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Linearizing some \( \mathbb{Z}/2\mathbb{Z} \) actions on affine space

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Let \( V \) be the affine space \( k^n \) over an algebraically closed field \( k \), \( G \) a linearly reductive group and \( A: G \times V \to V \) a group action with a fixed point, say the origin. Then for all \( g \in G \) let me denote by \( A(g) \) the corresponding automorphism of \( V \). We have

\[
A(g) = L(g) + D(g)
\]

where \( L(g), D(g) \in \text{End} \ V, L(g) \) linear and \( D(g) \) the sum of terms of higher degrees.

Let me recall the well known linearization problem: is the action \( A \) linearizable, i.e. conjugated to the linear action \( L: G \times V \to V \) (see e.g. \([B]\) and \([K]\))? Recently counter-examples have been found, see \([S]\) and \([K + S]\), so it is reasonable to study additional assumptions on the action \( A \). One of them is considered in the present paper.

First I want to define some morphism \( \sigma_A: V \to V \) which turns out to be a conjugating automorphism for \( A \), provided \( \sigma_A \) is invertible. It will be done using the Reynolds operator i.e. the equivariant projection \( \rho: \mathcal{C}(G) \to k \). For a finite dimensional \( k \)-space \( W \) we have the unique linear map \( \int_G: \text{Mor}(G, W) \to W \) such that for all linear maps \( f: W \to k \) the induced diagram

\[
\begin{array}{ccc}
\int_G : \text{Mor}(G, W) & \longrightarrow & W \\
\downarrow f_* & & \downarrow f \\
\rho: \mathcal{C}(G) & \longrightarrow & k
\end{array}
\]

is commutative. Now let \( \phi: G \to \text{End}(V) \) be such a map that the induced map \( G \times V \to V \) is an algebraic morphism. Then \( W := \text{lin hull} \ (\phi(G)) \) is finite dimensional, hence \( \int_G \phi \) is a well-defined element of \( \text{End}(V) \). Let us apply the above to the map \( \phi: G \ni g \mapsto L(g^{-1})A(g) \in \text{End}(V) \) and set \( \sigma = \sigma_A = \int \phi \) (compare \([J]\)). We have

\[
L(h)\sigma = \int_{g \in G} L(h)L(g^{-1})A(g) = \left( \int_{g \in G} L(hg^{-1})A(gh^{-1}) \right)A(h) = \sigma A(h)
\]

for all \( h \in G \).
So $\sigma$ invertible implies that $A(h) = \sigma^{-1}L(h)\sigma$. In particular the action $A$ is linearizable. Later we will give an example of an action $A$ which can be linearized but for which $\sigma_A$ is not invertible.

As mentioned in [J], the morphism $\sigma_A$ can be interpreted as an average deviation of $A$ from being linear.

**CONJECTURE (Kraft, Procesi).** Assume for some $d \geq 2$

$$A(g) = L(g) + H_d(g) + H_{d+1}(g) + \cdots + H_{2d-2}(g), \quad \text{for all } g,$$

where $H_m(g)$ is a homogeneous endomorphism of $V$ of degree $m$. Then $\sigma_A$ is invertible. In particular the action $A$ is linearizable.

**THEOREM.** The above conjecture is true in the following cases

1. $G$ linearly reductive, $d = 2$ and char $k \neq 2$,
2. $G$ diagonalizable, $d = 2$ and char $k$ arbitrary,
3. $G = \mathbb{Z}/2\mathbb{Z}$, $d$ arbitrary and char $k = 0$.

Cases 1 and 2 are the objects of [J].

**Proof for the case 3.** Let $I$ denote the identity map of $V$. We can write: $G = \{I, L + D\}$, where $L$ and $D$ are endomorphisms of $V$, $L$ linear and $D = H_d + \cdots + H_{2d-2}$. We have $L^2 = (L + D)^2 = I$. It follows that

$$LD + D(L + D) = 0. \quad (1)$$

Let me denote by $\tilde{H}_d$ the $d$-linear symmetric map from $V^d$ to $V$ corresponding to $H_d$. Then we have

$$D(L + D) = DL + d\tilde{H}_d(L, \ldots, L, H_d) + \cdots$$

where the first summand consists of terms of degrees $d, \ldots, 2d - 2$, the second is of degree $2d - 1$ and all further summands have higher degrees. Considering the possible cancellations in (1) we obtain:

$$-LD = DL = D(L + D). \quad (2)$$

By definition $\sigma = \frac{1}{2}(I + (I + LD)) = I - \frac{1}{2}DL$. We will prove that $I + \frac{1}{2}DL$ is the inverse of $\sigma$.

**LEMMA.** $D(I + mDL) = D$ for $m = 0, 1, 2, \ldots$

**Proof.** Suppose the above holds for some $m - 1, m > 0$. By (2), $D = D(I + DL)$. Therefore

$$D = D(I + (m - 1)DL)(I + DL) = D(I + DL + (m - 1)DL(I + DL)).$$
On the other hand $DL(I + DL) = -LD(I + DL) = -LD = DL$, and we are done.

Since $\text{char}(k) = 0$ the Lemma implies that $D(I + rDL) = D$ for all $r \in k$. Then taking $r = \frac{1}{2}$ we have

$$(I - \frac{1}{2}DL)(I + \frac{1}{2}DL) = I + \frac{1}{2}DL + \frac{1}{2}LD(I + \frac{1}{2}DL) = I,$$

and the same applies if we interchange the order of factors at the left hand side. Q.E.D.

**EXAMPLE OF A NON INVERTIBLE $\sigma$.** Let the linear endomorphism $L$ of $k^2$ be given by $L(x, y) = (x, -y)$ and an automorphism $\tau$ by $\tau(x, y) = (x - (x + y)^2, y + (x + y)^2)$ so that $\tau^{-1}(x, y) = (x + (x + y)^2, y - (x + y)^2)$. The automorphism $\tau^{-1}L\tau$ has order two, so it defines an action of the group of order two on $k^2$. The corresponding endomorphism $\sigma = \frac{1}{2}(I + L\tau^{-1}L\tau)$ takes $(x, y)$ to $(x - u + v, y + u + v)$, where $u = \frac{1}{2}(x + y)^2$, $v = \frac{1}{2}(x - y - 2(x + y)^2)^2$. Direct computation shows that the Jacobian determinant of $\sigma$ is

$$J(\sigma) = 1 - 4(x^2 + y^2) + 8(x^3 + y^3) + 24(x^2y + xy^2).$$

Therefore the endomorphism $\sigma$ is not invertible, while the considered group action can obviously be linearized.

**References**


