F. A. Bogomolov
A. N. Landia

2-cocycles and Azumaya algebras under birational transformations of algebraic schemes

Compositio Mathematica, tome 76, no 1-2 (1990), p. 1-5

<http://www.numdam.org/item?id=CM_1990__76_1-2_1_0>
2-Cocycles and Azumaya algebras under birational transformations of algebraic schemes

F.A. BOGOMOLOV & A.N. LANDIA

\textit{Steklov Mathematical Institute of the Academy of Sciences of USSR, Vavilov Street 42, Moscow 117333, USSR; Mathematical Institute of the Academy of Sciences of Georgian SSR, Z. Rukhadze Street 1, Tbilisi 380093, USSR}

Received 17 November 1988; accepted in revised form 23 November 1989

The basic question whether the injection $\text{Br}(X) \to H^2(X, \mathcal{O}_X^*)_{\text{tors}}$ is an isomorphism arose at the very definition of the Brauer group of an algebraic scheme $X$. Positive answers are known in the following cases:

1. the topological Brauer group $\text{Br}(X_{\text{top}}) \cong H^2(X, \mathcal{O}_X^*)_{\text{tors}} \cong H^3(X, \mathbb{Z})_{\text{tors}}$ (J.-P. Serre); in the etale (algebraic) case the isomorphism is proved for smooth projective surfaces (A. Grothendieck);
2. abelian varieties;
3. the union of two affine schemes (R. Hoobler, O. Gabber).

The first author has formulated a birational variant of the basic question, while considering the unramified Brauer group in [1]. The group $\text{Br}_r(K(X)) = \bigcap \text{Br}(A_v) \subseteq \text{Br}(K(X))$ (intersection taken over all discrete valuation subrings $A_v$ of the rational function field $K(X)$) is isomorphic to $H^2(\tilde{X}, \mathcal{O}^*)$, where $\tilde{X}$ is a nonsingular projective model of $X$, i.e. a nonsingular projective variety birationally equivalent to $X$.

QUESTION. Given a cocycle class $\gamma \in H^2(\tilde{X}, \mathcal{O}^*)$, is it possible to find a nonsingular projective model $\tilde{X}$ such that $\gamma$ is represented by a $\mathbb{P}^n$-bundle (i.e. by an Azumaya algebra) on $\tilde{X}$?

The case where $X$ is a nonsingular projective model of $V/G$, with $G$ a $\gamma$-minimal group and $V$ a faithful representation of $G$, was considered in [2]. O. Gabber in his letter to Bogomolov (12.1.1988) has given an affirmative answer to the question in the case of general algebraic spaces. In this paper we give a simple version of his proof for algebraic schemes.

Let $X$ be a scheme, $\gamma \in H^2(X, \mathcal{O}^*), \{U_i\}$ an affine cover of $X$. Then the restriction of $\gamma$ to each $U_i$ is represented by an Azumaya algebra $A_i$. If we would have isomorphisms $A_i|_{U_i \cap U_j} \cong A_j|_{U_i \cap U_j}$, we could glue the sheaves $\{A_j\}$ and get an...
Azumaya algebra on $X$, representing $\gamma$. But we have isomorphisms $A_{ij|U_i \cap U_j} \otimes \text{End}(E_{ij}) \cong A_{ji|U_i \cap U_j} \otimes \text{End}(E_{ji})$ for certain vector bundles $E_{ij}$, $E_{ji}$ on $U_i \cap U_j$.

**THEOREM.** Let $X$ be a noetherian scheme, $\gamma \in H^2(X, \mathcal{O}_X^*)$. There exists a proper birational morphism $\alpha: \tilde{X} \to X$ such that $\alpha^*(\gamma)$ is represented by an Azumaya algebra over $\tilde{X}$.

**Proof.** It is enough to consider $X$ which are connected. Suppose that $\{U_i\}$ is an affine open cover of $X$ and that $\gamma$ is non-trivial on at least one $U_i$. We will construct an Azumaya algebra on a birational model of $X$ by an inductive process which involves adjoining one by one proper preimages of the subsets $U_i$ and, by an appropriate birational change of the scheme and Azumaya algebra obtained, extending the new algebra to the union. We start with some affine open subset $U_0$ and an Azumaya algebra $A_0$ on it.

Now suppose by induction that we already have an Azumaya algebra $\tilde{A}_k$ on the scheme $X_k$, a Zariski-open subset of the scheme $\tilde{X}_k$, equipped with a proper birational map $\bar{\alpha}_k: \tilde{X}_k \to X$ such that $X_k = \tilde{\alpha}_k^{-1}(U_0 \cup \cdots \cup U_k)$. Let $U_{k+1}$ intersect $U_0 \cup \cdots \cup U_k$ and $\bar{U}_{k+1} = \tilde{\alpha}_k^{-1}(U_{k+1})$. Suppose that on $U_{k+1}$, $\gamma$ is represented by the Azumaya algebra $A_{k+1}$. In the same vein as above we have an isomorphism

$$\tilde{A}_{k|X_k \cap U_{k+1}} \otimes \text{End}(E_{k,k+1}) \cong \tilde{\alpha}_k^*(A_{k+1})|_{X_k \cap U_{k+1}} \otimes \text{End}(E_{k+1,k})$$

and we need to extend $E_{k,k+1}$ to $X_k$ and $E_{k+1,k}$ to $\bar{U}_{k+1}$ from their intersection. After this we will change $\tilde{A}_k$ and $\tilde{\alpha}_k^*(A_{k+1})$ by the other representatives $\tilde{A}_k \otimes \text{End}(E_{k,k+1}), \tilde{\alpha}_k^*(A_{k+1}) \otimes \text{End}(E_{k+1,k})$ of the same Brauer classes and glue these Azumaya algebras, hence the proof.

First, extend both sheaves $E$ as coherent sheaves. This can be done by the following

**LEMMA.** Let $X$ be a noetherian scheme, $U \subseteq X$ a Zariski-open subset, $E$ a coherent sheaf on $U$. Then there exists a coherent sheaf $E'$ on $X$ such that $E'|_U \cong E$. This is Ex. II.5.15 in [4].

Note that we can assume that in our inductive process we add neighborhoods $U_{k+1}$ of no more than one irreducible component (or an intersection of irreducible components) of $X$, different from those contained in $X_k$. Thus we assume $X_k \cap U_{k+1}$ to be connected and the rank of $E$ to be constant on $X_k \cap U_{k+1}$, hence $E'$ will be locally generated by $n$ elements, where $n$ is the rank of $E$.

**LEMMA** (see [3], Lemma 3.5). Let $X$ be a noetherian scheme, $E$ a coherent sheaf on $X$, locally free outside a Zariski closed subset $Z$ on $X$. Then there exists a coherent sheaf $I$ of ideals on $X$ such that the support of $\mathcal{O}_X/I$ is $Z$ with the following
property. Let $\alpha: \tilde{X} \to X$ be the blowing up of $X$ with center $I$, then the sheaf $i(E) := \text{the quotient of } \alpha^*(E) \text{ by the subsheaf of sections with support in } \alpha^{-1}(Z)$, is locally free on $\tilde{X}$.

Proof. The proof consists of two parts. First: to reduce the number of local generators to get this number constant on the connected components of $X$ (the minima are the values of the (local) rank function of $E$). Second, to force the kernel of the (local) presentations $\mathcal{O}_V \twoheadrightarrow E|_V \to 0$ to vanish for all neighborhoods from some cover $\{V\}$. Both parts are proved by indicating the suitable coherent sheaves of ideals and blowing up $X$ with respect to these sheaves. Let $\mathcal{O}_V \twoheadrightarrow E|_V \to 0$ be a local presentation of $E$. Then $\text{Ker}(f)$ is generated by all relations $\sum c_ia_i = 0$ where $\{a_i\}$ stand for the free basis of $\mathcal{O}_V$. The coherent sheaf of ideals in the first case is the sheaf defined locally as the ideal $I_V$ generated by all $c_i$ such that $\sum c_ia_i \in \text{Ker}(f)$ and in the second case as $J_X = \text{Ann}(\text{Ker}(f))$. As the number of generators is constant in the case we are interested in, we give the details only for the second part of the proof and refer to [3] for the first.

Let $\alpha: X' \to X$ be the blowing up of $X$ with respect to $J_X$ and let $\tilde{a}(E)$ be as in the statement of the Lemma. Let

$$0 \to (\text{Ker}(f))|_{V'} \to \mathcal{O}_{V'} \xrightarrow{f} \tilde{a}(E)|_{V'} \to 0$$

be the local presentation of $\tilde{a}(E)$. We have $\alpha^{-1}(\text{Ann}(f)) \subseteq \text{Ann}(\text{Ker}(f))$. Let $p \in Z', V' = \text{Spec}(A')$ an affine neighborhood of $p$ in $X'$ and let $\sum c_ia_i \in \text{Ker}(f)|_{V'}$ map to a nonzero element in $\text{Ker}(f)$. Denote by $\gamma$ a generator of the invertible sheaf $\alpha^{-1}(\text{Ann}(f))$ on $V'' = \text{Spec}(A'') \subseteq V'$ for suitable $A''$. It is clear that there exists for given $p$ and $V''$ a finite sequence of open affine neighborhoods $V_1', \ldots, V_s'$ such that $X' \setminus Z' = V_1', V'' = V_s'$ and $V_j' \cap V_{j+1}' \neq \emptyset$ for $j = 1, \ldots, s - 1$. So suppose $V' \cap (X' \setminus Z') \neq \emptyset$ and $q \in V'' \cap (X' \setminus Z)$. Then $c_i|_q = 0$ for $i = 1, \ldots, m$ and $q \in \text{Spec}(A'')$; hence $\gamma^kc_i = 0$ for $i = 1, \ldots, m$ for some $k$. Since $\gamma$ is not a zero divisor, we conclude that $c_i = 0$ for $i = 1, \ldots, m$. Thus (maybe after considering a finite sequence of points $q_1, \ldots, q_s$) we prove that $(\text{Ker}(f))_p$ is trivial for every $p \in X'$.

In this way we glue the two sheaves $\mathcal{A}_k$ and $A_{k+1}$ and get an Azumaya algebra on $\tilde{X}_k \cup \tilde{O}_{k+1}$. As the scheme $X$ is quasi-compact, we obtain an Azumaya algebra on $\tilde{X}$ after a finite number of such steps.

Now we have to show that this process can be done in such a way that the class $[A]$ of the Azumaya algebra $A$ constructed in this way is equal to $\tilde{a}^*(\gamma)$. Again this goes by induction on $k$. We have $X_{k+1} = U \cup V$ with $U = \tilde{a}_{k+1}^{-1}(U_0 \cup \cdots \cup U_k)$ and $V = \tilde{a}_{k+1}^{-1}(U_{k+1})$. We have the exact sequence

$$H^1(U \cap V, \mathcal{O}^*) \to H^2(X_{k+1}, \mathcal{O}^*) \to H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$$
and by induction hypothesis, $\tilde{\alpha}_{k+1}(\gamma) - [\tilde{A}_{k+1}]$ maps to zero in $H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$ so it comes from $\beta \in H^1(U \cap V, \mathcal{O}^*)$. By blowing up $X_{k+1}$ we may assume that $\beta$ is represented by a line bundle which extends to $U$. Then $\beta$ maps to zero in $H^2(X_{k+1}, \mathcal{O}^*)$, hence $\tilde{\alpha}_{k+1}(\gamma) - [\tilde{A}_{k+1}] = 0$. □

Note that we need not bother about the compatibility of isomorphisms, because at each step we choose a new isomorphism between the Azumaya algebra $A$ on $U_1 \cup \cdots \cup U_j$ from the preceding step and $A_k$ on $U_k$, modulo $\text{End}(E)$, $\text{End}(E_k)$.

COROLLARY 1. Let $G$ be a finite group, $V$ a faithful complex representation of $G$. Then there exists a nonsingular projective model $X$ of $V \rtimes G$ such that $\text{Br}(X) = H^2(X, \mathcal{O}^*)$.

Proof. The group $H^2(X, \mathcal{O}^*)$ is a birational invariant of nonsingular projective varieties and is isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$ if $X$ is a model of $V \rtimes G$ (see [1]). It remains to recall that the group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is finite. □

COROLLARY 2. Let $X$ be a noetherian scheme over $\mathbb{C}$, $Z$ a closed subscheme of $X$ and $\gamma \in H^2_{et}(X, \mathcal{O}^*)$. Then there exists a proper morphism $\alpha: X' \to X$ which is an isomorphism above $X \setminus Z$ and maps $\gamma$ to zero in $H^2_{et}(X', \mathcal{O}^*)$.

Proof. First, let's have $\alpha(\gamma)$ map to zero in $H^2(X, \mathcal{O}^*)$. To do this, desingularize $X$ by $X' \to X$. Then in the following exact sequence (in etale cohomology), $\beta$ will be injective:

$$H^1(X' \setminus Z', \mathcal{O}^*) \to H^2_{et}(X', \mathcal{O}^*) \to H^2(X', \mathcal{O}^*) \to H^2(X' \setminus Z', \mathcal{O}^*) \uparrow$$

The injectivity is due to the injectivity of $H^2(X', \mathcal{O}^*) \to H^2(K(X'), \mathcal{O}^*)$ for a nonsingular irreducible scheme $X'$.

Now $\gamma$ comes from $\gamma' \in H^1(X' \setminus Z', \mathcal{O}^*) = \text{Pic}(X' \setminus Z')$. It is obvious that Picard elements lift to Picard elements by the blowing ups from the theorem. Thus from the diagram

$$H^1(X'', \mathcal{O}^*) \to H^1(X'' \setminus Z'', \mathcal{O}^*) \to H^2_{et}(X'', \mathcal{O}^*) \uparrow \uparrow$$

we conclude that $\gamma$ becomes trivial on $Z''$ by $X'' \to X'$ which extends $\gamma'$ to $X''$. □

Now let us return to the problem of an isomorphism $\text{Br}(X) \to H^2(X, \mathcal{O}^*)$ for
nonsingular quasi-projective varieties. The theorem reduces the general problem to the following

**QUESTION.** Let $X'$ be a blowing up of a nonsingular variety $X$ along a smooth subvariety $S$ and let $A'$ be an Azumaya algebra on $X'$. Does there exist an Azumaya algebra $A$ on $X$ such that its inverse image on $X'$ is equivalent to $A'$?

In case the restriction of $A'$ to the pre-image of $S$ is trivial, the question reduces to the one, whether a vector bundle on this preimage can be extended to $X$ as a vector bundle. For example, if $\dim(X) = 2$ then $S$ is a point and its proper preimage is a $\mathbb{P}^1$ with self-intersection $-1$. Since the map $\text{Pic}(X') \to \text{Pic}(\mathbb{P}^1)$ is surjective, any vector bundle on $\mathbb{P}^1$ can be extended to $X'$.

Therefore we obtain a simple proof of the basic theorem in the case $\dim(X) = 2$ using the birational theorem.

In the case of $\dim(X) = 3$ the same procedure reduces the basic problem to the analogous problem of extending vector bundles from $\mathbb{P}^2$ and ruled surfaces to a variety of dimension three.

**References**


