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Homological properties of some Weyl algebra extensions

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Introduction

In this work we study rings of the form $A_1(K) \bigotimes_K S$, where K is a commutative field of characteristic zero, $A_1(K)$ the *Weyl algebra* in one variable and S a K-algebra satisfying the so called *Auslander-Gorenstein condition*. The K-algebra $A_1(K)$ is generated by two elements x and ∂ such that $\partial x - x\partial = 1$. We get a graded ring structure on $R = A_1(K) \bigotimes_K S$ where elements of S are homogeneous of degree zero, while x, respectively ∂ is homogeneous of degree -1, respectively $+1$.

Then, for a finitely generated and graded left(or right) R-module $M = \oplus M_v$ we are going to establish that the grade number of the R-module M is obtained via the grade numbers of the homogeneous components each of which is a finitely generated module over the polynomial ring in one variable over S.

Our main result occurs in Theorem 1.3 and we refer to §4 for some applications to modules over rings of differential operators.

Concerning *examples of Auslander-Gorenstein rings* we mention that if S is a commutative noetherian ring with finite injective dimension then S is Auslander-Gorenstein. Moreover, if S is a positively filtered ring (or more general a zariskian filtered ring) such that the associated graded ring is Auslander-Gorenstein then S is Auslander-Gorenstein. For details we refer to [Bj:2].

1. Announcements of the main results

Before we announce Theorem 1.3 and 1.4 below we need some preliminaries:

We consider the Weyl algebra $A_1(K)$ in one variable over a commutative field of characteristic zero. Recall that the K-algebra $A_1(K)$ is generated by two elements x and ∂ such that $\partial x - x\partial = 1$.

Denote by V_0 the K-subalgebra of $A_1(K)$ generated by x and $x\partial$. If $k \geqslant 1$ we set

$$V_{-k} = x^k V_0 \quad \text{and} \quad V_k = V_0 + \partial V_0 + \cdots + \partial^k V_0$$

It is easily seen that $\{V_k\}_{k\in\mathbb{Z}}$ is a filtration, i.e. $V_kV_m \subset V_{k+m}$, $\cap V_k = \{0\}$ and $\cup V_k = A_1(K)$.

The associated graded ring $\bigoplus V_k/V_{k-1}$ is denoted by $\mathrm{gr}(A_1)$. Let $\bar{x}$, respectively $\bar{\partial}$ denote the image of x in V_{-1}/V_{-2}, respectively ∂ in V_1/V_0. It is obvious that the K-algebra $\mathrm{gr}(A_1)$ is generated by $\bar{x}$ and $\bar{\partial}$. Moreover $\bar{\partial}\bar{x} - \bar{x}\bar{\partial} = 1$ in the ring $\mathrm{gr}(A_1)$. It follows that the K-algebras $\mathrm{gr}(A_1)$ and $A_1(K)$ are isomorphic. By definition $\mathrm{gr}(A_1)$ is a graded ring. If k is an integer we set $\mathrm{gr}_k(A_1) = V_k/V_{k-1}$. Observe that $\mathrm{gr}_0(A_1) = K[\bar{x}\bar{\partial}]$, i.e. $\mathrm{gr}_0(A_1)$ is the polynomial ring in one variable over K.

In this paper we study tensor product rings of the form $\mathrm{gr}(A_1) \otimes_K S$, where S is a K-algebra. Set $R = \mathrm{gr}(A_1) \otimes_K S$. The graded structure of $\mathrm{gr}(A_1)$ induces a graded structure of R such that

$$R_k = \mathrm{gr}_k(A_1) \otimes_K S.$$

We will assume that the ring S satisfies certain homological conditions and recall first:

Auslander-Gorenstein rings

Let A be a left and a right noetherian ring. We say that A has a finite injective dimension if there exists an integer n such that $\mathrm{Ext}^v_A(M, A) = 0$ for every $v > n$ and any left or right A-module M. We refer to the article [Za] for facts about noncommutative noetherian rings with finite injective dimension. Following [Bj:2, Definition 1.2] we give

1.1. DEFINITION. *A ring A is called an* Auslander-Gorenstein ring *if A is a left and a right noetherian ring with finite injective dimension and moreover: for every integer v, every finitely generated left or right A-module M and every submodule N of* $\mathrm{Ext}^v_A(M, A)$ *it follows that* $\mathrm{Ext}^i_A(N, A) = 0$ *for every $i < v$.*

Assume A is Auslander-Gorenstein. If M is a nonzero and finitely generated left or right A-module, then there exists a unique smallest integer k such that $\mathrm{Ext}^k_A(M, A)$ is nonzero. We set $k = j_A(M)$ and we call $j_A(M)$ the *grade number* of M. If M is the zero module we define $j_A(M) = \infty$.

1.2. REMARK. Let A be an Auslander-Gorenstein ring. It follows from the Auslander condition that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated A-modules then

$$j_A(M) = \inf\{j_A(M'), j_A(M'')\}.$$

For a proof see [Bj:2, Proposition 1.8].

REMARK. Let K be a commutative field and let S be an Auslander-Gorenstein K-algebra. Then every skew polynomial ring over S is Auslander-Gorenstein. For the proof we refer to [Ek; Theorem 4.2]. In particular, this gives that if S is an Auslander-Gorenstein ring, then so is $R = A_1(K) \otimes_K S$ since R may be regarded as a skew polynomial ring in two variables over S. Moreover, R_0 is a polynomial ring in one variable over S and hence an Auslander-Gorenstein ring.

Let us now consider an Auslander-Gorenstein K-algebra S and set $R = \mathrm{gr}(A_1) \otimes_K S$. Let $M = \oplus M_v$ be a graded and finitely generated R-module. It follows easily that every homogeneous component M_v is a finitely generated R_0-module. So the grade numbers $j_{R_0}(M_v)$ are defined for every $v \in \mathbb{Z}$. Now we have:

1.3. THEOREM. *Let S be an Auslander-Gorenstein K-algebra and set $R = \mathrm{gr}(A_1) \otimes_K S$. Let $M = \oplus M_v$ be a graded and finitely generated R-module. Then*

$$j_R(M) = \inf_{v \in \mathbb{Z}} \{j_{R_0}(M_v)\}$$

Before Theorem 1.4 below is announced we recall that if A is an Auslander-Gorenstein ring then a finitely generated A-module M is *pure* if and only if $j_A(N) = j_A(M)$ for every nonzero submodule N of M.

1.4. THEOREM. *Let S be an Auslander-Gorenstein K-algebra and M a finitely generated and graded R-module. Then the R-module M is pure if and only if each nonzero M_v is a pure R_0-module such that*

$$j_{R_0}(M_v) = j_R(M).$$

1.5. An application of Theorem 1.3

We are going to use Theorem 1.3 in order to compute projective dimensions of graded R-modules in the case $R = \mathrm{gr}(A_1) \otimes_K \mathscr{D}_1$, where $\mathscr{D}_1$ is the K-algebra given by the skew field extension of $A_1(K)$.

First we recall some general facts from [Na-Oy]. In general, let $R = \bigoplus_{v \in \mathbb{Z}} R_v$ be a $\mathbb{Z}$-graded ring. Assume that R is left and right noetherian with a finite global homological dimension. Denote by $G_f(R)$ the family of graded and finitely generated left R-modules.

1.6. DEFINITION. *The graded left global homological dimension of R is the smallest integer w such that* $\mathrm{proj.dim.}_R(M) \leqslant w$ *for every M in $G_f(R)$, where* $\mathrm{proj.dim.}_R$ *denotes the usual projective dimension in the category of R-modules.*

1.7. REMARK. Denote the graded left global homological dimension by l.gr.gl.(R). Replacing left by right we also obtain the graded right global homological dimension given by the smallest integer w' such that $\text{proj.dim.}_R(N) \leqslant w'$ for every finitely generated and graded right R-module N. By the results in chapter 1 of [Na-Oy] we have the equality

$$\text{l.gr.gl}(R) = \text{r.gr.gl}(R) \tag{1.8}$$

Let us only remark that (1.8) is a consequence of the fact that both l.gr.gl(R) and r.gr.gl(R) are equal to the so-called weak graded global dimension, which is the smallest integer w'' such that $\text{Tor}_v^R(N, M) = 0$ when $v > w''$, N is a graded right and M is a graded left R-module.

Using (1.8) we simply put l.gr.gl(R) = gr.gl(R) and refer to gr.gl(R) as the *graded global homological dimension.* By Theorem II.8.2 in [Na-Oy] we have

$$\text{gr.gl}(R) \leqslant \text{gl.dim.}(R) \leqslant \text{gr.gl}(R) + 1 \tag{1.9}$$

Now we are going to prove

1.10. THEOREM. *Let S be the skew field extension of $A_1(K)$. Then, with $R = \text{gr}(A_1) \otimes_K S$ we have*

$$\text{gl.dim.}(R) = 2 \quad \text{and} \quad \text{gr.gl}(R) = 1.$$

So Theorem 1.10 gives an example when the strict inequality gr.gl(R) < gl.dim.(R) holds.

Proof of Theorem 1.10. The equality gl.dim.$(R) = 2$ is proved by Hart in [H]. By (1.9) we get $1 \leqslant \text{gr.gl}(R) \leqslant 2$. So there remains to prove that $\text{gr.gl}(R) \leqslant 1$, i.e. we must show that $\text{proj.dim.}(M) \leqslant 1$ for every finitely generated and graded R-module. To prove this we argue by a contradiction, i.e. assume that there exists a finitely generated and graded R-module M such that proj.dim.$(M) > 1$. Notice that we then obtain proj.dim.$(M) = 2$ since gl.dim.$(R) = 2$. Then, standard homological algebra implies that $\text{Ext}_R^2(M, R)$ is non-zero. Set $N = \text{Ext}_R^2(M, R)$ and notice that N is a graded right R-module. Auslander's condition implies that $j_R(N) \geqslant 2$ and since gl.dim.$(R) = 2$ we must have $j_R(N) = 2$. Now $N = \oplus N_v$ and Theorem 1.3 gives

$$\text{(i)} \quad 2 = \inf_{v \in \mathbf{Z}} \{ j_{R_0}(N_v) \}$$

At this stage we get the contradiction since R_0 is the polynomial ring in one variable over the skew field $\mathscr{D}_1$. Thus, gl.dim.$(R_0) = 1$ so $j_{R_0}(N_v) \leqslant 1$ when $N_v \neq 0$ and (i) cannot hold.

2. Preliminary results

In this section we collect various results about Auslander-Gorenstein rings.

In general, let S be a ring and T one element of S such that T is neither a left nor a right zero-divisor. Moreover, we assume that $ST = TS$ and the twosided ideal generated by T is denoted by (T).

If M is a left S-module such that $TM = 0$, then we get a left $S/(T)$-module structure on M. Conversely, if N is a left $S/(T)$-module we see that N has a left S-module structure such that $TN = 0$. Let μ be the functor from the category of finitely generated left $S/(T)$-modules to the category of finitely generated left S-modules. Similarly we have a functor from the category of finitely generated right $S/(T)$-modules to the category of finitely generated right S-modules.

2.1. PROPOSITION. *Let M be a finitely generated $S/(T)$-module. Then* $\mathrm{Ext}_S^{v+1}(\mu(M), S) = \mu(\mathrm{Ext}_{S/(T)}^{v}(M, S/(T)))$ *for every* $v \geqslant 0$.

Proof. For the case T is a central element of S, this is Theorem 9.37 of [Rot]. When T is not central the same methods can still be applied to prove the proposition.

REMARK. Concerning the inequality

$$\mathrm{inj.dim}_{S/(T)}(S/(T)) \leqslant \mathrm{inj.dim}_S(S) - 1$$

we mention that equality holds if S is commutative. But in the noncommutative case we can have a strict inequality. For example consider the *Weyl algebra* $A_n(K)$ where K is a commutative field of characteristic zero. Using the positive *Bernstein filtration* we construct the *associated Rees ring* denoted by S. If T is the central element in S which corresponds to the identity in $A_n(K)$ put in degree one of the graded ring S, then $S/(1 - T)$ is isomorphic with $A_n(K)$. Moreover, $A_n(K)$ is a ring with a finite global homological dimension which is equal to its injective dimension. Hence we have that

$$n = \mathrm{inj.dim}_{S/(1-T)}(S/(1 - T))$$

On the other hand since $\mathrm{gr}(A_n(K))$ has global homological dimension $2n$, it follows that

$$\mathrm{inj.dim}_S(S) \geqslant 1 + \mathrm{inj.dim}_{S/(T)}(S/(T)) = 1 + 2n$$

Finally, the inequality

$$\mathrm{inj.dim}_S(S) \leqslant 1 + \\ + \sup\{\mathrm{inj.dim}_{S/(T)}(S/(T), \mathrm{inj.dim}_{S/(1-T)}(S/(1 - T))\}$$

is proved in [Ek] and hence we get $\text{inj.dim}_S(S) = 2n + 1$. So we get the strict inequality $n = \text{inj.dim}_{S/(1-T)}(S/(1 - T)) \ll \text{inj.dim}_S(S) = 2n + 1$ for every positive integer n.

2.2. PROPOSITION. *Let M be a finitely generated $S/(T)$-module. Then* $j_S(\mu(M)) = j_{S/(T)}(M) + 1$.

Proof. This follows immediately from Proposition 2.1.

2.3. THEOREM. *Let S be an Auslander-Gorenstein ring containing an element T as above. Then $S/(T)$ is an Auslander-Gorenstein ring.*

Proof. First we observe that the ring $S/(T)$ is left and right noetherian since S is.

Next it follows from Proposition 2.1 that

$$\text{inj.dim}_{S/(T)}(S/(T)) \leqslant \text{inj.dim}_S(S) - 1.$$

Now, let M be a finitely generated $S/(T)$-module and N a submodule of $\text{Ext}^v_{S/(T)}(M, S/(T))$ for some $v > 0$. Then by Proposition 2.1 $\mu(N)$ is an S-submodule of $\text{Ext}^{v+1}_S(\mu(M), S)$ so the Auslander condition of the ring S gives that $j_S(\mu(N)) \geqslant v + 1$. Finally Proposition 2.2 gives that $j_{S/(T)}(N) = j_S(\mu(N)) - 1 \geqslant v + 1 - 1 = v$ and the theorem is proved.

Let us consider a ring A and let $\rho: A \to A$ be a ring automorphism.

2.4. DEFINITION. *Let M be a left A-module and $v \in \mathbb{Z}$. Then $\eta^v(M)$ is the left A-module whose additive group is M and if $a \in A$ and $m \in \eta^v(M)$ then $a * m$ is equal to $\rho^v(a)m$ in the left A-module $\eta^v(M)$.*

It is obvious that $M \mapsto \eta(M)$ is an exact functor on the category of left A-modules and we leave out the detailed verification of the following:

2.5. LEMMA. *Let M be a finitely generated A-module. Then $j_A(M) = j_A(\eta(M))$.*

Next, the automorphism ρ enables us to construct the *skew polynomial ring*

$$B = A[x, \rho]$$

where $xa = \rho(a)x$ for every a in A.

The ring B is graded with $B_v = Ax^v$ for every $v \geqslant 0$ while $B_v = 0$ if $v < 0$.

From now on we assume that A is Auslander-Gorenstein. By [Ek, Theorem 4.2] B is also an Auslander-Gorenstein ring.

Let us now consider a finitely generated and positively graded B-module $M = \bigoplus_{v \geqslant 0} M_v$.

We note that every M_v is a finitely generated A-module.

2.6. LEMMA. *There exists an integer w such that $x\colon M_v \to M_{v+1}$ is bijective for every $v \geqslant w$.*

Proof. Set $K = \{m \in M : xm = 0\}$ and $N = M/xM$. We note that both K and N are graded and finitely generated B-modules such that $xK = xN = 0$. So we find an integer $v \geqslant w$ such that $v \geqslant w$ yields $K_v = N_v = 0$ and Lemma 2.6 follows.

2.7. COROLLARY. *Let M be a graded and finitely generated B-module. Then there exists an integer w such that $v \geqslant w$ yields $j_A(M_v) = j_A(M_w)$.*

Proof. Lemma 2.6 and the construction of the skew polynomial ring B show that if $v \geqslant w$ then the A-modules M_v and $\eta(M_{v+1})$ are isomorphic. Then Lemma 2.5 gives Corollary 2.7.

Using Corollary 2.7 we get

2.8. DEFINITION. *Let $M = \oplus M_v$ be a graded and finitely generated B-module. Then $\lim_{v \to \infty} j_A(M_v)$ exists and is denoted by $j_\infty(M)$.*

2.9. PROPOSITION. *Let M be a finitely generated and graded B-module. Then $j_B(M) = \inf_{v \in \mathbb{Z}}\{j_\infty(M), 1 + j_A(M_v)\}$.*

Proof. Choose w so that $x\colon M_v \to M_{v+1}$ is bijective for all $v \geqslant w$. Set $N = \bigoplus_{v \geqslant w} M_v$. It follows that the graded B-submodule N of M is isomorphic with the graded B-module $B \bigotimes_A M_v$.

Moreover $j_A(M_w) = j_\infty(M)$.

We have the exact sequence $0 \to N \to M \to M/N \to 0$ and the Auslander condition gives:

$$j_B(M) = \inf\{j_B(N), j_B(M/N)\}$$

The equality $j_B(N) = j_A(M_w)$ follows by standard homological algebra.

Finally, M/N is a graded B-module such that $x^{w+k}(M/N) = 0$ for some integer k. Then standard homological algebra gives:

$$j_B(M/N) = 1 + j_A(M/N).$$

The A-module M/N is isomorphic with $\bigoplus_{v<w} M_v$ and hence $j_A(M/N) = \inf_{v<w}\{j_A(M_v)\}$. Now we get Proposition 2.9.

3. Proof of the main results

In this section we set $R = \mathrm{gr}(A_1) \bigotimes_K S$. Here S is an Auslander-Gorenstein K-algebra. Let $M = \oplus M_v$ be a finitely generated and graded R-module.

3.1. LEMMA. *Every homogeneous component M_v is a finitely generated R_0-module.*

Proof. We notice that if $k \geqslant 1$ then $R_k = R_0\bar{\partial}^k$ and $R_{-k} = R_0\bar{x}^k$. Now Lemma 3.1 follows since M is a graded quotient of a free R-module of finite rank.

Next, $R_0 = \mathrm{gr}_0(A_1) \otimes_K S$ and $\mathrm{gr}_0(A_1)$ is the polynomial ring $K\nabla]$ with $\nabla = \bar{x}\bar{\partial}$. Hence $R_0 = S[\nabla]$, i.e. the polynomial ring in one variable over S.

The next result is a crucial step towards the proof of Theorem 1.3.

3.2. LEMMA. *For every $v \in \mathbb{Z}$ and $n \geqslant 1$ we have*

$$j_{S[\nabla]}(M_v) \geqslant \inf\{j_{S[\nabla]}(M_{v-n}), j_{S[\nabla]}(M_{v+n})\}.$$

Proof. Let $v \in \mathbb{Z}$ and $n \geqslant 1$. Set

$$K_{v,n} = \{m \in M_v : \bar{\partial}^n m = 0\} \quad \text{and} \quad T_{v,n} = \{m \in M_v : \bar{x}^n m = 0\}.$$

We notice that $K_{v,n}$ and $T_{v,n}$ both are $S[\nabla]$-submodules of M_v. Now we have:

SUBLEMMA. $K_{v,n} \cap T_{v,n} = 0$.

Proof of Sublemma. Assume the contrary and take $0 \neq m \in K_{v,n} \cap T_{v,n}$. Since $\mathrm{gr}(A_1)$ is a K-subalgebra of R we get the cyclic $\mathrm{gr}(A_1)$-module generated by m. By assumption $\bar{x}^n m = \bar{\partial}^n m = 0$ and this gives $\dim_K(\mathrm{gr}(A_1)m) < \infty$. This gives a contradiction since $\mathrm{gr}(A_1)$ is isomorphic to $A_1(K)$ and by the result from [G-R] there does not exist any non-zero $A_1(K)$-module whose underlying K-space is finite dimensional.

Proof continued. We have the exact sequence:

$$0 \to K_{v,n} \to M_v \to M_v/K_{v,n} \to 0$$

Next, consider the injective map $\bar{\partial}^n : M_v/K_{v,n} \to M_{v+n}$. In the ring R we have $\bar{\partial}^n \nabla = (\nabla + n)\bar{\partial}^n$. Let ρ be the automorphism on the ring $S[\nabla]$ defined by $\rho(s) = s$ for every $s \in S$ and $\rho(\nabla) = \nabla + 1$. Using the notation from 2.4 we have that $\eta^{-n}(M_v/K_{v,n})$ is isomorphic to a $S[\nabla]$-submodule of M_{v+n}. Since $j_{S[\nabla]}(\eta^{-n}(M_v/K_{v,n})) = j_{S[\nabla]}(M_v/K_{v,n})$ it follows from remark 1.2 that $j_{S[\nabla]}(M/K_{v,n}) \geqslant j_{S[\nabla]}(M_{v+n})$.

Next the sublemma implies that $K_{v,n}$ is isomorphic with an $S[\nabla]$-submodule of $M_v/T_{v,n}$. By similar methods as above we have

$$j_{S[\nabla]}(M_v/T_{v,n}) \geqslant j_{S[\nabla]}(M_{v-n}).$$

Now we get

$$\inf\{j_{S[\nabla]}(K_{v,n}), j_{S[\nabla]}(M_v/K_{v,n})\}$$
$$\geqslant \inf\{j_{S[\nabla]}(M_{v,-n}), j_{S[\nabla]}(M_{v+n})\}$$

and then Remark 1.2 gives Lemma 3.2.

3.3. The τ-filtration on R

If $v \geqslant 0$ we set $\tau_v(R) = \bigoplus_{|k|\leqslant v} R_k$. We see that $\{\tau_v(R)\}_{v\geqslant 0}$ is a positive filtration on the ring R. Let $\mathrm{gr}_\tau(R) = \oplus\, \tau_v(R)/\tau_{v-1}(R)$ be the associated graded ring. The image of $\bar{x}$ in $\tau_1(R)/\tau_0(R)$ is denoted by X. Similarly, Y is the image of $\bar{\partial}$ in $\tau_1(R)/\tau_0(R)$.

In order to describe the ring $\mathrm{gr}_\tau(R)$ we also notice that if we set $\tau(v) = \tau_v(R)/\tau_{v-1}(R)$ then $\tau(0) = \tau_0(R) = R_0$ and hence the R_0-element ∇ is identified with an element in $\tau(0)$, i.e. ∇ is homogeneous of degree zero in the graded ring $\mathrm{gr}_\tau(R)$. Moreover, since $\bar{x}\cdot\bar{x}\cdot\bar{\partial} - \bar{x}\cdot\bar{\partial}\cdot\bar{x} = -\bar{x}$ holds in R, it follows that

$$\nabla X - X\nabla = -X \tag{3.4}$$

holds in $\mathrm{gr}_\tau(R)$. Similarly, we obtain

$$\nabla Y - Y\nabla = Y \tag{3.5}$$

Let us now consider the ring $S[\nabla]$. Denote by ρ_1 the automorphism on $S[\nabla]$ defined by $\rho_1(s) = s$ for every $s\in S$ and $\rho_1(\nabla) = \nabla + 1$, then $Xa = \rho_1(a)X$ holds in $\mathrm{gr}_\tau(R)$ for every $a\in S[\nabla] = \tau(0)$. Similarly, if ρ_2 is the $S[\nabla]$-automorphism defined by $\rho_2(s) = s$ and $\rho_2(\nabla) = \nabla - 1$, then $Ya = \rho_2(a)Y$ holds in $\mathrm{gr}_\tau(R)$ for every $a\in S[\nabla]$. Now we notice that the automorphisms ρ_1 and ρ_2 commute and hence we can construct the skew polynomial ring $S[\nabla][z_1, z_2; \rho_1, \rho_2]$. Denote by (z_1z_2) the two-sided ideal generated by z_1z_2. With these notations we have:

3.6. LEMMA. *The ring* $\mathrm{gr}_\tau(R)$ *is isomorphic with*

$$S[\nabla][z_1, z_2; \rho_1, \rho_2]/(z_1z_2).$$

Proof. We notice that (3.4) and (3.5) yield a surjective K-algebra morphism: $\varphi\colon S[\nabla][z_1, z_2; \rho_1, \rho_2] \to \mathrm{gr}_\tau(R)$ which is the identity on $S[\nabla]$ while z_1 is mapped into X and z_2 into Y. Next, in the ring R we have $\bar{x}\bar{\partial}$ in R_0 and then XY belongs to $\tau_0(R)$ so the image of XY in $\tau_2(R)/\tau_1(R)$ is zero. Thus $XY = 0$ in $\mathrm{gr}_\tau(R)$.

Similarly $YX = 0$ in $\mathrm{gr}_\tau(R)$. So the map φ induces a surjective K-algebra morphism ψ from the quotient ring $S[\nabla][z_1, z_2; \rho_1, \rho_2]/(z_1 z_2)$ into $\mathrm{gr}_\tau(R)$. There remains only to see that ψ is injective. To prove the injectivity, we notice that if $v \geqslant 1$ then we have a direct sum decomposition

$$\tau(v) = R_v \oplus R_{-v}.$$

Moreover, $R_v = \bar{\partial}^w R_0$ and $R_{-v} = \bar{x}^v R_0$ for $v > 0$. Then it easily follows that ψ is injective.

3.7. COROLLARY. *The ring* $\mathrm{gr}_\tau(R)$ *is Auslander-Gorenstein.*

Proof. By the remark after Remark 1.2 the skew polynomial ring $S[\nabla][z_1, z_2; \rho_1, \rho_2]$ is Auslander-Gorenstein. With $T = z_1 z_2$ we apply Theorem 2.3 and then Lemma 3.6 gives Corollary 3.7.

Let us now consider a finitely generated and graded R-module $M = \oplus M_v$. Set

$$\Gamma_v = \bigoplus_{|k| \leqslant v} M_k$$

We notice that $\tau_j(R)\Gamma_v \subset \Gamma_{j+v}$ and hence $\{\Gamma_v\}$ is a filtration on M when R is equipped with the positive τ-filtration. Set $\Gamma(v) = \Gamma_v/\Gamma_{v-1}$ and then we notice that

$$\Gamma(v) = M_v \oplus M_{-v}, \qquad v \geqslant 1 \text{ while } \Gamma(0) = M_0 \tag{3.9}$$

3.10. LEMMA. $\oplus\, \Gamma_v/\Gamma_{v-1}$ is a finitely generated $\mathrm{gr}_\tau(R)$-module.

Proof. From Lemma 3.1 it follows that $\Gamma(v)$ is a finitely generated R_0-module for every v. Moreover, since M is a graded and finitely generated R-module it follows that there exists a non-negative integer w such that $v \geqslant w$ yields $M_{v+1} = \bar{\partial} M_v$ and $M_{-v-1} = \bar{x} M_{-v}$. Then Lemma 3.10 follows easily.

At this stage we are going to use some results concerned with filtered rings. First, the τ-filtration on the Auslander-Gorenstein ring R is positive, i.e. $\tau_v = 0$ if $v < 0$. Then the material in [Bj:2] yields

$$j_R(M) = j_{\mathrm{gr}_\tau(R)}(\mathrm{gr}_\Gamma(M)) \tag{3.11}$$

where $\mathrm{gr}_\Gamma(M) = \oplus\, \Gamma_v/\Gamma_{v-1}$ is the finitely generated and graded $\mathrm{gr}_\tau(R)$-module from Lemma 3.10.

Hence Theorem 1.3 will follow if we have proved:

3.12. PROPOSITION. We have that

$$j_{\mathrm{gr}_\tau(R)}(\mathrm{gr}_\Gamma(M)) = \inf_{v \in \mathbb{Z}} \{j_{R_0}(M_v)\}.$$

Proof. By (3.9) we have

$$\mathrm{gr}_\Gamma(M) = \bigoplus_{v \geqslant 0} \Gamma(v)$$

where $\Gamma(v) = M_v \oplus M_{-v}$ if $v \geqslant 1$ and $\Gamma(0) = M_0$.

Put $N_+ = \bigoplus_{v \geqslant 1} M_v$. We have that the $\mathrm{gr}_\tau(R)$-elements X and Y from (3.4) and (3.5) which are images of $\bar{x}$ respectively $\bar{\partial}$. Since $\bar{x}M_v \subset M_{v-1}$ hold for every $v \geqslant 1$, it follows that $XN_+ = 0$. Also, since $\bar{\partial}M_v \subset M_{v+1}$ we obtain $YN_+ \subset N_+$ and conclude that N_+ is a graded $\mathrm{gr}_\tau(R)$-submodule of $\mathrm{gr}_\Gamma(M)$ annihilated by X.

Next, put $B = S[\nabla][z_1, z_2; \rho_1, \rho_2]$ and recall from Lemma 3.6 that $\mathrm{gr}_\tau(R) = B/(z_1, z_2)$. So we can consider N_+ as a left B-module and since $XN_+ = 0$ we obtain $z_1 N_+ = 0$.

Now Proposition 2.2 gives

(a) $j_B(N_+) = 1 + j_{B/(z_1)}(N_+)$

Next, $B/(z_1)$ is isomorphic with $S[\nabla][z_2; \rho_2]$ which implies that N_+ also is a graded $S[\nabla][z_2; \rho_2]$-module. Now Proposition 2.9 gives

(b) $j_{B/(z_1)}(N_+) = \inf_{v \geqslant 1} \{j_\infty(N_+), 1 + j_{S[\nabla]} M_v)\}$

where $j_\infty(N_+) = \lim_{v \to +\infty} j_{S[\nabla]}(M_v)$.

Let us also consider the element $z_1 z_2$ in the ring B. Another application of Proposition 2.9 gives

(c) $j_B(N_+) = j_{B/(z_1 z_2)}(N_+) + 1 = j_{\mathrm{gr}_\tau(R)}(N_+) + 1$

Finally, (a), (b) and (c) yield

(d) $j_{\mathrm{gr}_\tau(R)}(N_+) = \inf_{v \geqslant 1} \{j_\infty(N_+), 1 + j_{S[\nabla]}(M_v)\}$

Let us now consider the quotient module $\mathrm{gr}_\Gamma(M)/N_+$ and denote it by N_-. We notice that N_- is a graded $\mathrm{gr}_\tau(R)$-module with $N_- = \bigoplus_{v \geqslant 0} N_-(v)$ where the $S[\nabla]$-module $N_-(v)$ is equal to M_{-v} for every $v \geqslant 0$. Since $\bar{\partial}M_{-v} \subset M_{-v+1}$ for every $v \geqslant 0$ we easily get $YN_- = 0$. Repeating the steps (a), (b) and (c) reversing the roles between X and T we obtain.

(e) $j_{\mathrm{gr}_\tau(R)}(N_-) = \inf_{v \geqslant 0} \{j_\infty(N_-), 1 + j_{S[\nabla]}(M_{-v}\}$

where $j_\infty(N_-) = \lim_{v \to +\infty} j_{S[\nabla]}(M_{-v})$.

At this stage we are going to use Lemma 3.2 to finish the proof of Proposition 3.12. Namely, Lemma 3.2 yields

(f) $\inf\{j_\infty(N_+), j_\infty(N_-)\} = \inf_{v \in \mathbb{Z}} \{j_{S[\nabla]}(M_v)\}$

Then (d), (e) and (f) yield

(g) $\inf\{j_{\mathrm{gr}_\tau(R)}(N_+), j_{\mathrm{gr}_\tau(R)}(N_-)\} = \inf_{v \in \mathbb{Z}} \{j_{S[\nabla]}(M_v)\}$

Finally, Remark 1.2 and the exact sequence

$$0 \to N_+ \to \mathrm{gr}_\Gamma(M) \to N_- \to 0$$

yield

$$j_{\mathrm{gr}_\tau(R)}(\mathrm{gr}_\Gamma(M)) = \inf\{j_{\mathrm{gr}_\tau(R)}(N_+), j_{\mathrm{gr}_\tau(R)}(N_-)\}$$

So the equality in (g) gives Proposition 3.12.

We have now finished the proof of Theorem 1.3. Let us now consider a finitely generated left R_0-module N. We obtain a graded left R-module $\mathcal{N} = R \otimes_{R_0} N$, where we remark that if k is some integer then the homogeneous component R_k in the ring R is an R-module and thus $R_k \otimes_{R_0} N$ is a left R_0-module which is equal to the k-th homogeneous component of the graded R-module $\mathcal{N}$. Now Theorem 1.3 gives

$$j_R(\mathcal{N}) = \inf_{v \in \mathbb{Z}} \{j_{R_0}(\mathcal{N}_v)\} \tag{3.13}$$

Concerning the right hand side in (3.13) we have the following:

3.14. LEMMA. $\inf_{v \in \mathbb{Z}} \{j_{R_0}(\mathcal{N}_v)\} = j_{R_0}(N)$.

Proof. Recall that $R_0 = S[\nabla]$ and let ρ be the ring-automorphism on $S[\nabla]$ defined by $\rho(s) = s$ for every $s \in S$ and $\rho(\nabla) = \nabla + 1$. Next, if $v \geqslant 1$ then $R_v = \bar{\partial}^v R_0$ and using the η-functor from Definition 2.4 we find that the R-

module N_v is isomorphic to $\eta^{-v}(N)$ and hence $j_{R_0}(N_v) = j_{R_0}(N)$. Next, we have $R_{-v} = \bar{x}^v R_0$ and find that the R_0-modules N_{-v} and $\eta^v(N)$ are isomorphic. So again $j_{R_0}(N_{-v}) = j_{R_0}(N)$ and Lemma 3.14 follows.

3.15. *Proof of Theorem* 1.4. Let $M = \oplus M_v$ be a finitely generated and graded R-module. Assume first that every non-zero m_v is a pure R_0-module with $j_{R_0}(M_v) = j_R(M)$. Then we show that the R-module M is pure. To prove this we apply the theory in [Ek: §3] which implies that M is a pure R-module if $j_R(N) = j_R(M)$ for every non-zero and *graded* submodule N of M. If $N = \oplus N_v$ is a graded R-submodule then the hypothesis yields $j_{R_0}(N_v) = j_R(M)$ when $N_v \neq 0$. Now we see that Theorem 1.3 gives $j_R(N) = j_R(M)$.

Conversely, assume that M is a pure R-module. Then, if some non-zero homogeneous component M_v fails to be a pure R_0-module with $j_{R_0}(M_v) = j_R(M)$, it follows that there exists an integer v and a non-zero R_0-submodule N of M_v such that $j_{R_0}(N) > j_R(M)$. Using N, we construct the graded left R-module

$$\mathcal{N} = R \otimes_{R_0} N$$

where $\mathcal{N}_k = R_{k-v} \otimes_{R_0} N_v$ for every k. Lemma 3.14 gives $j_R(\mathcal{N}) = j_{R_0}(N)$. Next, we notice that there exists an R-linear map $\varphi\colon \mathcal{N} \to M$ such that $\varphi(\mathcal{N})$ is the graded R-submodule of M generated by N. The purity of M gives $j_R(\varphi(\mathcal{N})) = j_R(M)$. On the other hand we have $j_R(\varphi(\mathcal{N})) \geqslant j_R(N) > j_R(M)$ which is a contradiction.

4. Applications

Let $R = \mathrm{gr}(A_1) \otimes_K S$ where S is an Auslander-Gorenstein K-algebra and let $M = \oplus M_v$ be a finitely generated and graded R-module. In Proposition 3.1 we noticed that every homogeneous component M_v is a finitely generated $\mathrm{gr}_0(R)$-module where $\mathrm{gr}_0(R)$ is the polynomial ring $S[\nabla]$.

In the special case when every M_v is a finitely generated S-module we say that the graded R-module M is *strongly finitely generated.*

We shall give examples of strongly finitely generated R-modules in 4.5. But first we establish the following:

4.1. THEOREM. *Let $M = \oplus M_v$ be a graded and strongly finitely generated R-module. Then $j_R(M) = 1 + \inf_{v \in \mathbb{Z}} \{j_S(M_v)\}$.*

Proof. In general, if N is an $S[\nabla]$-module such that its underlying S-module is finitely generated then

$$j_S(N) = j_{S[\nabla]}(N) - 1$$

by Proposition 2.2. Now we see that Theorem 1.3 gives Theorem 4.1.

4.2. V-filtrations on rings of differential operators

Following example 1.5.5 in Chapter II of [Sch] we consider a non-singular complex analytic hypersurface Y of a complex manifold X. Let $\mathcal{T}_Y$ be the ideal of $\mathcal{O}_X$ whose sections vanish on Y, that is, the defining ideal of Y in X. If $k \in \mathbb{Z}$ we set

$$F_k\mathcal{D}_X = \{P \in \mathcal{D}_X : P\mathcal{T}_Y^v \subset \mathcal{T}_Y^{v-k} \text{ for all } v\}$$

Then $\{F_k\mathcal{D}_X\}$ is a filtration on $\mathcal{D}_X$, called *the V-filtration of* $\mathcal{D}_X$ *along* Y. If $\dim(X) = n+1$ and $(x_1, \ldots, x_n, t)$ are local coordinates such that $Y = \{(x,t): t = 0\}$ then $F_0\mathcal{D}_X$ is the subring of $\mathcal{D}_X$ generated by the zero-order differential operators and $D_{x_1}, \ldots, D_{x_n}$ and tD_t. If $k \geqslant 1$ we have

$$F_{-k}\mathcal{D}_X = t^k F_0\mathcal{D}_X$$

and

$$F_k\mathcal{D}_X = D_t^k F_0\mathcal{D}_X + D_t^{k-1}F_1\mathcal{D}_X + \cdots + D_t F_{k-1}\mathcal{D}_X$$

Moreover, let $g\mathcal{D}_X = \oplus F_k\mathcal{D}_X / F_{k-1}\mathcal{D}_X$ be the associated graded sheaf of rings. Then

$$G\mathcal{D}_X = \mathrm{gr}(A_1(C)) \otimes_C i^{-1}\mathcal{D}_Y$$

where $i\colon Y \to X$ is the closed embedding and $\mathrm{gr}(A_1(C))$ is the graded one-dimensional Weyl algebra where $\deg(t) = -1$ and $\deg(D_t) = 1$. If $(x_0, 0)$ is a point in Y then the stalk $\mathcal{D}_Y(x_0)$ is the ring $\mathcal{D}_n$ of differential operators with coefficients in the local ring $\mathcal{O}_n$. We see that the stalk

$$G\mathcal{D}_X(x_0, 0) = \mathrm{gr}(A_1(C)) \otimes_C \mathcal{D}_n$$

and recall that $\mathcal{D}_n$ is an Auslander-Gorenstein K-algebra whose global homological dimension is equal to n. So with $S = \mathcal{D}_n$ we can apply the main results to stalks of coherent and graded $G\mathcal{D}_X$-modules.

To get specific examples we first consider a coherent left $\mathcal{D}_X$-module $\mathcal{M}$. Working locally if necessary we assume that $\mathcal{M}$ has a good V-filtration which by definition consists of an increasing sequence $\{F_k\mathcal{M}\}$ such that there exists locally a finite set of generators $m_1, \ldots, m_s$ of the $\mathcal{D}_X$-module $\mathcal{M}$ and integers $k_1 \cdots k_s$ with

$$F_k\mathcal{M} = F_{k-k_1}(\mathcal{D}_X)m_1 + \cdots + F_{k-k_s}(\mathcal{D}_X)m_s$$

for every integer k. Then $G\mathcal{M} = \oplus F_k\mathcal{M}/F_{k-1}\mathcal{M}$ is a coherent and graded $G\mathcal{D}_X$-module. So for every $(x_0, 0)$ in Y we get the grade number

$$j_{G\mathcal{D}_X(x_0,0)}(G\mathcal{M}(x_0, 0))$$

Theorem 1.3 shows that this grade number equals

$$\inf_{v\in\mathbb{Z}} \{j_{\mathrm{gr}_0(G\mathcal{D}_X(x_0,0))}(F_v\mathcal{M}(x_0, 0)/F_{v-1}\mathcal{M}(x_0, 0)\}$$

Next, following Definition 1.6.1 in Chapter III of [Sch] we say that the coherent $\mathcal{D}_X$-module $\mathcal{M}$ is *elliptic* along Y if the following holds: the $\mathcal{D}_X$-module $\mathcal{M}$ is generated by a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0$ such that locally there exists polynomials $b = \nabla^n + a_1(x, t)\nabla^{m-1} + \cdots + a_m(x, t)$ where $\nabla = tD_t$ and $a_v(x, t)\in\mathcal{O}_X$ and $b\mathcal{M}_0 \subset F_{-1}(\mathcal{D}_X)\mathcal{M}_0$.

Following [Sch, p. 134] we have the result below:

4.3. PROPOSITION. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is elliptic along the hypersurface Y. Then for every good V-filtration $\{F_k\mathcal{M}\}$ and $(x_0, 0)\in Y$, it follows that $\oplus F_k\mathcal{M}(x_0, 0)/F_{k-1}\mathcal{M}(x_0, 0)$ is a strongly finitely generated $G\mathcal{D}_X(x_0, 0)$-module.*

Now Theorem 4.1 can be applied and identifying the stalk $\mathcal{D}_Y(x_0)$ with the ring $\mathcal{D}_n$ we obtain

4.4. COROLLARY. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is elliptic along Y and $\{F_k\mathcal{M}\}$ a good V-filtration. Then* $j_{G\mathcal{D}_X(x_0,0)}(G\mathcal{M}(x_0, 0)) = \inf_{k\in\mathbb{Z}} \{j_{\mathcal{D}_Y(x_0)}(F_k\mathcal{M}(x_0, 0)/F_{k-1}\mathcal{M}(x_0, 0))\}$

for every $(x_0, 0)\in Y$.

4.5. REMARK. In the special case when $\mathcal{M}$ is a coherent and pure $\mathcal{D}_X$-module, i.e. when every stalk $\mathcal{M}(x_0)$ is a pure $\mathcal{D}_X(x_0)$-module where x_0 belongs to $\mathrm{Supp}\,\mathcal{M}$, it follows by the general result in [Bj: 2, Theorem 3.8] that there locally exists a good V-filtration $\{F_k\mathcal{M}\}$ such that if $G\mathcal{M} = \oplus F_k\mathcal{M}/F_{k-1}\mathcal{M}$ and $(x_0, 0)\in\mathrm{Supp}\,G\mathcal{M}$ then $G\mathcal{M}(x_0, 0)$ is a pure $G\mathcal{D}_X(x_0, 0)$-module with

$$j_{G\mathcal{D}_X(x_0,0)}(G\mathcal{M}(x_0, 0)) = j_{\mathcal{D}_X(x_0)}(\mathcal{M}(x_0)).$$

So in the special case when $\mathcal{M}$ is pure along Y we can use such a special good V-filtration and applying Theorem 1.4 it follows first that if $(x_0, 0)\in\mathrm{Supp}\,G\mathcal{M}$, then

$$j_{\mathrm{gr}_0 G\mathcal{D}_X(x_0,0)}(F_k\mathcal{M}(x_0, 0)/F_{k-1}\mathcal{M}(x_0, 0))$$

is equal to $j_{\mathscr{D}_X(x_0)}(\mathscr{M}(x_0)$ for every k such that $(x_0, 0)$ belongs to $\text{Supp}(F_k\mathscr{M}/F_{k-1}\mathscr{M})$.

Finally, if $\mathscr{M}$ is both pure and elliptic then we first notice that Theorem 1.4 and the actual proof of Theorem V.1 implies that if $(x_0, 0) \in \text{Supp}(F_k\mathscr{M}/F_{k-1}\mathscr{M})$, then the $\mathscr{D}_Y(x_0)$-module $F_k\mathscr{M}(x_0, 0)/F_{k-1}\mathscr{M}(x_0, 0)$ is pure with grade number

$$j_{\mathscr{D}_Y(x_0)}(F_k\mathscr{M}(x_0, 0)/F_{k-1}\mathscr{M}(x_0, 0)) = j_{\mathscr{D}_X(x_0)}(\mathscr{M}) - 1 \tag{4.6}$$

Let us remark that if $\mathscr{M}$ is a *holonomic* $\mathscr{D}_X$-module then $\mathscr{M}$ is elliptic along every non-singular hypersurface Y. For a detailed proof of this fact we refer to the article [L-S].

Also, any holonomic $\mathscr{D}_X$-module is pure and the equality in (4.6) yields

$$j_{\mathscr{D}_Y(x_0)}(F_k\mathscr{M}(x_0, 0)/F_{k-1}\mathscr{M}(x_0, 0)) = j_{\mathscr{D}_X(x_0)}(\mathscr{M}(x_0)) - 1 = \dim Y$$

where $(x_0, 0) \in \text{Supp}(F_k\mathscr{M}/F_{k-1}\mathscr{M})$. This means that $F_k\mathscr{M}/F_{k-1}\mathscr{M}$ is a holonomic $\mathscr{D}_Y$-module for every integer k. We remark that this conclusion is already contained in the work [L-S] while our main results yield new facts concerned with non-holonomic $\mathscr{D}_X$-modules and their associated $G_{\mathscr{D}_X}$-modules obtained by good V-filtrations.

References

[Bj: 1] Björk, J.-E., Filtered Noetherian Rings. Article in Noetherian rings and their applications. *Math. Surveys and Monographs.* AMS 24 (1987) 51–98.

[Bj: 2] Björk, J.-E., The Auslander condition on noetherian rings. Sem Dubreil-Malliavin. *Lecture Notes in Math.* 1404 (1987–88) 137–173.

[Bj: 3] Björk, J.-E., *Rings of Differential Operators.* North-Holland Math. Libr. Series Vol. 21 (1979).

[Ek] Ekström, E. K., The Auslander condition on graded and filtered Noetherian rings. Sem. Dubreil-Malliavin. Springer, *Lecture Notes in Math.* 1404 (1987–88) 220–245.

[G-R] Gabriel, P. and Rentschler, R., Sur la dimension des anneaux et ensembles ordonné. *C.R. Acad. Sci. Ser. A* 265 (1967) 712–715.

[G-H-L] Goodearl, K. R., Hodge, T. J. and Lenagan, T. H., Krull and Global dimensions of Weyl algebras over Division rings. *J. of Algebra* 91 (1984) 334–359.

[H] Hart, R., A note on tensor product of algebras. *J. of Algebra* 21 (1972) 422–427.

[Lau] Laumon, G., $\mathscr{D}$-modules filtrés. *Soc. Math. France Asterisque.* 130 (1985) 334–359.

[L-S] Laurent, Y. and Schapira, P., Images inverses des modules differentiels. *Compositio Math.* 61 (1987) 229–251.

[M-R] McConnell, J. C. and Robson, J. C., *Noncommutative Noetherian rings.* Wiley series in pure and applied Mathematics (1987).

[Na-Oy] Nastasescu, C. and Van Oystaeyen, F., *Graded ring theory.* North-Holland. Math. Libr. series. (1982).

[Rot] Rotman, J. J., *An introduction to homological algebra.* Academic Press, New York (1982).

[Sch] Schapira, P., *Microdifferential systems in the complex domain.* Grundlehren der Math. 269 Springer Verlag (1985).

[Za] Zaks, A., Injective dimension of semi-primary rings. *J. of Algebra* 13 (1969) 73–86.