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## On the moduli of curves with theta-characteristics

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### 0. Introduction

Let  $C$  be a smooth complete connected curve of genus  $g$  (i.e.  $C$  is a compact connected Riemann surface of genus  $g$ ) over the field of complex numbers  $\mathbf{C}$ . We denote by  $K_C$  the canonical line bundle of  $C$ .

DEFINITION. A line bundle  $\mathcal{L}$  is called a *theta-characteristic* on  $C$  if  $\mathcal{L}^2 := \mathcal{L} \otimes \mathcal{L} \simeq K_C$ .

If  $\mathcal{L}$  is a line bundle on  $C$  then ‘deg  $\mathcal{L}$ ’ denotes its degree. Since  $\deg K_C = 2g - 2$ , we have  $\deg \mathcal{L} = g - 1$  for any theta-characteristic  $\mathcal{L}$  on  $C$ . If  $g \geq 1$ , then  $\text{Pic}(C) \simeq \text{Pic}^0(C) \times \mathbf{Z}$  (where  $\text{Pic}(C)$  is the Picard group of  $C$ ;  $\text{Pic}^0(C)$  is the identity component of  $\text{Pic}(C)$ ) and  $\text{Pic}^0(C)$  is a complex torus of dimension  $g$ . Thus if  $g \geq 1$ , then there are  $2^{2g}$  theta-characteristics on  $C$ .

DEFINITION. Let  $\mathcal{L}$  be a line bundle on  $C$ . Then  $\mathcal{L}$  is said to be even (resp. odd) theta-characteristic if  $h^0(\mathcal{L}) := \dim_{\mathbf{C}} H^0(C, \mathcal{L})$  is even (resp. odd).

Among the  $2^{2g}$  theta-characteristics on  $C$ ,  $2^{g-1}(2^g + 1)$  (resp.  $2^{g-1}(2^g - 1)$ ) of them are even (resp. odd) [see [M] p. 190].

It follows from a theorem of Mumford [see [M] p. 184] and the fact about the monodromy action [see [ACGH] p. 294] that the moduli of curves with theta-characteristics (i.e., the variety parametrizing isomorphism classes of  $(C, \mathcal{L})$ ,  $C$  a curve of genus  $g$  as above and  $\mathcal{L}$  a theta-characteristic on  $C$ ) has exactly two connected components  $\mathcal{M}_g^+$  and  $\mathcal{M}_g^-$  corresponding to even and odd theta-characteristics respectively. If  $\mathcal{M}_g$  denotes the moduli of genus  $g$  curves, then we have covering projections  $\mathcal{M}_g^+ \rightarrow \mathcal{M}_g$  and  $\mathcal{M}_g^- \rightarrow \mathcal{M}_g$  of degree  $2^{g-1}(2^g + 1)$  and  $2^{g-1}(2^g - 1)$  respectively.

Let  $\mathcal{M}_g^r \subset \mathcal{M}_g$  be the closure of the locus of all curves  $C$ , such that on  $C$  there is a theta-characteristic  $\mathcal{L}$  with  $h^0(\mathcal{L}) = r$ , with its natural subscheme structure (see §1). It follows that  $\mathcal{M}_g^r$  is the locus of the curves  $C$  possessing a theta-characteristic  $\mathcal{L}$  with  $h^0(\mathcal{L}) \geq r$  and  $h^0(\mathcal{L}) \equiv r(2)$ . Note if  $r$  is 0 or 1  $\mathcal{M}_g^r = \mathcal{M}_g$ . In this note, following a suggestion of M. V. Nori, we give a method to compute Zariski tangent spaces to  $\mathcal{M}_g^r$  in the moduli-stack. Using the above method we find the dimension of the Zariski tangent spaces to  $\mathcal{M}_g^r$  at a hyperelliptic curve.

We give an example of a  $\mathcal{M}_g^r$  which is not reduced as a scheme. Also we give example of  $\mathcal{M}_g^r$  which is not irreducible.

We refer to Teixidor I Bigas. M. [T], for a detailed study of the above moduli for small  $r$ .

It is a great pleasure for me to thank Prof. Madhav. V. Nori, for his help and constant encouragement.

**1. Method to compute Zariski tangent space**

For the standard facts about moduli [see [T] p. 100]. Let  $C$  be a curve of genus  $g$  as in the introduction. Let  $U$  be a neighbourhood of  $[C]$  in a suitable cover of the moduli space  $\mathcal{M}_g$  of genus  $g$  curves. Note that the tangent space to  $U$  at  $[C]$  can be identified with  $H^1(C, T_C)$ , where  $T_C$  is the tangent bundle of  $C$ . Let  $\mathbf{C}$  be the corresponding universal curve over  $U$ , i.e., we have a proper smooth morphism

$$\pi: \mathbf{C} \rightarrow U,$$

such that for each point  $x \in U$ ,  $C_x := \pi^{-1}(x)$  is a smooth curve of genus  $g$  with suitable universal properties. Let  $L$  be a line bundle on  $\mathbf{C}$  such that for  $x \in U$ ,  $L_x := L|_{\pi^{-1}(x)}$  is a theta-characteristic on  $C_x$ . Since  $\deg L_x = g - 1$ , by Riemann-Roch theorem [see [H] p. 295] we see that  $h^0(L_x) = h^1(L_x)$ , where  $h^0(L_x) := \dim_{\mathbf{C}} H^0(C_x, L_x)$  and  $h^1(L_x) := \dim_{\mathbf{C}} H^1(C_x, L_x)$ . Set  $\mathcal{L} = L|_{[C]}$ , where  $[C]$  is the point of  $U$  corresponding  $C$ . If  $h^0(\mathcal{L}) = r$ , then by using semi-continuity theorem [see [H] p. 281–291] we get a morphism (changing  $U$  by a suitable neighbourhood of  $[C]$ , if necessary)

$$\theta: U \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L})) \simeq M_r(\mathbf{C}),$$

such that the scheme-theoretic inverse image of the origin is the locus of curves in  $U$  corresponding to  $\mathcal{M}_g^r$  defined in the introduction. We are interested in the tangent space mapping

$$\Theta: H^1(C, T_C) \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L})),$$

of  $\theta$  at  $[C] \in U$ .

First note that by Serre’s duality theorem [See [H] p. 295]  $H^0(C, \mathcal{L})$  (resp.  $H^0(C, K_C^2)$ ), where  $K_C^2 := K_C \otimes K_C$  is naturally dual to  $H^1(C, \mathcal{L})$  (resp.  $H^1(C, T_C)$ ).

**THEOREM 1.** *The mapping*

$$\Theta^\vee: H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}) \rightarrow H^0(C, K_C^2),$$

defined by

$$(fe_1, ge_1) \mapsto (fdg - gdf)e_1^2,$$

(where  $fe_1, ge_1 \in H^0(C, \mathcal{L})$ ), is dual to  $\Theta$  (up to a scalar multiple).

*Proof.* Let  $t \in H^1(C, T_C)$ . First we describe the homomorphism

$$\Theta(t): H^0(C, \mathcal{L}) \rightarrow H^1(C, \mathcal{L}).$$

Choose an affine covering  $\{U_1, U_2\}$  of  $C$  such that

$$\mathcal{L}(U_1) \simeq \mathcal{O}_C(U_1)e_1 \quad \text{and} \quad \mathcal{L}(U_2) \simeq \mathcal{O}_C(U_2)e_2$$

and also we have

$$\begin{aligned} K_C(U_1) &\simeq \mathcal{O}_C(U_1)e_1^2 = \mathcal{O}_C(U_1)h \, da \\ K_C(U_2) &\simeq \mathcal{O}_C(U_2)e_2^2 = \mathcal{O}_C(U_2)h_1 \, db, \end{aligned}$$

where  $\mathcal{O}_C$  is the structure sheaf of  $C$  and  $a, b, h, h_1$  are rational functions on  $C$ . If  $\alpha_{12} \in \mathcal{O}_C(U_1 \cap U_2)^*$  is the transition function of  $\mathcal{L}$ , then by our assumption on  $\mathcal{L}$ ,  $\alpha_{12}^2$  is the transition function for  $K_C$ , where  $\mathcal{O}_C(U_1 \cap U_2)^*$  is the group of invertible elements of  $\mathcal{O}_C(U_1 \cap U_2)$ . Now  $t \in H^1(C, T_C)$  gives an infinitesimal deformation  $C_t[\varepsilon]$  as follows: let  $D: \mathcal{O}_C(U_1 \cap U_2) \rightarrow \mathcal{O}_C(U_1 \cap U_2)$  be the derivation corresponding to 't', then  $C_t[\varepsilon]$  is defined by gluing

$$\text{Spec} \left( \mathcal{O}_C(U_1) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)} \right) \quad \text{and} \quad \text{Spec} \left( \mathcal{O}_C(U_2) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)} \right)$$

along  $\text{Spec}(\mathcal{O}_C(U_1 \cap U_2) \otimes \mathbf{C}[\varepsilon]/(\varepsilon^2))$  by the function

$$f \mapsto f + \varepsilon D(f).$$

If  $K_{C_t[\varepsilon]}$  is the relative cononical bundle of  $C_t[\varepsilon] \rightarrow \text{Spec}(\mathbf{C}[\varepsilon]/(\varepsilon^2))$ , then it is easy to verify that  $K_{C_t[\varepsilon]}$  is given by the transition function

$$\alpha_{12}^2 \left( 1 + \varepsilon \left( \frac{d(D(a))}{da} + \frac{D(h)}{h} \right) \right) \in \left( \mathcal{O}_C(U_1 \cap U_2) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)} \right)^*.$$

Then

$$\alpha_{12} \left( 1 + \frac{1}{2} \varepsilon \left( \frac{d(D(a))}{da} + \frac{D(h)}{h} \right) \right)$$

gives transition function for a line bundle  $\mathcal{L}_1$  on  $C_t[\varepsilon]$  such that  $\mathcal{L}_1^2 \simeq K_{C_t[\varepsilon]}$  and  $\mathcal{L}_1|_C \simeq \mathcal{L}$ . Also on  $C_t[\varepsilon]$  we have an exact sequence

$$0 \rightarrow \varepsilon\mathcal{L} \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L} \rightarrow 0.$$

From this exact sequence we get a coboundary homomorphism

$$\psi: H^0(C, \mathcal{L}) \rightarrow H^1(C, \mathcal{L}).$$

Using Čech-cohomology with respect to the covering  $\{U_1, U_2\}$ , we get

$$\psi(fe_1) = \left( \frac{f}{2} \left( \frac{d(D(a))}{da} + \frac{D(h)}{h} \right) + D(f) \right) e_1,$$

where  $fe_1 \in H^0(C, \mathcal{L})$ . But  $\Theta(t)$  is nothing but  $\psi$ . Note that if  $fe_1, ge_1 \in H^0(C, \mathcal{L})$  then cup product of  $\psi(fe_1)$  and  $fe_1$  gives an element

$$\psi(fe_1)ge_1 \in H^1(C, K_C).$$

But

$$\begin{aligned} \psi(fe_1)ge_1 &= \left( \frac{gf}{2} \left( \frac{d(D(a))}{da} + \frac{D(h)}{h} \right) + D(f)g \right) e_1^2 \\ &= \frac{fgh}{2} d(D(a)) + \frac{gfD(h)}{2} da + (ghD(f)) da \\ &= \frac{fgh}{2} d(D(a)) + \frac{fgD(a)}{2} dh + ghD(a) df \\ &= \frac{D(a)h}{2} (f dg - g df) + \frac{1}{2}d(fghD(a)) \end{aligned}$$

(In the above equation we have used the fact  $D(h)da = D(a)dh$  and  $D(a)df = D(f)da$ ). So if  $p \in C - U_1$ , then

$$\text{res}_p(\psi(fe_1)ge_1) = \text{res}_p \left( \frac{D(a)h}{2} (f df - g dg) \right).$$

On the other hand the derivation  $D$  corresponding to  $t \in H^1(C, T_C)$  induces (by cup product) a homomorphism

$$D: H^0(C, K_C^2) \rightarrow H^1(C, K_C).$$

Composing the homomorphism with

$$\text{res} := \sum \text{res}_p: H^1(C, K_C) \rightarrow \mathbf{C},$$

where summation is over all  $p \in C - U_1$ , gives that the homomorphism

$$\begin{aligned} H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}) &\rightarrow H^0(C, K_C^2) \\ (fe_1, ge_1) &\mapsto (fdg - gdf)h \, da, \end{aligned}$$

is dual to the homomorphism (up to a scalar multiple)

$$\Theta: H^1(C, T_C) \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L})).$$

This proves the theorem.

**COROLLARY 1.** *Image of  $\Theta$  is contained in the set of alternating matrices.*

*Proof.* The corollary follows immediately from the theorem because  $\Theta^\vee$  is clearly zero on symmetric tensors.

**COROLLARY 2.** [See [Ha] p. 616]. *If  $\mathcal{M}_g^r \neq \emptyset$ , then every irreducible component of  $\mathcal{M}_g^r$  has codimension at most  $(r(r - 1))/2$  in  $\mathcal{M}_g$ .*

*Proof.* From the corollary (1) above and the definition of  $\mathcal{M}_g^r$ , corollary (2) follows immediately.

## 2. Examples

First we compute the tangent space map described above, at hyperelliptic curve. If  $C$  is a hyperelliptic curve of genus  $g$ , then  $C$  is the normalization of the plane curve

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i),$$

( $a_i \in \mathbf{C}$  and  $a_i \neq a_j$  for  $1 \leq i, j \leq 2g + 2$ ). Then

$$H^0(C, K_C) = \mathbf{C} \frac{dx}{y} \oplus \mathbf{C}x \frac{dx}{y} \oplus \cdots \oplus \mathbf{C}x^{g-1} \frac{dx}{y},$$

and

$$\begin{aligned}
 H^0(C, K_C^2) &= \mathbf{C} \left( \frac{dx}{y} \right)^2 \oplus \mathbf{C}x \left( \frac{dx}{y} \right)^2 \oplus \dots \\
 &\quad \oplus \mathbf{C}x^{2g-2} \left( \frac{dx}{y} \right)^2 \oplus \mathbf{C}y \left( \frac{dx}{y} \right)^2 \oplus \dots \\
 &\quad \oplus \mathbf{C}x^{g-3}y \left( \frac{dx}{y} \right)^2.
 \end{aligned}$$

Given integer  $r$  ( $0 \leq r \leq [(g + 1)/2]$ ), then set  $s = (g - 1) - 2(r - 1)$ . If

$$\mathcal{L} = \pi^* \mathcal{O}_{\mathbf{P}^1}(r - 1) \otimes \mathcal{O}_C \left( \sum_{k=1}^s t_{i_k} \right),$$

where  $\pi: C \rightarrow \mathbf{P}^1$  is the covering ramified precisely over  $a_i$  ( $1 \leq i \leq 2g + 2$ ) and  $t_{i_k} \in \pi^{-1}(\{a_1, \dots, a_{2g+2}\})$ , then  $\mathcal{L}$  is a theta-characteristic on  $C$  with  $h^0(\mathcal{L}) = r$ . Conversely every theta-characteristic  $\mathcal{L}$  on  $C$  with  $h^0(\mathcal{L}) = r$ , is of the above form. Fix a theta-characteristic  $\mathcal{L}$  on  $C$  with  $h^0(\mathcal{L}) = r$ , then

$$\Theta^\vee: H^0(C, \mathcal{L}) \otimes H^0(C, \mathcal{L}) \rightarrow H^0(C, K_C)$$

is induced by

$$(x^a e_1, x^b e_1) \rightarrow (x^a dx^b - x^b dx^a) e_1^2 = (b - a)x^{a+b-1}y \left( \frac{dx}{y} \right)^2,$$

(where  $x^a e_1, x^b e_1 \in H^0(C, \mathcal{L})$ ). Now if  $r \geq 2$  it is easy to see that image of  $\Theta^\vee$  is a  $2r - 3$  dimensional subspace of  $H^0(C, K_C^2)$ . So by the above theorem if  $r \geq 2$  it follows that at  $(C, \mathcal{L})$  the tangent mapping

$$\Theta: H^1(C, T_C) \rightarrow \text{Hom}_{\mathbf{C}}(H^0(C, \mathcal{L}), H^1(C, \mathcal{L}))$$

has rank  $(2r - 3)$ , hence the  $\ker(\Theta)$  is of codimension  $2r - 3$  in  $H^1(C, T_C)$ . Thus we have proved the following:

**THEOREM 2.** *In a suitable covering space of  $\mathcal{M}_g$ , the Zariski tangent space to  $\mathcal{M}_g^r$  ( $r \geq 2$ ) at  $(C, \mathcal{L})$  has dimension  $3g - 2r$ , where  $C$  is an hyperelliptic curve and  $\mathcal{L}$  is a theta-characteristic on  $C$  with  $h^0(\mathcal{L}) = r$ .*

**THEOREM 3.**  $\mathcal{M}_8^4$  is non-reduced scheme of dimension 15 and  $(\mathcal{M}_8^4)_{\text{red}}$  is the locus of hyperelliptic curves.

*Proof.* By Theorem (2), if  $(C, \mathcal{L}) \in \mathcal{M}_8^4$  is such that  $C$  is hyperelliptic curve of

genus 8 and  $\mathcal{L}$  a theta-characteristic on  $C$  with  $h^0(\mathcal{L}) = 4$ , then the Zariski tangent space at  $(C, \mathcal{L})$  is of dimension 16. On the other hand we show that if  $C$  is a curve of genus 8 with a theta-characteristic  $\mathcal{L}$  such that  $h^0(\mathcal{L}) = 4$ , then  $C$  is hyperelliptic, this will prove the theorem.

CLAIM. *If  $(C, \mathcal{L}) \in \mathcal{M}_8^4$  then  $C$  hyperelliptic.*

*Proof* (of the claim). Suppose  $\mathcal{L}$  has a base point  $p$ , then  $\mathcal{L}(-p)$  is a degree 6 line bundle on a genus 8 curve with 4 linearly independent sections, hence by Clifford's theorem [see [H] p. 343]  $C$  is hyperelliptic, so claim is proved if  $\mathcal{L}$  has a base point. Hence we can assume that  $\mathcal{L}$  has no base point. Let

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^3$$

be the corresponding morphism. Since  $h^0(\mathcal{L}^2) = 8$  and  $h^0(\mathcal{O}_{\mathbf{P}^3}(2)) = 10$ , there are at least two linearly independent quadrics vanishing on  $\phi_{\mathcal{L}}(C)$ . This implies, since  $\phi_{\mathcal{L}}(C)$  is not contained in any hyperplane, degree of  $\phi_{\mathcal{L}}(C)$  in  $\mathbf{P}^3$  is  $\leq 4$ . But  $\deg \mathcal{L} = 7$ , so  $\mathcal{L}$  must have a base point which contradicts our assumption on  $\mathcal{L}$ . This proves the claim.

Since locus of hyperelliptic curves is a 15 dimensional subvariety of  $\mathcal{M}_8$  theorem follows.

Next we will describe moduli  $\mathcal{M}_g^3$  for small  $g$ . Note that  $\mathcal{M}_g^3 = \emptyset$  for  $1 \leq g \leq 4$  by Clifford's theorem. For  $g \geq 5$  we have the following:

**THEOREM** [see [T] p. 113]. *The locus  $\mathcal{M}_g^3$  has pure codimension in  $\mathcal{M}_g$  if  $g \geq 5$ , and a generic point of any of its components is a curve  $C$  which has only one  $\mathcal{L}$  with  $\mathcal{L}^2 \simeq \mathcal{K}_C$  such  $h^0(\mathcal{L}) = 3$  if  $g \geq 6$ . Moreover this theta-characteristic gives a birational morphism of  $C$  into  $\mathbf{P}^2$  if  $g \geq 6$ .*

- (1) *When  $g = 5$ , it follows by Clifford's theorem that  $\mathcal{M}_5^3$  is precisely the locus of hyperelliptic curves.*
- (2) *Next  $g = 6$ . Let  $C$  be a curve of genus 6 and  $\mathcal{L}$  be a theta-characteristic on  $C$  with  $h^0(\mathcal{L}) = 3$ . If  $\mathcal{L}$  has no base point then clearly  $\mathcal{L}$  gives embedding of  $C$  in  $\mathbf{P}^2$ . Locus  $(\mathcal{M}_6^3)^0$  smooth plane curves of degree 6 is locally closed in moduli  $\mathcal{M}_6$  of genus 6 curves and again by Clifford's theorem it follows that  $\mathcal{M}_6^3 = (\mathcal{M}_6^3)^0 \cup \mathcal{H}_6$ , where  $\mathcal{H}_6$  is the locus of hyperelliptic curves.*

**THEOREM 4.**  *$\mathcal{M}_7^3$  is an irreducible subvariety of dimension 15 in the moduli-space  $\mathcal{M}_7$ .*

*Proof.* By tangent space computations it follows that  $\mathcal{M}_7^3$  has dimension  $\geq 15$ . It follows by Clifford's theorem that if  $C \in \mathcal{M}_7^3$  and if the corresponding theta-characteristic  $\mathcal{L}$  on  $C$  has a base point then  $C$  must be a hyperelliptic curve. But moduli of hyperelliptic curves is of dimension 13, hence  $C$  cannot be a general



member of  $\mathcal{M}_7^3$ . So on a general member  $C \in \mathcal{M}_7^3$  there exists theta-characteristic  $\mathcal{L}$  with  $h^0(\mathcal{L}) = 3$  and  $\mathcal{L}$  does not have base points. Let

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^2$$

be the corresponding morphism. Then the image curve can have degree 2, 3 or 6. But again by dimension count we get that if  $C$  is general then image  $C$  is a degree 6 in  $\mathbf{P}^2$ , hence  $\phi_{\mathcal{L}}$  is birational onto its image. Since  $\mathcal{L}^2 \simeq K_C$ , we see that image of  $C$  under  $\phi_{\mathcal{L}}$  is a degree 6 curve having exactly three ordinary double points lying on a line and no other singularities. Now fix a line  $l \subset \mathbf{P}^2$  and three distinct points  $p_1, p_2, p_3$  on  $l$  then the exact sequence

$$0 \rightarrow \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2 \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow \bigoplus_{i=1}^3 \frac{\mathcal{O}_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2} \rightarrow 0,$$

where  $m_{\mathbf{P}^2, p_i}$  is the ideal sheaf of the point  $p_i \in \mathbf{P}^2$ , after tensoring with  $\mathcal{O}_{\mathbf{P}^2}(6)$  gives the following cohomology exact sequence

$$\begin{aligned} 0 \rightarrow H^0\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right) &\rightarrow H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6)) \\ \rightarrow \bigoplus_{i=1}^3 H^0\left(\mathbf{P}^2, \frac{\mathcal{O}_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2}(6)\right) &\rightarrow H^1\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right) \rightarrow 0. \end{aligned}$$

But it is easy to see that

$$H^1\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right) = 0.$$

Now using the fact that  $p_i (1 \leq i \leq 3)$  lie on a line and Bertini's theorem [See [H], p. 274] we get an open set

$$U \subset \mathbf{P}\left(H^0\left(\mathbf{P}^2, \prod_{i=1}^3 m_{\mathbf{P}^2, p_i}^2(6)\right)\right),$$

such that if  $C \in U$  then  $C$  is irreducible plane curve of degree 6 and has double points at  $p_i (1 \leq i \leq 3)$  and no other singularities. Note that  $\dim U = 18$  and general member is a nodal curve. If we vary  $l \subset \mathbf{P}^2$  and  $p_i \in l (1 \leq i \leq 3)$ , we get a 23 dimensional irreducible locally closed subvariety  $W$  of  $\mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6)))$  such that if  $C \in W$  then  $C$  has exactly three ordinary double points all of them lie on a line and has no other singularities. On  $W$ ,  $\text{PGL}(3)$  acts with finite stabilizer at each of its points. Now the quotient  $V$  of  $W$  by  $\text{PGL}(3)$  gives a dense open subset of  $\mathcal{M}_7^3$ . Since dimension of  $V$  is 15 and  $V$  is irreducible theorem follows immediately.

**THEOREM 5.**  $\mathcal{M}_8^3$  is an irreducible subvariety of dimension 18 in the moduli space  $\mathcal{M}_8$ .

*Proof.* By the theorem quoted above of Teixidor I Bigas, each irreducible component of  $\mathcal{M}_8^3$  is 18 dimensional and whose general member  $C$  has a theta-characteristic  $\mathcal{L}$  such that  $\mathcal{L}$  gives a birational morphism

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^2.$$

As above using the fact that  $\mathcal{L}^2 \simeq K_C$ , we get  $\phi_{\mathcal{L}}(C)$  is a curve of degree 7 and has exactly 7 ordinary double points all of them lie on smooth conic and has no other singularities. Fix a smooth conic  $E \subset \mathbf{P}^2$  and 7 distinct points  $p_1, \dots, p_7$  on it. Consider the exact sequence

$$0 \rightarrow \prod_{i=1}^7 m_{p_i}^2 \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow \bigoplus_{i=1}^7 \frac{\mathcal{O}_{\mathbf{P}^2}}{m_{p_i}^2} \rightarrow 0,$$

where  $m_{p_i}$  is the ideal sheaf of  $p_i$  in  $\mathbf{P}^2$ . It is easy to see that

$$H^1 \left( \mathbf{P}^2, \prod_{i=1}^7 m_{p_i}^2(7) \right) = 0.$$

Hence from the above exact sequence, after tensoring with  $\mathcal{O}_{\mathbf{P}^2}(7)$  we get a cohomology exact sequence

$$0 \rightarrow H^0 \left( \mathbf{P}^2, \prod_{i=1}^7 m_{p_i}^2(7) \right) \rightarrow H^0 \left( \mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(7) \right) \rightarrow \bigoplus_{i=1}^7 \frac{\mathcal{O}_{\mathbf{P}^2}}{m_{p_i}^2} \rightarrow 0.$$

Again it is easy to see that there exists an open set

$$U \subset \mathbf{P} \left( H^0 \left( \mathbf{P}^2, \prod_{i=1}^7 m_{p_i}^2(7) \right) \right)$$

such that every curve parametrized by  $U$  is irreducible and has ordinary double points exactly at the points  $p_1, \dots, p_7$  and no other singularities. Now varying the conic and the 7 points on it we get a 26 dimensional irreducible locally closed sub variety  $W$  of  $\mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(7)))$ , on which  $\text{PGL}(3)$  acts with finite stabilizer at each of its points. The quotient  $V$ , of  $W$  by  $\text{PGL}(3)$  is a dense open subset of  $\mathcal{M}_8^3$ . Thus  $\mathcal{M}_8^3$  is a irreducible codimension 3 subvariety of  $\mathcal{M}_8$ .

**THEOREM 6.**  $\mathcal{M}_9^3$  has exactly two irreducible components each of dimension 21 in  $\mathcal{M}_9$ .

*Proof.* Again by the theorem of Teixidor I Bigas, each irreducible component

of  $\mathcal{M}_3^3$  is 21 dimensional and whose general members is a curve  $C$  with a theta-characteristic  $\mathcal{L}$  which give rise to a birational morphism

$$\phi_{\mathcal{L}}: C \rightarrow \mathbf{P}^2.$$

As above the fact that  $\mathcal{L}$  is theta-characteristic gives that  $\phi_{\mathcal{L}}(C)$  is a curve of degree 8 and has exactly 12 ordinary double points all of which lie on degree 3 curve and has no other singularities. Also note that the above 12 points on the degree 3 curve has the property twice the sum of these 12 points is the zeros of a section of  $\mathcal{O}_{\mathbf{P}^2}(8)$  restricted to the degree 3 curve. Since we are interested in an open subset of  $\mathcal{M}_3^3$ , we look at curves  $C$  as above with corresponding singularities of  $\phi_{\mathcal{L}}(C)$  lie on a smooth degree 3 curve. We fix a smooth degree 3 curve  $E \subset \mathbf{P}^2$  and 12 distinct points  $p_1, \dots, p_{12}$  on it such that  $2 \sum_{i=1}^{12} p_i \in \mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2}(8)|_E))$ . Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(5)) & \rightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(8)) & \rightarrow & H^0(E, \mathcal{O}_E(8)) \rightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \rightarrow & H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(5)) & \rightarrow & V & \rightarrow & \mathbf{C}\sigma \rightarrow 0 \end{array}$$

where  $\sigma$  is a section of  $\mathcal{O}_E(8)$  corresponding to  $2 \sum_{i=1}^{12} p_i$ ,  $V$  is the inverse image of  $\mathbf{C}\sigma$  in  $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(8))$ ,  $\dim_{\mathbf{C}} V = 22$ . From  $V$  we have the following mapping

$$V \rightarrow \bigoplus_{i=1}^{12} \frac{m_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2}$$

whose image is the 12 dimensional subspace

$$\bigoplus_{i=1}^{12} \frac{m_{E, p_i}}{m_{E, p_i}^2}.$$

So

$$W = \ker \left( V \rightarrow \bigoplus_{i=1}^{12} \frac{m_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2} \right)$$

is a 10 dimensional subspace of  $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(8))$ . Again by Bertini's theorem  $\mathbf{P}(W)$  contains an open set  $U_{(E, (p_i))}$  such that if  $C$  is a curve corresponding to a point of  $U_{(E, (p_i))}$  then  $C$  irreducible degree 8 curve which has ordinary double points at  $p_i$  ( $1 \leq i \leq 12$ ) and has no other singularities. Now the variety  $H$  parametrizing  $(E, \sum_{i=1}^{12} p_i)$ ,  $E \subset \mathbf{P}^2$  degree 3 smooth curve,  $p_1, \dots, p_{12}$  distinct points on  $E$  such that  $2 \sum_{i=1}^{12} p_i \in \mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2}(8)|_E))$  is 20 dimensional. Note that  $H$  has two

connected (irreducible) components (see, Introduction) of dimension 20 corresponding to two types of points  $p_1, \dots, p_{12}$  namely whether  $\sum_{i=1}^{12} p_i$  is in  $\mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2(4)}|_E))$  or not. The above construction gives a variety  $X$  fibred over  $H$  with fibres the 9 dimensional variety  $U_{(E,(p_i))}$ . On  $X = X_1 \cup X_2$ ,  $\mathbf{PGL}(3)$  acts with finite stabilizer at each of its point and the quotient  $W$  is a dense open subset of  $\mathcal{M}_9^3$ . This proves that  $\mathcal{M}_9^3$  has two irreducible components.

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