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JERZY URBANOWICZ

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Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, II

JERZY URBANOWICZ

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950 Warszawa, Poland

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Abstract. The paper is a continuation of my earlier paper on this subject. We prove analogous congruences, as in that paper, but modulo larger powers of 2.

0. Introduction

For the discriminant d of a real quadratic field, denote $k_2(d) = B_{2,(\frac{d}{\cdot})}$ if $d \neq 5, 8$ and $k_2(5) = k_2(8) = 4$. The Birch-Tate conjecture for real quadratic fields F with the discriminant d states that

$$|K_2 O_F| = k_2(d).$$

Here K_2, O_F, $(\frac{d}{\cdot})$ and $B_{k,\chi}$ denote the Milnor functor, the ring of integers of F, the Kronecker symbol and the kth Bernoulli number respectively. For the discriminant d of an imaginary quadratic field, let $h(d)$ denote the class number of this field.

We have found in [8] some connections between $k_2(d)$ and class numbers of appropriate imaginary quadratic fields of the Lerch-Berndt type (see [5] and [1]). From the obtained formulas we have got appropriate congruences for $k_2(d)$ modulo powers of 2 (4, 8, 16, 32 and 64). It is the purpose of this paper to prove analogous congruences modulo larger powers of 2.

For the discriminant D of a quadratic field and a square-free natural number $N > 1$ prime to D, let

$$\varphi_D(N) = \prod_{p \mid N} \left(p - \left(\frac{D}{p}\right)\right),$$

and

$$\psi_D(N) = \prod_{p \mid N} \left(1 - \left(\frac{D}{p}\right)\right),$$

where the products are taken over all prime factors of N. Set $\varphi_D(1) = \psi_D(1) = 1$. Note that $\varphi_D(N)$ and $\psi_D(N)$ are products of Euler's factors. Denote by φ Euler's totient function.

Let d and e, $e \mid d$ be the odd discriminants of quadratic fields. For $D \in \{-8e, -4e, e, 8e\}$, set

$$K(d, D) = -\left(\frac{e}{|d/e|}\right) \varphi_D(|d/e|) k_2(D), \quad \text{if } D > 0,$$

and

$$H(d, D) = \left(\frac{e}{|d/e|}\right) \psi_D(|d/e|) h(D), \quad \text{if } D < 0.$$

We shall prove the following:

THEOREM. *Let d be the odd discriminant of a quadratic field having n prime factors. We have in the above notation:*

$$\begin{aligned}
&\sum_{\substack{e \mid d \\ e>1 \\ e \equiv 1 (\mathrm{mod}\, 4)}} \left(K(d, 8e) - \left(34 - \left(\frac{e}{2}\right)\right) K(d, e)\right) + \\
&\quad + \sum_{\substack{e \mid d \\ e<0 \\ e \equiv 1 (\mathrm{mod}\, 4)}} \left(K(d, -8e) + \left(\frac{e}{2}\right) K(d, -4e)\right) + K(d) \\
&\quad \equiv 2|d| \Bigg(\sum_{\substack{e \mid d \\ e>1 \\ e \equiv 1 (\mathrm{mod}\, 4)}} \left(\left(\frac{e}{2}\right) H(d, -4e) + H(d, -8e)\right) \\
&\quad + \sum_{\substack{e \mid d \\ e<0 \\ e \equiv 1 (\mathrm{mod}\, 4)}} \left(\left(1 - \left(\frac{e}{2}\right)\right) H(d, e) - H(d, 8e)\right) + H(d)\Bigg)
\end{aligned}$$

(mod 2^{n+6}). *Here*

$$H(d) = \tfrac{1}{4}(-1)^n \varphi(|d|) + A(|d|) + \Theta(d),$$

where $A(|d|)$ is defined in Lemma 3 (*see the section* 2), *and*

$$\Theta(d) = \begin{cases} -\frac{13}{3} H(d, -3), & \text{if } 3 \mid d, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover

$$K(d) = \tfrac{11}{2}\varphi(|d|) - 2\varphi_8(|d|) + \upsilon(d),$$

where

$$\upsilon(d) = \begin{cases} 28K(d, 5), & \text{if } 5 \mid d, \\ 0, & \text{otherwise.} \end{cases}$$

For illustration we give the following:

COROLLARY. *Let p, $p > 5$ be a prime number and let $f_s(x)$ for $s \in \{1, 3, 5, 7\}$* be the polynomial defined as follows:

$$f_s(x) = ax^2 + bx + c,$$

where

$$a = -\frac{1}{2},$$

$$b = -\left(\frac{-4}{s}\right)\left(1 + 2\left(\frac{8}{s}\right)\right)$$

$$c = \frac{1}{2}\left(11 - 4\left(\frac{8}{s}\right)\right).$$

We have for $p \equiv s \pmod 8$, $s \in \{1, 3, 5, 7\}$:

(i) *if $p \equiv 1 \pmod 4$, then:*

$$-k_2(8p) + \left(34 - \left(\frac{8}{p}\right)\right) k_2(p) \equiv 2p\left(\left(\frac{8}{p}\right) h(-4p) + h(-8p)\right) + f_s(p) \pmod{128},$$

(ii) *if $p \equiv 3 \pmod 4$, then:*

$$-k_2(8p) - \left(\frac{8}{p}\right) k_2(4p) \equiv 2p\left(\left(1 - \left(\frac{8}{p}\right)\right) h(-p) - h(-8p)\right) + f_s(p) \pmod{128}.$$

This corollary (Theorem for $n = 1$) can be a consequence of Theorems 1, 2 [8], too. All the congruences obtained for $n \geqq 2$ are new.

Let us note that in our notation Theorem of [4] states:

$$\sum_{\substack{e\mid d\\ e>1\\ e\equiv 1 \pmod 4}} \left(\left(\frac{e}{2}\right) H(d,\ -4e) + H(d,\ -8e)\right)$$
$$+ \sum_{\substack{e\mid d\\ e<0\\ e\equiv 1 \pmod 4}} \left(\left(5-\left(\frac{e}{2}\right)\right) H(d,\ e) - H(d,\ 8e)\right) + H_1(d) \equiv 0 \pmod{2^{n+2}},$$

where

$$H_1(d) = \tfrac{1}{2}(-1)^n \varphi(|d|) + A(|d|) + \tfrac{12}{13}\Theta(d).$$

In the present paper we extend the results of [4] and [3] using ideas of [4]. Similar problems were also dealt with in [2] and [7].

1. Lemmas of the Lerch-Berndt type

Let e be the odd discriminant of a quadratic field. Denote for $k = 1, 2, \dots, 8$

$$I_k = ((k-1)|e|/8,\ k|e|/8).$$

Let

$$T_k = \sum_{l\in I_k} \left(\frac{e}{l}\right) \quad \text{and} \quad S_k = \sum_{l\in I_k} \left(\frac{e}{l}\right) l.$$

LEMMA 1 ([5], [1]). *We have*:

$$T_1 = \begin{cases} \frac{1}{4}\left(\frac{e}{2}\right) h(-4e) + \frac{1}{4}h(-8e), & \textit{if } e>0,\\ \frac{1}{4}\left(5-\frac{e}{2}\right) h(e) - \frac{1}{4}h(8e) - \lambda(e), & \textit{if } e<0, \end{cases}$$

$$T_2 = \begin{cases} \frac{1}{4}\left(2-\frac{e}{2}\right) h(-4e) - \frac{1}{4}h(-8e), & \textit{if } e>0,\\ \frac{3}{4}\left(-1+\frac{e}{2}\right) h(e) + \frac{1}{4}h(8e) + \lambda(e), & \textit{if } e<0, \end{cases}$$

$$T_3 = \begin{cases} \frac{1}{4}\left(-2-\frac{e}{2}\right) h(-4e) + \frac{1}{4}h(-8e), & \textit{if } e>0,\\ \frac{3}{4}\left(1-\frac{e}{2}\right) h(e) + \frac{1}{4}h(8e) - \lambda(e), & \textit{if } e<0, \end{cases}$$

$$T_4 = \begin{cases} \frac{1}{4}\left(\frac{e}{2}\right) h(-4e) - \frac{1}{4}h(-8e), & \text{if } e > 0, \\ \frac{3}{4}\left(1 - \frac{e}{2}\right) h(e) - \frac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

where $\lambda(e) = 1$, if $e = -3$, and $\lambda(e) = 0$, otherwise.

Moreover we have for $k = 5, 6, 7, 8$

$$T_k = \left(\frac{e}{-1}\right) T_{9-k}.$$

□

LEMMA 2 ([8]). *We have*:

$$S_1 = \begin{cases} \frac{1}{64} k_2(8e) - \frac{1}{64}\left(34 - \frac{e}{2}\right) k_2(e) \\ \quad + \frac{1}{32}e\left(\left(\frac{e}{2}\right) h(-4e) + h(-8e)\right) + 7\omega(e), & \text{if } e > 0, \\ \frac{1}{64}k_2(-8e) + \frac{1}{64}\left(\frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{1}{32}e\left(\left(1 - \frac{e}{2}\right) h(e) - h(8e)\right) - v(e), & \text{if } e < 0, \end{cases}$$

$$S_2 = \begin{cases} -\frac{1}{64}k_2(8e) + \frac{3}{64}\left(2 - 3\left(\frac{e}{2}\right)\right) k_2(e) \\ \quad + \frac{1}{32}e\left(\left(\left(4 - \frac{e}{2}\right)\right) h(-4e) - h(-8e)\right) - 3\omega(e), & \text{if } e > 0, \\ -\frac{1}{64}k_2(-8e) + \frac{1}{64}\left(4 - \frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{1}{32}e\left(5\left(-1 + \frac{e}{2}\right) h(e) + h(8e)\right) + 5v(e), & \text{if } e < 0, \end{cases}$$

$$S_3 = \begin{cases} -\frac{1}{64}k_2(8e) - \frac{3}{64}\left(2 - 3\,\frac{e}{2}\right) k_2(e) \\ \quad + \frac{1}{32}e\left(\left(-4 - 3\frac{e}{2}\right) h(-4e) + 3h(-8e)\right) + 3\omega(e), & \text{if } e > 0, \\ \frac{1}{64}k_2(-8e) - \frac{1}{64}\left(4 + \frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{1}{32}e\left(7\left(1 - \frac{e}{2}\right) h(e) + 3h(8e)\right) - 7v(e), & \text{if } e < 0, \end{cases}$$

$$S_4=\begin{cases}\frac{1}{64}k_2(8e)-\frac{15}{64}\left(2-\frac{e}{2}\right)k_2(e)\\ \quad+\frac{1}{32}e\left(3\left(\frac{e}{2}\right)h(-4e)-3h(-8e)\right)+9\omega(e), & \textit{if } e>0,\\ -\frac{1}{64}k_2(-8e)+\frac{1}{64}\left(\frac{e}{2}\right)k_2(-4e)\\ \quad-\frac{1}{32}e\left(13\left(1-\frac{e}{2}\right)h(e)-3h(8e)\right)-13\nu(e), & \textit{if } e<0,\end{cases}$$

where $\omega(e)=\frac{1}{4}$, *if* $e=5$, $\omega(e)=0$, *otherwise, and* $\nu(e)=\frac{1}{8}$, *if* $e=-3$, $\nu(e)=0$, *otherwise. Moreover we have for* $k=5, 6, 7, 8$

$$S_k=eT_{9-k}-\left(\frac{e}{-1}\right)S_{9-k}. \qquad \square$$

2. Lemma of the Nagell type

Let N be a natural number. In [6] (see also [4]) the explicit formulas for

$$\sum_{\substack{0<k<N/8\\(k,N)=1}} 1$$

are found. Now, we shall determine the sum

$$J(N)=\sum_{\substack{0<k<N/8\\(k,N)=1}} k$$

in the cases that are of our interest.

We shall prove the following:

LEMMA 3. *Take the notation of Introduction. Let* x, y, z, u *be respectively the numbers of prime divisors of an odd natural square-free number* N, $N>1$, *of the form* $8t+1$, $8t-1$, $8t+3$, $8t-3$. *Set* $n=x+y+z+u$. *Then:*

$$J(N)=(-1)^n\frac{\varphi(N)}{128}((-1)^nN-11)+(-1)^n\frac{\varphi_8(N)}{32}+(-1)^n\frac{N}{32}A(N),$$

where

$$A(N)=\begin{cases}0, & \textit{if } x>0 \textit{ or } x=0,\ y\geqq 0,\ z>0,\ u>0,\\ 2^n, & \textit{if } x=0,\ y\geqq 0,\ z=0,\ u>0,\\ 2^{n-1}, & \textit{if } x=0,\ y\geqq 0,\ z>0,\ u=0,\\ 3\cdot 2^{n-1}, & \textit{if } x=0,\ y>0,\ z=0,\ u=0.\end{cases}$$

Proof. We have

$$J(N) = \sum_{0<k<N/8} k \sum_{f\mid(k.N)} \mu(f) = \sum_{f\mid N} \mu(f) \sum_{\substack{0<k<N/8\\ f\mid k}} k = \sum_{f\mid N} \mu(f)f \sum_{k=1}^{[N/8f]} k$$

$$= \sum_{f\mid N} \mu(f)f \frac{[N/8f]([N/8f]+1)}{2} = \frac{1}{2}(-1)^n N \sum_{f\mid N} \frac{\mu(f)}{f} [f/8]([f/8]+1)$$

$$= \frac{(-1)^n N}{128} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \mu(f) \frac{(f-s)(f-s+8)}{f}$$

$$= \frac{(-1)^n N}{128} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \mu(f)\left(f + 2(4-s) + \frac{s^2-8s}{f}\right)$$

$$= \frac{(-1)^n N}{128} \sum_{f\mid N} \mu(f)f + \frac{(-1)^n N}{128} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} 2(4-s) \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \mu(f) +$$

$$+ \frac{(-1)^n N}{128} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} (s^2-8s) \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \frac{\mu(f)}{f}.$$

Therefore by the table

s	$4-s$	s^2-8s
1	3	-7
3	1	-15
5	-1	-15
7	-3	-7

we obtain

$$J(N) = \frac{(-1)^n N}{128} \sum_{f\mid N} \mu(f)f + \frac{(-1)^n N}{64} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \mu(f)\left(\frac{-8}{s}\right) +$$

$$+ \frac{(-1)^n N}{64} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \mu(f)\left(\left(\frac{-8}{s}\right) + \left(\frac{-4}{s}\right)\right) - \frac{7(-1)^n N}{128} \sum_{f\mid N} \frac{\mu(f)}{f} -$$

$$- \frac{(-1)^n N}{32} \sum_{\substack{s=1\\ s\text{ odd}}}^{7} \sum_{\substack{f\mid N\\ f\equiv s(\mathrm{mod}\,8)}} \frac{\mu(f)}{f}\left(1 - \left(\frac{8}{s}\right)\right) = \frac{(-1)^n N}{128} \sum_{f\mid N} \mu(f)f +$$

$$+ \frac{(-1)^n N}{64} \sum_{f\mid N} \mu(f)\left(\frac{-8}{f}\right) + \frac{(-1)^n N}{64} \sum_{f\mid N} \mu(f)\left(\left(\frac{-8}{f}\right) + \left(\frac{-4}{f}\right)\right) -$$

$$- \frac{7(-1)^n N}{128} \sum_{f\mid N} \frac{\mu(f)}{f} - \frac{(-1)^n N}{32} \sum_{f\mid N} \frac{\mu(f)}{f}\left(1 - \left(\frac{8}{f}\right)\right).$$

Now we shall use the well-known formula

$$\sum_{d \mid N} \mu(d)F(d) = \prod_{p \mid N} (1 - F(p)) \tag{2.1}$$

that is true for a multiplicative arithmetical function F. In this formula the sum and the product are extended over all divisors and over all prime factors of N respectively.

We get

$$\begin{aligned} J(N) = {} & \frac{(-1)^n N}{128} \prod_{p \mid N} (1 - p) + \frac{(-1)^n N}{32} \prod_{p \mid N} \left(1 - \left(\frac{-8}{p}\right)\right) \\ & + \frac{(-1)^n N}{64} \prod_{p \mid N} \left(1 - \left(\frac{-4}{p}\right)\right) - \frac{11(-1)^n N}{128} \prod_{p \mid N} \left(1 - \frac{1}{p}\right) \\ & + \frac{(-1)^n N}{32} \prod_{p \mid N} \left(1 - \left(\frac{8}{p}\right) p^{-1}\right) \\ = {} & \frac{(-1)^n \varphi(N)}{128} ((-1)^n N - 11) + \frac{(-1)^n \varphi_8(N)}{32} + \frac{(-1)^n N}{32} A(N), \end{aligned}$$

where

$$A(N) = \prod_{p \mid N} \left(1 - \left(\frac{-8}{p}\right)\right) + \frac{1}{2} \sum_{p \mid N} \left(1 - \left(\frac{-4}{p}\right)\right).$$

Finally, it suffices to observe that

$$\prod_{p \mid N} \left(1 - \left(\frac{-8}{p}\right)\right) = \begin{cases} 2^n, & \text{if } x = z = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\prod_{p \mid N} \left(1 - \left(\frac{-4}{p}\right)\right) = \begin{cases} 2^n, & \text{if } x = u = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and Lemma 3 follows. □

3. Proof of Theorem

We shall start with the following congruence (see (2.3) of [4]):

$$\sum_{e \mid d}^{*} (-1)^{\tau(e)} \left(\sum_{\substack{0 < k < |d|/8 \\ (k,d) = 1}} \left(\frac{e}{k}\right) k \right) \equiv 0 \pmod{2^n}, \tag{3.1}$$

where the asterisk over the sum means that it is taken over all divisors, negative or positive, of d such that $e \equiv 1 \pmod 4$. Here $\tau(e)$ denotes the number of distinct prime factors of e.

Denote for $e \mid d$, $e \equiv 1 (\mathrm{mod}\ 4)$, $e \neq 1$

$$S = S(d, e) = \sum_{\substack{0 < k < |d|/8 \\ (k,d) = 1}} \left(\frac{e}{k}\right) k.$$

We have (for details see (2.7), (2.8), (2.9) and the page 268 of [4]):

$$S = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \sum_{\substack{0 < k < |d|/8 \\ f \mid k}} \left(\frac{e}{k}\right) k = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{0 < 1 < |d|/8f} \left(\frac{e}{l}\right) l.$$

Hence we obtain

$$S = R + U, \tag{3.2}$$

where for $t = \left[\frac{|d/e|}{8f}\right]$, $t' = \left[\frac{|d|}{8f}\right]$

$$\begin{aligned} R = R(d, e) &= \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{l=1}^{t|e|} \left(\frac{e}{l}\right) l \\ &= -\rho(e) h(e) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f t |e|, \end{aligned}$$

$$\rho(e) = \begin{cases} 0, & \text{if } e > 0, \\ \frac{1}{3}, & \text{if } e = -3, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$U = U(d, e) = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{l = t|e| + 1}^{t'} \left(\frac{e}{l}\right) l,$$

because

$$\sum_{l=1}^{t|e|} \left(\frac{e}{l}\right) l = \begin{cases} 0, & \text{if } e > 0, \\ -t, & \text{if } e = -3, \\ teh(e), & \text{otherwise.} \end{cases}$$

First, we shall deal with $U(d, e)$. We find

$$\sum_{l=t|e|+1}^{t'} \left(\frac{e}{l}\right) l = \sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right)(m + t|e|)$$
$$= \sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) m + t|e| \sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right).$$

Moreover we have

$$\sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) = \sum_{k=1}^{s} T_k,$$

where $(|d/e|/f) \equiv s \pmod 8$, $s \in \{1, 3, 5, 7\}$ (for details see p. 268 of [4]) and

$$\sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) m = \sum_{k=1}^{s} S_k,$$

where T_k and S_k are defined in the section 2.

Therefore

$$U = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\sum_{k=1}^{s} S_k + t|e| \sum_{k=1}^{s} T_k\right)$$
$$= \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{k=1}^{s} S_k$$
$$+ \frac{1}{8} \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) (|d| - s|e|f) \sum_{k=1}^{s} T_k$$
$$= \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\sum_{k=1}^{s} S_k - \frac{s|e|}{8} \sum_{k=1}^{s} T_k\right)$$
$$+ \frac{1}{8} |d| \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) \sum_{k=1}^{s} T_k. \tag{3.3}$$

As for $R(d, e)$ for $e < 0$ we have

$$R = -\frac{1}{8} \rho(e) h(e) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) (|d| - s|e|f). \tag{3.4}$$

Now we shall use Lemmas 1 and 2. We have for $s = 1, 3, 5$ and 7

$$\sum_{k=1}^{s} T_k = \begin{cases} \frac{1}{4}\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right) h(-4e) + \frac{1}{4}\left(\frac{-8}{s}\right) h(-8e), & \text{if } e > 0, \\ \frac{1}{4}\left(5 - \left(\frac{e}{2}\right)\right) h(e) - \frac{1}{4}\left(\frac{8}{s}\right) h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

(see (2.22) of [4] or Lemma 1), and from Lemma 2

$$\sum_{k=1}^{s} S_k = \begin{cases} \frac{1}{64}\left(\frac{8}{s}\right) k_2(8e) - \frac{1}{64}\left(34 - \left(\frac{e}{2}\right)\right) k_2(e) + \\ \quad + \frac{se}{32}\left(\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right) h(-4e) + \left(\frac{-8}{s}\right) h(-8e)\right) + 7\omega(e), & \text{if } e > 0, \\ \frac{1}{64}\left(\frac{-8}{s}\right) k_2(-8e) + \frac{1}{64}\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{se}{32}\left(\left(1 - \left(\frac{e}{2}\right)\right) h(e) - \left(\frac{e}{2}\right) h(8e)\right) - sv(e), & \text{if } e < 0. \end{cases}$$

From the last two formulas we get that

$$\sum_{k=1}^{s} S_k - \frac{s|e|}{8} \sum_{k=1}^{s} T_k$$

$$= \begin{cases} \frac{1}{64}\left(\frac{8}{s}\right) k_2(8e) - \frac{1}{64}\left(34 - \left(\frac{e}{2}\right)\right) k_2(e) + 7\omega(e), & \text{if } e > 0, \\ \frac{1}{64}\left(\frac{-8}{s}\right) k_2(-8e) + \frac{1}{64}\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right) k_2(-4e) + \frac{se}{8} h(e) + 2sv(e), & \text{if } e < 0. \end{cases}$$

Hence, from (3.2) and (3.3) we obtain for $e > 0$:

$$\begin{aligned} 64(-1)^{\tau(d/e)} S = (-1)^{\tau(d/e)} \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\left(\frac{8}{|d/e|/f}\right) k_2(8e) - \right. \\ \left. - \left(34 - \left(\frac{e}{2}\right)\right) k_2(e) + 64 \cdot 7\omega(e)\right) + 2|d|(-1)^{\tau(d/e)} \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) \\ \times \left(\left(\frac{-4}{|d/e|/f}\right)\left(\frac{e}{2}\right) h(-4e) + \left(\frac{-8}{|d/e|/f}\right) h(-8e)\right). \end{aligned}$$

Therefore we have for $e > 0$:

$$\begin{aligned} 64(-1)^{\tau(d/e)} S = b_1(d, e) k_2(8e) + b_2(d, e) k_2(e) + b_5(d, e) + \\ + 2|d|(c_1(d, e) h(-4e) + c_2(d, e) h(-8e)), \end{aligned} \tag{3.5}$$

where in view of (2.1)

$$b_1(d, e) = (-1)^{\tau(d/e)} \left(\frac{8}{|d/e|}\right) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{8e}{f}\right) f$$
$$= (-1)^{\tau(d/e)} \left(\frac{8}{|d/e|}\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{8e}{p}\right) p\right) = \left(\frac{e}{|d/e|}\right) \varphi_{8e}(|d/e|),$$

$$b_2(d, e) = -(-1)^{\tau(d/e)} \left(34 - \left(\frac{e}{2}\right)\right) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f$$
$$= -(-1)^{\tau(d/e)} \left(34 - \left(\frac{e}{2}\right)\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{e}{p}\right) p\right)$$
$$= -\left(34 - \left(\frac{e}{2}\right)\right)\left(\frac{e}{|d/e|}\right) \varphi_e(|d/e|),$$

$$b_5(d, e) = 0 \quad \text{if } e \neq 5, \quad \text{and} \quad b_5(d, 5) = 16 \cdot 7 \left(\frac{5}{|d/5|}\right) \varphi_5(|d/5|).$$

$c_1(d, e)$ and $c_2(d, e)$ are the same as in [4] i.e.

$$c_1(d, e) = \left(\frac{e}{2}\right)\left(\frac{e}{|d/e|}\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{-4e}{p}\right)\right)$$
$$= \left(\frac{e}{2}\right)\left(\frac{e}{|d/e|}\right) \psi_{4e}(|d/e|),$$

$$c_2(d, e) = \left(\frac{e}{|d/e|}\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{-8e}{p}\right)\right) = \left(\frac{e}{|d/e|}\right) \psi_{8e}(|d/e|).$$

Similarly, from (3.2), (3.3) and (4.3) we find for $e < 0$:

$$S = U + R = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\frac{1}{64} \left(\frac{-8}{|d/e|/f}\right) k_2(-8e) + \right.$$
$$\left. + \frac{1}{64} \left(\frac{-4}{|d/e|/f}\right)\left(\frac{e}{2}\right) k_2(-4e) + \frac{se}{8} h(e) + 2sv(e)\right) +$$
$$+ \frac{1}{32} |d| \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right)\left(\left(5 - \left(\frac{e}{2}\right)\right) h(e) - \right.$$
$$\left. - \left(\frac{8}{|d/e|/f}\right) h(8e) - 4\lambda(e)\right) -$$
$$- \frac{1}{8} \rho(e) h(e) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) (|d| + sef)$$

$$=\frac{1}{64}\,k_2(-8e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)f\left(\frac{-8}{|d/e|/f}\right)+$$

$$+\frac{1}{64}\left(\frac{e}{2}\right)k_2(-4e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)f\left(\frac{-4}{|d/e|/f}\right)+$$

$$+\frac{1}{8}\,eh(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)fs+2\nu(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)fs+$$

$$+\frac{1}{32}\,|d|\left(\left(5-\left(\frac{e}{2}\right)\right)h(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)-\right.$$

$$-h(8e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)\left(\frac{8}{|d/e|/f}\right)-$$

$$\left.-4\lambda(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)-4\rho(e)h(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)\right)-$$

$$-\frac{1}{8}\,e\rho(e)h(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)fs.$$

Since for any $e<0$

$$\frac{1}{8}\,h(e)e\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)fs+2\nu(e)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)fs-$$

$$-\frac{1}{8}\,\rho(e)h(e)e\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)sf=0$$

we obtain for $e<0$

$$64(-1)^{\tau(d/e)}S=b_3(d,e)k_2(-8e)+b_4(d,e)k_2(-4e)+$$

$$+2|d|(c_3(d,e)h(e)+c_4(d,e)h(8e)+c_5(d,e)), \tag{3.6}$$

where in virtue of (2.1)

$$b_3(d,e)=(-1)^{\tau(d/e)}\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)f\left(\frac{-8}{|d/e|/f}\right)$$

$$=(-1)^{\tau(d/e)}\left(\frac{-8}{|d/e|}\right)\prod_{p\,|\,|d/e|}\left(1-\left(\frac{-8e}{p}\right)p\right)$$

$$=\left(\frac{e}{|d/e|}\right)\varphi_{-8e}(|d/e|),$$

$$b_4(d,e)=(-1)^{\tau(d/e)}\left(\frac{e}{2}\right)\sum_{f\,|\,|d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)f\left(\frac{-4}{|d/e|/f}\right)$$
$$=(-1)^{\tau(d/e)}\left(\frac{e}{2}\right)\left(\frac{-4}{|d/e|}\right)\prod_{p\,|\,|d/e|}\left(1-\left(\frac{-4e}{p}\right)p\right)$$
$$=\left(\frac{e}{2}\right)\left(\frac{e}{|d/e|}\right)\varphi_{-4e}(|d/e|).$$

Moreover similarly as in [4]

$$c_3(d,e)=\left(1-\left(\frac{e}{2}\right)\right)\left(\frac{e}{|d/e|}\right)\prod_{p\,|\,|d/e|}\left(1-\left(\frac{e}{p}\right)\right)=\left(1-\left(\frac{e}{2}\right)\right)\left(\frac{e}{|d/e|}\right)\psi_e(|d/e|),$$
$$c_4(d,e)=-\left(\frac{e}{|d/e|}\right)\prod_{p\,|\,|d/e|}\left(1-\left(\frac{8e}{p}\right)\right)=-\left(\frac{e}{|d/e|}\right)\psi_{8e}(|d/e|),$$
$$c_5(d,e)=0\quad\text{if } e\neq -3,\quad\text{and}\quad c_5(d,-3)=-\frac{13}{3}\left(\frac{-3}{|d/3|}\right)\psi_{-3}(|d/e|).$$

Now we get from (3.1)

$$(-1)^n\sum_{\substack{e\,|\,d\\ e>1\\ e\equiv 1(\mathrm{mod}\,4)}}(-1)^{\tau(e)}64S(d,e)+(-1)^n\sum_{\substack{e\,|\,d\\ e<0\\ e\equiv 1(\mathrm{mod}\,4)}}(-1)^{\tau(e)}64S(d,e)+$$
$$+(-1)^n 64J(|d|)\equiv 0 \pmod{2^{n+6}}.$$

Hence, from (3.5), (3.6) and Lemma 3, Theorem follows. □

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