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2-Cocycles and Azumaya algebras under birational transformations of algebraic schemes

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The basic question whether the injection \( \text{Br}(X) \rightarrow H^2(X, O^*_X)_{\text{tors}} \) is an isomorphism arose at the very definition of the Brauer group of an algebraic scheme \( X \). Positive answers are known in the following cases:

1. the topological Brauer group \( \text{Br}(X_{\text{top}}) \cong H^2(X, O^*_{\text{top}})_{\text{tors}} \cong H^3(X, \mathbb{Z})_{\text{tors}} \) (J.-P. Serre); in the etale (algebraic) case the isomorphism is proved for
2. smooth projective surfaces (A. Grothendieck);
3. abelian varieties;
4. the union of two affine schemes (R. Hoobler, O. Gabber).

The first author has formulated a birational variant of the basic question, while considering the unramified Brauer group in [1]. The group \( \text{Br}_r(K(X)) = \bigcap \text{Br}(A_v) \cong \text{Br}(K(X)) \) (intersection taken over all discrete valuation subrings \( A_v \)) of the rational function field \( K(X) \) is isomorphic to \( H^2(\mathcal{X}, O^*) \), where \( \mathcal{X} \) is a nonsingular projective model of \( X \), i.e. a nonsingular projective variety birationally equivalent to \( X \).

QUESTION. Given a cocycle class \( \gamma \in H^2(\mathcal{X}, O^*) \), is it possible to find a nonsingular projective model \( \mathcal{X} \) such that \( \gamma \) is represented by a \( \mathbb{P}^n \)-bundle (i.e. by an Azumaya algebra) on \( \mathcal{X} \)?

The case where \( X \) is a nonsingular projective model of \( V/G \), with \( G \) a \( \gamma \)-minimal group and \( V \) a faithful representation of \( G \), was considered in [2]. O. Gabber in his letter to Bogomolov (12.1.1988) has given an affirmative answer to the question in the case of general algebraic spaces. In this paper we give a simple version of his proof for algebraic schemes.

Let \( X \) be a scheme, \( \gamma \in H^2(X, O^*) \), \( \{U_i\} \) an affine cover of \( X \). Then the restriction of \( \gamma \) to each \( U_i \) is represented by an Azumaya algebra \( A_i \). If we would have isomorphisms \( A_i|_{U_i \cap U_j} \cong A_j|_{U_i \cap U_j} \), we could glue the sheaves \( \{A_j\} \) and get an
Azumaya algebra on \( X \), representing \( y \). But we have isomorphisms \( A_{i|U_i \cap U_j} \otimes \text{End}(E_{ij}) \cong A_{j|U_i \cap U_j} \otimes \text{End}(E_{ji}) \) for certain vector bundles \( E_{ij}, E_{ji} \) on \( U_i \cap U_j \).

**THEOREM.** Let \( X \) be a noetherian scheme, \( y \in H^2(X, \mathcal{O}_X^*) \). There exists a proper birational morphism \( \alpha: \tilde{X} \to X \) such that \( \alpha^*(y) \) is represented by an Azumaya algebra on \( \tilde{X} \).

**Proof.** It is enough to consider \( X \) which are connected. Suppose that \( \{U_i\} \) is an affine open cover of \( X \) and that \( y \) is non-trivial on at least one \( U_i \). We will construct an Azumaya algebra on a birational model of \( X \) by an inductive process which involves adjoining one by one proper preimages of the subsets \( U_i \) and, by an appropriate birational change of the scheme and Azumaya algebra obtained, extending the new algebra to the union. We start with some affine open subset \( U_0 \) and an Azumaya algebra \( A_0 \) on it.

Now suppose by induction that we already have an Azumaya algebra \( \tilde{A}_k \) on the scheme \( X_k \), a Zariski-open subset of the scheme \( \tilde{X}_k \), equipped with a proper birational map \( \tilde{\alpha}_k: \tilde{X}_k \to X \) such that \( X_k = \tilde{\alpha}_k^{-1}(U_0 \cup \cdots \cup U_k) \). Let \( U_{k+1} \) intersect \( U_0 \cup \cdots \cup U_k \) and \( \tilde{U}_{k+1} = \tilde{\alpha}_k^{-1}(U_{k+1}) \). Suppose that on \( U_{k+1} \), \( y \) is represented by the Azumaya algebra \( A_{k+1} \). In the same vein as above we have an isomorphism

\[
\tilde{A}_{k|X_k \cap U_{k+1}} \otimes \text{End}(E_{k,k+1}) \cong \tilde{\alpha}_k^*(A_{k+1})_{|X_k \cap U_{k+1}} \otimes \text{End}(E_{k+1,k})
\]

and we need to extend \( E_{k,k+1} \) to \( X_k \) and \( E_{k+1,k} \) to \( \tilde{U}_{k+1} \) from their intersection. After this we will change \( \tilde{A}_k \) and \( \tilde{\alpha}_k^*(A_{k+1}) \) by the other representatives \( \tilde{A}_k \otimes \text{End}(E_{k,k+1}), \tilde{\alpha}_k^*(A_{k+1}) \otimes \text{End}(E_{k+1,k}) \) of the same Brauer classes and glue these Azumaya algebras, hence the proof.

First, extend both sheaves \( E \) as coherent sheaves. This can be done by the following

**LEMMA.** Let \( X \) be a noetherian scheme, \( U \subseteq X \) a Zariski-open subset, \( E \) a coherent sheaf on \( U \). Then there exists a coherent sheaf \( E' \) on \( X \) such that \( E'|_U \cong E \). This is Ex. II.5.15 in [4].

Note that we can assume that in our inductive process we add neighborhoods \( U_{k+1} \) of no more than one irreducible component (or an intersection of irreducible components) of \( X \), different from those contained in \( X_k \). Thus we assume \( X_k \cap U_{k+1} \) to be connected and the rank of \( E \) to be constant on \( X_k \cap U_{k+1} \), hence \( E' \) will be locally generated by \( n \) elements, where \( n \) is the rank of \( E \).

**LEMMA** (see [3], Lemma 3.5). Let \( X \) be a noetherian scheme, \( E \) a coherent sheaf on \( X \), locally free outside a Zariski closed subset \( Z \) on \( X \). Then there exists a coherent sheaf \( I \) of ideals on \( X \) such that the support of \( \mathcal{O}_X/I \) is \( Z \) with the following
property. Let $\alpha : \tilde{X} \to X$ be the blowing up of $X$ with center $I$, then the sheaf $\bar{\alpha}(E) := \text{the quotient of } \alpha^*(E) \text{ by the subsheaf of sections with support in } \alpha^{-1}(Z)$, is locally free on $\tilde{X}$.

Proof. The proof consists of two parts. First: to reduce the number of local generators to get this number constant on the connected components of $X$ (the minima are the values of the (local) rank function of $E$). Second, to force the kernel of the (local) presentations $\mathcal{O}_v^\mathcal{P} \to E|_v \to 0$ to vanish for all neighborhoods from some cover $\{V\}$. Both parts are proved by indicating the suitable coherent sheaves of ideals and blowing up $X$ with respect to these sheaves. Let $\mathcal{O}_v^\mathcal{P} \to E|_v \to 0$ be a local presentation of $E$. Then $\text{Ker}(f)$ is generated by all relations $\Sigma_{i=1}^m c_i a_i = 0$ where $\{a_i\}$ stand for the free basis of $\mathcal{O}_v^\mathcal{P}$. The coherent sheaf of ideals in the first case is the sheaf defined locally as the ideal $I_v$ in $\mathcal{O}_v^\mathcal{P}$ generated by all $c_i$ such that $\Sigma_{i=1}^m c_i a_i \in \text{Ker}(f)$ and in the second case as $J_X = \text{Ann}(\text{Ker}(f))$. As the number of generators is constant in the case we are interested in, we give the details only for the second part of the proof and refer to [3] for the first.

Let $\alpha : X' \to X$ be the blowing up of $X$ with respect to $J_X$ and let $\bar{\alpha}(E)$ be as in the statement of the Lemma. Let

$$0 \to (\text{Ker}(f))|_v \to \mathcal{O}_v^\mathcal{P} \xrightarrow{f} \bar{\alpha}(E)|_v \to 0$$

be the local presentation of $\bar{\alpha}(E)$. We have $\alpha^{-1}(\text{Ann}(f)) \subseteq \text{Ann}(\text{Ker}(f))$. Let $p \in Z'$, $V' = \text{Spec}(A')$ an affine neighborhood of $p$ in $X'$ and let $\Sigma_{i=1}^m c_i a_i \in \text{Ker}(f)|_v$ map to a nonzero element in $\text{Ker}(f)_p$. Denote by $\gamma$ a generator of the invertible sheaf $\alpha^{-1}(\text{Ann}(f))$ on $V'' = \text{Spec}(A'') \subseteq V'$ for suitable $A''$. It is clear that there exists for given $p$ and $V''$ a finite sequence of open affine neighborhoods $V'_1, \ldots, V'_s$ such that $X' \setminus Z' = V'_1, V'' = V'_{s}$ and $V'_j \cap V'_{j+1} = \emptyset$ for $j = 1, \ldots, s - 1$. So suppose $V' \cap (X' \setminus Z') \neq \emptyset$ and $q \in V'' \cap (X' \setminus Z)$. Then $(c_i)_{\alpha} = 0$ for $i = 1, \ldots, m$ and $q \in V'' \cap (X' \setminus Z)$. Then $\gamma^k c_i = 0$ for $i = 1, \ldots, m$ for some $k$. Since $\gamma$ is not a zero divisor, we conclude that $c_i = 0$ for $i = 1, \ldots, m$. Thus (maybe after considering a finite sequence of points $q_1, \ldots, q_s$) we prove that $(\text{Ker}(f))_p$ is trivial for every $p \in X'$.

In this way we glue the two sheaves $\tilde{A}_k$ and $A_{k+1}$ and get an Azumaya algebra on $\tilde{X}_k \cup \tilde{U}_{k+1}$. As the scheme $X$ is quasi-compact, we obtain an Azumaya algebra on $\tilde{X}$ after a finite number of such steps.

Now we have to show that this process can be done in such a way that the class $[A]$ of the Azumaya algebra $A$ constructed in this way is equal to $\bar{\alpha}^*(\gamma)$. Again this goes by induction on $k$. We have $X_{k+1} = U \cup V$ with $U = \tilde{\alpha}_{k+1}^{-1}(U_0 \cup \cdots \cup U_k)$ and $V = \tilde{\alpha}_{k+1}^{-1}(U_{k+1})$. We have the exact sequence

$$H^1(U \cap V, \mathcal{O}^\ast) \to H^2(X_{k+1}, \mathcal{O}^\ast) \to H^2(U, \mathcal{O}^\ast) \oplus H^2(V, \mathcal{O}^\ast)$$
and by induction hypothesis, $\alpha_{k+1}(\gamma) - [A_{k+1}]$ maps to zero in $H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$ so it comes from $\beta \in H^1(U \cap V, \mathcal{O}^*)$. By blowing up $X_{k+1}$ we may assume that $\beta$ is represented by a line bundle which extends to $U$. Then $\beta$ maps to zero in $H^2(X_{k+1}, \mathcal{O}^*)$, hence $\alpha_{k+1}(\gamma) - [A_{k+1}] = 0$.

Note that we need not bother about the compatibility of isomorphisms, because at each step we choose a new isomorphism between the Azumaya algebra $A$ on $U_1 \cup \cdots \cup U_j$ from the preceding step and $A_k$ on $U_k$, modulo $\text{End}(E)$, $\text{End}(E_k)$.

**COROLLARY 1.** Let $G$ be a finite group, $V$ a faithful complex representation of $G$. Then there exists a nonsingular projective model $X$ of $V/G$ such that $\text{Br}(X) = H^2(X, \mathcal{O}^*)$.

*Proof.* The group $H^2(X, \mathcal{O}^*)$ is a birational invariant of nonsingular projective varieties and is isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$ if $X$ is a model of $V/G$ (see [1]). It remains to recall that the group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is finite. \qed

**COROLLARY 2.** Let $X$ be a noetherian scheme over $\mathbb{C}$, $Z$ a closed subscheme of $X$ and $\gamma \in H^2_Z(X, \mathcal{O}^*)$. Then there exists a proper morphism $\alpha : X' \to X$ which is an isomorphism above $X \setminus Z$ and maps $\gamma$ to zero in $H^2_{Z-1}(X', \mathcal{O}^*)$.

*Proof.* First, let’s have $\alpha(\gamma)$ map to zero in $H^2(X, \mathcal{O}^*)$. To do this, desingularize $X$ by $X' \to X$. Then in the following exact sequence (in etale cohomology), $\beta$ will be injective:

$$
\begin{array}{ccc}
H^1(X' \setminus Z', \mathcal{O}^*) & \to & H^2_Z(X', \mathcal{O}^*) \\
\uparrow & & \uparrow \\
H^2(X, \mathcal{O}^*) & \to & H^2(X', \mathcal{O}^*)
\end{array}
$$

The injectivity is due to the injectivity of $H^2(X', \mathcal{O}^*) \to H^2(K(X'), \mathcal{O}^*)$ for a nonsingular irreducible scheme $X'$.

Now $\gamma$ comes from $\gamma' \in H^1(X' \setminus Z', \mathcal{O}^*) = \text{Pic}(X' \setminus Z')$. It is obvious that Picard elements lift to Picard elements by the blowing ups from the theorem. Thus from the diagram

$$
\begin{array}{ccc}
H^1(X^\prime, \mathcal{O}^*) & \to & H^1(X'' \setminus Z'', \mathcal{O}^*) \\
\uparrow & & \uparrow \\
H^1(X' \setminus Z', \mathcal{O}^*) & \to & H^2_Z(X', \mathcal{O}^*)
\end{array}
$$

we conclude that $\gamma$ becomes trivial on $Z''$ by $X'' \to X'$ which extends $\gamma'$ to $X''$. \qed

Now let us return to the problem of an isomorphism $\text{Br}(X) \to H^2(X, \mathcal{O}^*)$ for
nonsingular quasi-projective varieties. The theorem reduces the general problem to the following

**QUESTION.** Let $X'$ be a blowing up of a nonsingular variety $X$ along a smooth subvariety $S$ and let $A'$ be an Azumaya algebra on $X'$. Does there exist an Azumaya algebra $A$ on $X$ such that its inverse image on $X'$ is equivalent to $A'$?

In case the restriction of $A'$ to the pre-image of $S$ is trivial, the question reduces to the one, whether a vector bundle on this preimage can be extended to $X$ as a vector bundle. For example, if $\dim(X) = 2$ then $S$ is a point and its proper preimage is a $\mathbb{P}^1$ with self-intersection $-1$. Since the map $\text{Pic}(X') \to \text{Pic}(\mathbb{P}^1)$ is surjective, any vector bundle on $\mathbb{P}^1$ can be extended to $X'$.

Therefore we obtain a simple proof of the basic theorem in the case $\dim(X) = 2$ using the birational theorem.

In the case of $\dim(X) = 3$ the same procedure reduces the basic problem to the analogous problem of extending vector bundles from $\mathbb{P}^2$ and ruled surfaces to a variety of dimension three.

**References**


