducing the m.f.r., say,

$$0 \longrightarrow F_3 \xrightarrow{\gamma_2} F_2 \xrightarrow{\gamma_1} F_1 \xrightarrow{\gamma_0} F_0 \longrightarrow F \longrightarrow 0$$

of $F = H^0 F(*)$.

From the second row of the display we obtain the m.f.r. of $Q = H^0 2(*)$:

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S(-3) \otimes W & = & S(-3) \otimes W \\
\downarrow{\beta_1} & & \downarrow{\beta_1} \\
S(-2) \otimes V \otimes W & = & S(-2) \otimes V \otimes W \\
\downarrow{\beta_0} & & \downarrow{\beta_0} \\
0 \to S(-1) \otimes V_1 \to (S(-1) \otimes V_1) \oplus (S(-1) \otimes V_1 \otimes U) \to S(-1) \otimes V_1 \otimes U \to 0 \\
\| & & \| \\
0 \to S(-1) \otimes V_1 \longrightarrow H^0 (\Lambda^2 F)(-3) \otimes W & \longrightarrow & Q \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]
The third column of the display gives rise to the commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
F_3 \\
\downarrow \\
F_2 \xrightarrow{x_3'} S(-3) \otimes W \\
\downarrow \\
F_1 \xrightarrow{x_2'} S(-2) \otimes V \otimes W \\
\downarrow \\
F_0 \xrightarrow{x_1'} S(-1) \otimes V_1 \otimes U \xrightarrow{\alpha_0} S \otimes V_3 \rightarrow H^1(F(\ast)) \rightarrow 0
\end{array}
\]

with exact columns and bottom row.

The induced

\[
0 \rightarrow F_3 \xrightarrow{\gamma_2} F_2 \xrightarrow{\gamma_1} F_1 \oplus (S(-3) \otimes W) \xrightarrow{\alpha_0} F_0 \oplus (S(-2) \otimes V \otimes W) \xrightarrow{(\alpha_1', \beta_0')} \rightarrow S(-1) \otimes V_1 \otimes U \xrightarrow{\alpha_0} S \otimes V_3 \rightarrow H^1(F(\ast)) \rightarrow 0
\]

is exact and it is minimal, iff \(\alpha'_1, \alpha'_2, \alpha'_3\) have no entries in \(C \setminus \{0\}\). But since \(H^0(F(1)) = 0\) [H-M], these maps have only entries in degrees \(\geq 1\).

**Remark 2** (i) Let us describe \((M(F))\) more explicitly by choosing convenient bases of \(V_1, V_3 = V^\ast, W\) and forgetting the \(N\)-module structure (compare the proof of [H-M, Lemma 2.5].)

Choose the basis \(e_0, \ldots, e_4\) of \(V = \text{Map}(\mathbb{Z}_5, \mathbb{C})\) given by \(e_i(j) = \delta_{ij}\) and its dual basis \(z_0, \ldots, z_4 \in V^\ast\).
Define

\[
A = (a_{ij})_{0 \leq i, j \leq 4}
\]

by

\[
\begin{align*}
  a_{i0} &= e_i + 2 \wedge e_i + 3 \\
  a_{i1} &= e_i + 1 \wedge e_i + 4
\end{align*}
\]
i mod 5.

Then \( w_0, w_1, \) given by \( w_j(e_i) = a_{ij} \) is a basis of \( W. \) Identifying \( W \cong \mathbb{C}^2, \) the invariant form on \( W \) becomes the standard symplectic form \( Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) on \( \mathbb{C}^2. \)

We thus may rewrite \((M(F))\) as

\[
(M(F)) \quad S(1) \xrightarrow{Q \cdot A^{-1}} 2(L^2 \mathcal{F})(-3) \xrightarrow{A} 50,
\]

the matrices operating by exterior multiplication.

(ii) From the explicit form of \((M(F))\) we can compute \( \alpha_0 \) explicitly. Choose a convenient basis of \( \Lambda^2 V \otimes W = (V_1 \oplus U). \) Then

\[
15S(-1) \xrightarrow{\gamma_0} 5S \rightarrow H^1 \mathcal{F}(\ast) \rightarrow 0
\]

is the matrix

\[
\alpha_0 = \begin{pmatrix}
0 & z_3 & 0 & 0 & z_2 & 0 & 0 & z_1 & z_4 & 0 & z_0 & 0 & 0 & 0 & 0 \\
z_3 & 0 & z_4 & 0 & 0 & 0 & 0 & z_2 & z_0 & 0 & z_1 & 0 & 0 & 0 & 0 \\
0 & z_4 & 0 & z_0 & 0 & z_1 & 0 & 0 & z_3 & 0 & 0 & z_2 & 0 & 0 & 0 \\
0 & 0 & z_0 & 0 & z_1 & z_4 & z_2 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 & 0 \\
z_2 & 0 & 0 & z_1 & 0 & 0 & z_0 & z_3 & 0 & 0 & 0 & 0 & 0 & z_4 & 0
\end{pmatrix}.
\]

Resolving it (use e.g. [B-S]), we obtain the m.f.r. of \( H^1 \mathcal{F}(\ast). \) Its shape is

\[
0 \rightarrow 2S(-8) \rightarrow 20S(-6) \rightarrow 35S(-5) \oplus 2S(-3) \rightarrow (15S(-4) \oplus 4S(-3)) \oplus 10S(-2) \rightarrow 15S(-1) \rightarrow 5S \rightarrow H^1 \mathcal{F}(\ast) \rightarrow 0.
\]

(iii) Consider the induced m.f.r. of \( F \) and its dual

\[
\cdots \rightarrow 35S(-5) \xrightarrow{\gamma_0} 15S(-4) \oplus 4S(-3) \xrightarrow{\Gamma} 15S(3) \oplus 4S(2) \xrightarrow{\gamma_0} 35S(4) \rightarrow \cdots
\]

\[F \simeq F^\vee(-1)\]

\[
\Gamma \text{ can be computed by resolving } \gamma_0 \text{ (use again [B-S]). We thus obtain explicit}
\]
bases for the spaces of sections $H^0\mathcal{F}(m)$. Especially we get the equations of the zero-schemes of sections of $\mathcal{F}(3)$, including the abelian surfaces in $\mathbb{P}_4$.

(iv) Let $\pi: \mathbb{P}_4 \to \mathbb{P}_4$ be a finite morphism and $d^4$ its degree. Then $\pi^*\mathcal{F}$ is a stable 2-bundle with Chern-classes $c_1 = -d$, $c_2 = 4d^2$. Proposition 2 and the above remarks also hold for $\pi^*\mathcal{F}$: Replace $(K)$ by $\pi^*(K)$, $(M(F))$ by $\pi^*(M(\mathcal{F})) = (M(\pi^*\mathcal{F}))$ and $z_0, \ldots, z_4$ in $a_0$ by $f_0, \ldots, f_4$, where $f_0, \ldots, f_4$ are the forms of degree $d$ defining $\pi$.

Acknowledgement

I'd like to thank R. Hartshorne for helpful discussions.

References