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2-Cocycles and Azumaya algebras under birational transformations of algebraic schemes

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The basic question whether the injection $\mathrm{Br}(X) \to H^2(X, \mathcal{O}_X^*)_{\mathrm{tors}}$ is an isomorphism arose at the very definition of the Brauer group of an algebraic scheme X. Positive answers are known in the following cases:

1. the topological Brauer group $\mathrm{Br}(X_{\mathrm{top}}) \cong H^2(X, \mathcal{O}_{\mathrm{top}}^*)_{\mathrm{tors}} \cong H^3(X, \mathbb{Z})_{\mathrm{tors}}$ (J.-P. Serre); in the etale (algebraic) case the isomorphism is proved for
2. smooth projective surfaces (A. Grothendieck);
3. abelian varieties;
4. the union of two affine schemes (R. Hoobler, O. Gabber).

The first author has formulated a birational variant of the basic question, while considering the unramified Brauer group in [1]. The group $\mathrm{Br}_V(K(X)) = \cap \mathrm{Br}(A_v) \subseteq \mathrm{Br}(K(X))$ (intersection taken over all discrete valuation subrings A_v of the rational function field $K(X)$) is isomorphic to $H^2(\tilde{X}, \mathcal{O}^*)$, where $\tilde{X}$ is a nonsingular projective model of X, i.e. a nonsingular projective variety birationally equivalent to X.

QUESTION. Given a cocycle class $\gamma \in H^2(\tilde{X}, \mathcal{O}^*)$, is it possible to find a nonsingular projective model $\bar{X}$ such that γ is represented by a $\mathbb{P}^n$-bundle (i.e. by an Azumaya algebra) on $\bar{X}$?

The case where X is a nonsingular projective model of V/G, with G a γ-minimal group and V a faithful representation of G, was considered in [2]. O. Gabber in his letter to Bogomolov (12.1.1988) has given an affirmative answer to the question in the case of general algebraic spaces. In this paper we give a simple version of his proof for algebraic schemes.

Let X be a scheme, $\gamma \in H^2(X, \mathcal{O}^*)$, $\{U_i\}$ an affine cover of X. Then the restriction of γ to each U_i is represented by an Azumaya algebra A_i. If we would have isomorphisms $A_{i|U_i \cap U_j} \cong A_{j|U_i \cap U_j}$, we could glue the sheaves $\{A_j\}$ and get an

Azumaya algebra on X, representing γ. But we have isomorphisms $A_{i|U_i \cap U_j} \otimes \operatorname{End}(E_{ij}) \cong A_{j|U_i \cap U_j} \otimes \operatorname{End}(E_{ji})$ for certain vector bundles E_{ij}, E_{ji} on $U_i \cap U_j$.

THEOREM. *Let X be a noetherian scheme, $\gamma \in H^2(X, \mathcal{O}_X^*)$. There exists a proper birational morphism $\alpha: \tilde{X} \to X$ such that $\alpha^*(\gamma)$ is represented by an Azumaya algebra on $\tilde{X}$.*

Proof. It is enough to consider X which are connected. Suppose that $\{U_i\}$ is an affine open cover of X and that γ is non-trivial on at least one U_i. We will construct an Azumaya algebra on a birational model of X by an inductive process which involves adjoining one by one proper preimages of the subsets U_i and, by an appropriate birational change of the scheme and Azumaya algebra obtained, extending the new algebra to the union. We start with some affine open subset U_0 and an Azumaya algebra A_0 on it.

Now suppose by induction that we already have an Azumaya algebra $\bar{A}_k$ on the scheme X_k, a Zariski-open subset of the scheme $\bar{X}_k$, equipped with a proper birational map $\bar{\alpha}_k: \bar{X}_k \to X$ such that $X_k = \bar{\alpha}_k^{-1}(U_0 \cup \cdots \cup U_k)$. Let U_{k+1} intersect $U_0 \cup \cdots \cup U_k$ and $\bar{U}_{k+1} = \bar{\alpha}^{-1}(U_{k+1})$. Suppose that on U_{k+1}, γ is represented by the Azumaya algebra A_{k+1}. In the same vein as above we have an isomorphism

$$\bar{A}_{k|X_k \cap \bar{U}_{k+1}} \otimes \operatorname{End}(E_{k,k+1}) \cong \bar{\alpha}_k^*(A_{k+1})|_{X_k \cap \bar{U}_{k+1}} \otimes \operatorname{End}(E_{k+1,k})$$

and we need to extend $E_{k,k+1}$ to X_k and $E_{k+1,k}$ to $\bar{U}_{k+1}$ from their intersection. After this we will change $\bar{A}_k$ and $\bar{\alpha}_k^*(A_{k+1})$ by the other representatives $\bar{A}_k \otimes \operatorname{End}(E_{k,k+1})$, $\bar{\alpha}_k^*(A_{k+1}) \otimes \operatorname{End}(E_{k+1,k})$ of the same Brauer classes and glue these Azumaya algebras, hence the proof.

First, extend both sheaves E as coherent sheaves. This can be done by the following

LEMMA. *Let X be a noetherian scheme, $U \subseteq X$ a Zariski-open subset, E a coherent sheaf on U. Then there exists a coherent sheaf E' on X such that $E'|_U \cong E$.* This is Ex. II.5.15 in [4].

Note that we can assume that in our inductive process we add neighborhoods U_{k+1} of no more than one irreducible component (or an intersection of irreducible components) of X, different from those contained in X_k. Thus we assume $X_k \cap U_{k+1}$ to be connected and the rank of E to be constant on $X_k \cap U_{k+1}$, hence E' will be locally generated by n elements, where n is the rank of E.

LEMMA (see [3], Lemma 3.5). *Let X be a noetherian scheme, E a coherent sheaf on X, locally free outside a Zariski closed subset Z on X. Then there exists a coherent sheaf I of ideals on X such that the support of $\mathcal{O}_X/I$ is Z with the following* *property. Let $\alpha: \tilde{X} \to X$ be the blowing up of X with center I, then the sheaf $\bar{\alpha}(E) := $ the quotient of $\alpha^*(E)$ by the subsheaf of sections with support in $\alpha^{-1}(Z)$, is locally free on $\tilde{X}$.*

Proof. The proof consists of two parts. First: to reduce the number of local generators to get this number constant on the connected components of X (the minima are the values of the (local) rank function of E). Second, to force the kernel of the (local) presentations $\mathcal{O}_V^m \to E|_V \to 0$ to vanish for all neighborhoods from some cover $\{V\}$. Both parts are proved by indicating the suitable coherent sheaves of ideals and blowing up X with respect to these sheaves. Let $\mathcal{O}_V^m \xrightarrow{f} E|_V - 0$ be a local presentation of E. Then $\operatorname{Ker}(f)$ is generated by all relations $\sum_{i=1}^m c_i a_i = 0$ where $\{a_i\}$ stand for the free basis of $\mathcal{O}_V^m$. The coherent sheaf of ideals in the first case is the sheaf defined locally as the ideal I_V in $\mathcal{O}_V$ generated by all c_i such that $\sum_{i=1}^m c_i a_i \in \operatorname{Ker}(f)$ and in the second case as $J_X = \operatorname{Ann}(\operatorname{Ker}(f))$. As the number of generators is constant in the case we are interested in, we give the details only for the second part of the proof and refer to [3] for the first.

Let $\alpha: X' \to X$ be the blowing up of X with respect to J_X and let $\bar{\alpha}(E)$ be as in the statement of the Lemma. Let

$$0 \to (\operatorname{Ker}(\bar{f}))|_{V'} \to \mathcal{O}_{V'}^m \xrightarrow{\bar{f}} \bar{\alpha}(E)|_{V'} \to 0$$

be the local presentation of $\bar{\alpha}(E)$. We have $\alpha^{-1}(\operatorname{Ann}(f)) \subseteq \operatorname{Ann}(\operatorname{Ker}(\bar{f}))$. Let $\mathfrak{p} \in Z'$, $V' = \operatorname{Spec}(A')$ an affine neighborhood of $\mathfrak{p}$ in X' and let $\sum_{i=1}^m c_i a_i \in \operatorname{Ker}(\bar{f})|_{V'}$ map to a nonzero element in $\operatorname{Ker}(\bar{f})_{\mathfrak{p}}$. Denote by γ a generator of the invertible sheaf $\alpha^{-1}(\operatorname{Ann}(f))$ on $V'' = \operatorname{Spec}(A'') \subseteq V'$ for suitable A''. It is clear that there exists for given $\mathfrak{p}$ and V'' a finite sequence of open affine neighborhoods $V_1'', \dots, V_s''$ such that $X' \backslash Z' = V_1''$, $V'' = V_s''$ and $V_j'' \cap V_{j+1}'' \neq \emptyset$ for $j = 1, \dots, s-1$. So suppose $V' \cap (X' \backslash Z') \neq \emptyset$ and $\mathfrak{q} \in V'' \cap (X \backslash Z)$. Then $(c_i)_{\mathfrak{q}} = 0$ for $i = 1, \dots, m$ and $\mathfrak{q} \in \operatorname{Spec}(A_\gamma'')$ hence $\gamma^k c_i = 0$ for $i = 1, \dots, m$ for some k. Since γ is not a zero divisor, we conclude that $c_i = 0$ for $i = 1, \dots, m$. Thus (maybe after considering a finite sequence of points $\mathfrak{q}_1, \dots, \mathfrak{q}_s$) we prove that $(\operatorname{Ker}(f))_{\mathfrak{p}}$ is trivial for every $\mathfrak{p} \in X'$. □

In this way we glue the two sheaves $\bar{A}_k$ and A_{k+1} and get an Azumaya algebra on $\tilde{X}_k \cup \tilde{U}_{k+1}$. As the scheme X is quasi-compact, we obtain an Azumaya algebra on $\tilde{X}$ after a finite number of such steps.

Now we have to show that this process can be done in such a way that the class $[A]$ of the Azumaya algebra A constructed in this way is equal to $\bar{\alpha}^*(\gamma)$. Again this goes by induction on k. We have $X_{k+1} = U \cup V$ with $U = \bar{\alpha}_{k+1}^{-1}(U_0 \cup \cdots \cup U_k)$ and $V = \bar{\alpha}_{k+1}^{-1}(U_{k+1})$. We have the exact sequence

$$H^1(U \cap V, \mathcal{O}^*) \to H^2(X_{k+1}, \mathcal{O}^*) \to H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$$

and by induction hypothesis, $\bar{\alpha}^*_{k+1}(\gamma) - [\bar{A}_{k+1}]$ maps to zero in $H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$ so it comes from $\beta \in H^1(U \cap V, \mathcal{O}^*)$. By blowing up X_{k+1} we may assume that β is represented by a line bundle which extends to U. Then β maps to zero in $H^2(X_{k+1}, \mathcal{O}^*)$, hence $\bar{\alpha}^*_{k+1}(\gamma) - [\bar{A}_{k+1}] = 0$. □

Note that we need not bother about the compatibility of isomorphisms, because at each step we choose a new isomorphism between the Azumaya algebra A on $U_1 \cup \cdots \cup U_j$ from the preceding step and A_k on U_k, modulo $\mathrm{End}(E)$, $\mathrm{End}(E_k)$.

COROLLARY 1. *Let G be a finite group, V a faithful complex representation of G. Then there exists a nonsingular projective model X of V/G such that* $\mathrm{Br}(X) = H^2(X, \mathcal{O}^*)$.

Proof. The group $H^2(X, \mathcal{O}^*)$ is a birational invariant of nonsingular projective varieties and is isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$ if X is a model of V/G (see [1]). It remains to recall that the group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is finite. □

COROLLARY 2. *Let X be a noetherian scheme over $\mathbb{C}$, Z a closed subscheme of X and $\gamma \in H^2_Z(X, \mathcal{O}^*)$. Then there exists a proper morphism $\alpha\colon X' \to X$ which is an isomorphism above $X \backslash Z$ and maps γ to zero in $H^2_{\alpha^{-1}(Z)}(X', \mathcal{O}^*)$.*

Proof. First, let's have $\alpha(\gamma)$ map to zero in $H^2(X, \mathcal{O}^*)$. To do this, desingularize X by $X' \to X$. Then in the following exact sequence (in etale cohomology), β will be injective:

$$\begin{array}{ccccccc}
H^1(X'\backslash Z', \mathcal{O}^*) & \to & H^2_{Z'}(X', \mathcal{O}^*) & \to & H^2(X', \mathcal{O}^*) & \xrightarrow{\beta} & H^2(X'\backslash Z', \mathcal{O}^*) \\
 & & & & \uparrow & & \\
 & & & & H^2_Z(X, \mathcal{O}^*) & &
\end{array}$$

The injectivity is due to the injectivity of $H^2(X', \mathcal{O}^*) \to H^2(K(X'), \mathcal{O}^*)$ for a nonsingular irreducible scheme X'.

Now γ comes from $\gamma' \in H^1(X'\backslash Z', \mathcal{O}^*) = \mathrm{Pic}(X'\backslash Z')$. It is obvious that Picard elements lift to Picard elements by the blowing ups from the theorem. Thus from the diagram

$$\begin{array}{ccccc}
H^1(X'', \mathcal{O}^*) \to & H^1(X''\backslash Z'', \mathcal{O}^*) & \to & H^2_{Z''}(X'', \mathcal{O}^*) \\
 & \uparrow & & \uparrow \\
 & H^1(X'\backslash Z', \mathcal{O}^*) & \to & H^2_{Z'}(X', \mathcal{O}^*)
\end{array}$$

we conclude that γ becomes trivial on Z'' by $X'' \to X'$ which extends γ' to X''. □

Now let us return to the problem of an isomorphism $\mathrm{Br}(X) \to H^2(X, \mathcal{O}^*)$ for nonsingular quasi-projective varieties. The theorem reduces the general problem to the following

QUESTION. Let X' be a blowing up of a nonsingular variety X along a smooth subvariety S and let A' be an Azumaya algebra on X'. Does there exist an Azumaya algebra A on X such that its inverse image on X' is equivalent to A'?

In case the restriction of A' to the pre-image of S is trivial, the question reduces to the one, whether a vector bundle on this preimage can be extended to X as a vector bundle. For example, if $\dim(X) = 2$ then S is a point and its proper preimage is a $\mathbb{P}^1$ with self-intersection -1. Since the map $\mathrm{Pic}(X') \to \mathrm{Pic}(\mathbb{P}^1)$ is surjective, any vector bundle on $\mathbb{P}^1$ can be extended to X'.

Therefore we obtain a simple proof of the basic theorem in the case $\dim(X) = 2$ using the birational theorem.

In the case of $\dim(X) = 3$ the same procedure reduces the basic problem to the analogous problem of extending vector bundles from $\mathbb{P}^2$ and ruled surfaces to a variety of dimension three.

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