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On periods and quasi-periods of Drinfeld modules

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Introduction

Let \mathcal{C} be a smooth projective, geometrically irreducible curve over a finite field \mathbb{F}_q , $q = p^n$. We fix a closed point ∞ on \mathcal{C} , and consider the ring A of functions on \mathcal{C} regular away from ∞ . We set k to be the function field of \mathcal{C} and k_∞ its completion at ∞ . After taking algebraic closure, we obtain the field \bar{k}_∞ whose elements will be called “numbers”. We fix an embedding $\bar{k} \subset \bar{k}_\infty$ throughout.

We are interested in transcendental numbers (i.e. elements in \bar{k}_∞ transcendental over k) which arise naturally from algebro-geometric objects defined over \bar{k} . Thus our aim is to develop a theory in characteristic p which is analogous to the classical transcendence theory of abelian integrals. The algebro-geometric objects we have in mind are the Drinfeld A -modules (elliptic modules) introduced by V.G. Drinfeld in [5], 1973. One can associate periods to such Drinfeld A -modules of characteristic ∞ , and we have shown in [10] that if a given Drinfeld A -module is defined over \bar{k} , then all its periods are transcendental. This result is parallel to the well-known theorem of Siegel-Schneider, on elliptic integrals of first kind.

Our purpose here is twofold. First, we shall extend our previous work to deal with higher-dimension Drinfeld modules. More specifically, we shall study the transcendence properties of the abelian t -modules. We shall prove in particular that, for period vectors of abelian t -modules defined over \bar{k} , at least one coordinate component is transcendental.

The second purpose is to extend transcendence theory to periods of the second kind. Just recently, basing on an idea of P. Deligne, a very interesting theory of quasi-periods for Drinfeld modules emerges from the work of G. Anderson [2]. With this we shall prove that all quasi-periods are transcendental, once the (dimension one) Drinfeld A -module in question is defined over \bar{k} . This parallels completely the classical work of Schneider on elliptic integrals of the second kind.

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For brevity, we shall restrict ourselves here only to the case of dimension one quasi-periods theory.

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1. Background of Drinfeld modules

Following [10], we first introduce the concept of E_q -functions. We denote by $d(a)$ the additive valuation of $a \in k$ which equals the order of pole of a at ∞ times the degree of ∞ . As usual we extend this valuation to \bar{k}_∞ . If $\alpha \in \bar{k} \subset \bar{k}_\infty$, the maximum of the valuations of all the conjugates of α is said to be the size of α , noted by $\overline{|\alpha|}$.

Let K be a finite extension of k . We say that an entire function $f: \bar{k}_\infty \rightarrow \bar{k}_\infty$ is an E_q -function with respect to K if it has the following properties:

(i) It is additive and it has the form

$$f(z) = \sum_{h=0}^{\infty} b_h z^{q^h}, \quad b_h \in K \quad \text{with } \overline{|\overline{b_h}|} \ll 1.$$

(ii) It has finite growth order, i.e. there exists real $\rho > 0$ such that

$$\max_h (d(b_h) + q^h r) \leq q^{\rho r} \quad \text{for all rationals } r \text{ large.}$$

(iii) There exists a sequence (a_h) in A satisfying

- (1) $d(a_h) \leq hq^h$
- (2) For all $j \leq h$, $a_h b_j$ are integral over A .
- (3) If $q^{h_1} + \dots + q^{h_s} < q^N$, then $a_{h_1} \dots a_{h_s} | a_N$.

These E_q -functions behave like classical functions satisfying algebraic differential equations, and we have proved the following basic theorem in [10]:

THEOREM 1.1. *Let K/k be a finite extension. Let f_1, f_2 be E_q -functions with respect to K which are algebraically independent over \bar{k} . Then there are only finitely many points at which f_1, f_2 , simultaneously assume values in K .*

All interesting examples of E_q -functions are related to Drinfeld's theory. Let τ be the Frobenius map $X \mapsto X^q$. Let $\bar{k}_\infty\{\tau\}$ be the non-commutative polynomial ring generated by τ over \bar{k}_∞ under composition (i.e. the ring of \mathbb{F}_q -linear endomorphisms of the additive group \mathbb{G}_a). Recall that a Drinfeld A -module ϕ is a \mathbb{F}_q -linear ring homomorphism from the Dedekind ring A into $\bar{k}_\infty\{\tau\}$ such that

for a suitable positive integer n and all $a \neq 0$ in A

$$\phi(a) = a\tau^0 + \sum_{j=1}^{nd(a)} \phi(a)_j \tau^j, \quad \phi(a)_{nd(a)} \neq 0.$$

The integer n is said to be the rank of ϕ . What makes such a homomorphism more significant is the fact that \mathbb{G}_a , together with the A -action given by ϕ , can be parametrized by an unique entire exponential function e_ϕ , in the sense that the following identities are satisfied:

$$e_\phi(az) = \phi(a)(e_\phi(z)), \quad \text{for all } a \in A$$

$$e'_\phi(z) \equiv 1.$$

One can deduce from here that once the Drinfeld A -module ϕ is defined over \bar{k} (i.e. all the coefficients $\phi(a)_j$ lie in $\bar{k} \subset \bar{k}_\infty$), then $e_\phi: \bar{k}_\infty \rightarrow \bar{k}_\infty$ is a E_q -function with respect to some finite extension of k , cf. Theorem 3.3 in [10].

Let L_ϕ be the zero set of the exponential function e_ϕ . This is always a finitely generated discrete A -submodule of \bar{k}_∞ (considered as $\text{Lie } \mathbb{G}_a$). Its projective A -rank equals the rank of the Drinfeld A -module ϕ . We call L_ϕ the period lattice, and any non-zero element in it is called a period of the Drinfeld A -module ϕ . By applying Theorem 1.1, we have shown in [10] that all the periods are transcendental if ϕ is defined over \bar{k} .

As an illustration, we shall extract one more application of Theorem 1.1 to periods. Recall that if ϕ_1, ϕ_2 are two Drinfeld A -modules, a morphism from ϕ_1 to ϕ_2 is an element $P \in \bar{k}_\infty \{\tau\}$ satisfying $P \circ \phi_1(a) = \phi_2(a) \circ P$, for all $a \in A$. A non-zero morphism is called an isogeny. If there exists isogeny from ϕ_1 to ϕ_2 , there also exists isogeny from ϕ_2 to ϕ_1 , and we say ϕ_1 is isogenous to ϕ_2 . Given isogenous Drinfeld A -modules ϕ_1 and ϕ_2 , they must have the same rank. If both of them are defined over \bar{k} , then one can always find an isogeny P with coefficients in \bar{k} . It follows $P' \in \bar{k} = \bar{k}\tau^0$, $P' \neq 0$ and $P' L_{\phi_1} \subset L_{\phi_2}$. Thus given any period ω_1 of ϕ_1 , there exists period ω_2 of ϕ_2 such that the ratio ω_1/ω_2 is algebraic. This, however, will never happen if ϕ_1 is not isogenous to ϕ_2 .

THEOREM 1.2. *Let ϕ_1 and ϕ_2 be Drinfeld A -modules defined over \bar{k} . Suppose there exists $\omega_1 \in L_{\phi_1} - \{0\}$ and $\omega_2 \in L_{\phi_2} - \{0\}$ such that $\omega_1/\omega_2 \in \bar{k}$. Then ϕ_1 is isogenous to ϕ_2 .*

Proof. Let K be a common field of definition for ϕ_1 and ϕ_2 , finite over k . Let $\omega_2 = \lambda\omega_1$. Then the functions $f_1(z) = e_{\phi_1}(z), f_2(z) = e_{\phi_2}(\lambda z)$ are E_q -functions with respect to $K(\lambda)$. Since

$$f_1(a\omega_1) = f_2(a\omega_1) = 0 \in K(\lambda), \quad \text{for all } a \in A,$$

Theorem 1.1 implies that $e_{\phi_1}(z)$ and $e_{\phi_2}(\lambda z)$ are algebraically dependent functions over \bar{k} .

By a well-known theorem of E. Artin (cf. [7], Chap. VIII), one can then find non-trivial algebraic relations of the form

$$\sum_{i=0}^l \alpha_i e_{\phi_1}(z)^{p^i} + \sum_{j=0}^m \beta_j e_{\phi_2}(\lambda z)^{p^j} \equiv 0.$$

Thus, if $\omega \in L_{\phi_1}$, $\omega_1 \neq 0$ and $a \in A$, all the values $e_{\phi_2}(\lambda a \omega)$ must be among the finitely many roots of the additive equation

$$\sum_{j=0}^m \beta_j X^{p^j} = 0.$$

Hence there exists $a \neq 0$ in A such that $a\lambda\omega \in L_{\phi_2}$. Let ω run over a finite set of generators of L_{ϕ_1} . We then get $a_0 \in A$, $a_0 \neq 0$ such that $a_0\lambda L_{\phi_1} \subset L_{\phi_2}$. Similarly, one can also get $a_1 \neq 0$ in A such that $a_1\lambda^{-1}L_{\phi_2} \subset L_{\phi_1}$. This shows that the two Drinfeld A -modules ϕ_1 and ϕ_2 have the same rank. Also, multiplication by $a_1\lambda$ induces an isogeny from ϕ_1 to ϕ_2 . □

The above proof actually leads to a more general theorem.

THEOREM 1.3. *Let ϕ_1 and ϕ_2 be non-isogenous Drinfeld A -modules defined over \bar{k} . Let $u_1, u_2 \in \bar{k}_\infty - \{0\}$ satisfying $e_{\phi_1}(u_1) \in \bar{k}$ and $e_{\phi_2}(u_2) \in \bar{k}$. Then u_1/u_2 is transcendental.*

2. Abelian t -modules and transcendence

We shall consider abelian t -modules introduced by G. Anderson in [1]. Let T be a non-constant element in A . Let \tilde{K} be either \bar{k} or \bar{k}_∞ , viewed as $\mathbb{F}_q[t]$ -algebra via $t \mapsto T$. By a t -module defined over \tilde{K} , we mean a pair consisting of an algebraic group E defined over \tilde{K} and an \mathbb{F}_q -linear ring homomorphism $\phi: \mathbb{F}_q[t] \rightarrow \text{End}_{\mathbb{F}_q} E$ such that the following properties are satisfied:

- (i) There is an isomorphism of E onto \mathbb{G}_a^d which identifies $\phi(\mathbb{F}_q)$ with scalar multiplications on \mathbb{G}_a^d .
- (ii) $(\phi(t)_* - T I)^N \text{Lie}(E) = 0$ for some integer $N > 0$.

We let \mathbb{F}_q act on E by $\phi(\mathbb{F}_q)$, and let $\text{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$ be the \tilde{K} -vector space of \mathbb{F}_q -linear algebraic group homomorphisms over \tilde{K} . We say that the t -module (E, ϕ) is an abelian t -module if there exists a finite-dimensional subspace W in $\text{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$ such that

$$\text{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a) = \sum_{j=0}^{\infty} W \circ \phi(t^j).$$

Let $(E_1, \phi_1), (E_2, \phi_2)$ be two t -modules. A \mathbb{F}_q -linear morphism $f: E_1 \rightarrow E_2$ which commutes with the t -action is said to be a morphism of the t -modules. To each t -module $E = (E, \phi)$, one can associate functorially an exponential map

$$\exp_E: \text{Lie } E(\bar{k}_\infty) \rightarrow E(\bar{k}_\infty).$$

Expressed in terms of a given coordinate system (i.e. fixed isomorphism of E onto \mathbb{G}_a^d over \bar{K} identifying $\phi(\mathbb{F}_q)$ with scalars), this exponential map becomes an entire \mathbb{F}_q -linear map e_E from \bar{k}_∞^d to \bar{k}_∞^d satisfying the equation

$$e_E(\phi(t)_*(z)) = \phi(t)(e_E(z)).$$

We let t act on $\text{Lie } E(\bar{k}_\infty)$ via $\phi(t)_*$. The $\ker(\exp_E)$ is always a discrete $\mathbb{F}_q[t]$ -submodule in $\text{Lie } E(\bar{k}_\infty)$. We call $\ker(\exp_E)$ the period lattice of the t -module $E = (E, \phi)$, and any non-zero element in it is called a period vector of E . If (E, ϕ) is abelian, then its period lattice is always free of finite rank over $\mathbb{F}_q[t]$ (cf. Anderson [1], Lemma 2.4.1).

EXAMPLES:

- (I) The trivial t -module. Let $E = \mathbb{G}_a$ and let t act as scalar multiplication by T . This is not an abelian t -module. The exponential here is just $e(z) = z$.
- (II) Any Drinfeld A -module can be considered as abelian t -module with $E = \mathbb{G}_a$ and Drinfeld's exponential as the exponential map. In fact, all one-dimensional abelian t -modules arise in this way.
- (III) A very interesting class of higher dimensional abelian t -modules is given by the tensor powers of the Carlitz module $E_c^{\otimes m}$. The underlying algebraic group of $E_c^{\otimes m}$ is \mathbb{G}_a^m . The homomorphism ϕ is given by

$$\phi(t): \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} \mapsto \begin{pmatrix} T & 1 & 0 \\ & \ddots & 1 \\ 0 & & T \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 0 \end{pmatrix} \begin{pmatrix} X_1^q \\ \vdots \\ X_m^q \end{pmatrix},$$

where the first square matrix on the right-hand side is the standard Jordan block, the second square matrix is the elementary one with the lower left corner equal to 1. The exponential map for $E = E_c^{\otimes m}$ is thus characterized by the condition

$$e_E \left(\begin{pmatrix} T & 1 & 0 \\ & \ddots & 1 \\ 0 & & T \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} \right) = \phi(t)(e_E(z)).$$

By a Theorem of Anderson-Thakur [3], the period lattice of $E_c^{\otimes m}$ is a rank one $\mathbb{F}_q[t]$ -module generated by $\omega(m)$, with the last coordinate component of $\omega(m)$ equal to $\tilde{\pi}^m$, where $\tilde{\pi}$ is the period of the Carlitz module given by

$$\tilde{\pi} = (T - T^q)^{1/(q-1)} \prod_{i=1}^{\infty} \left(1 - \frac{T^{q^i} - T}{T^{q^{i+1}} - T} \right).$$

If m is a power of p , then one simply has $\omega(m) = (0, \dots, 0, \tilde{\pi}^m)$.

To study general abelian t -modules, we consider the \tilde{K} -vector space $\text{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$ as finitely generated $\tilde{K}[t]$ -module, via the t -action $f \mapsto f \circ \phi(t)$. It is also finitely generated as $\tilde{K}[t^n]$ -module, for any $n > 0$.

LEMMA 2.1. *Let (E, ϕ) be an abelian t -module over \tilde{K} . Let $n > 0$ be an integer. Let $E_1 \neq \{0\}$ be a connected algebraic subgroup of E over \tilde{K} . Suppose E_1 is invariant under $\phi(\mathbb{F}_q[t^n])$. Then $(E_1, \phi|_{\mathbb{F}_q[t^n]})$ is an abelian t^n -module over \tilde{K} .*

Proof. Since E_1 is connected, we can always find \mathbb{F}_q -linear isomorphism between E_1 and \mathbb{G}_a^d for some $d > 0$. It remains only to show that $\text{Hom}_{\mathbb{F}_q}(E_1, \mathbb{G}_a)$ is a quotient $\tilde{K}[t^n]$ -module of $\text{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_a)$. This follows from the fact that any \mathbb{F}_q -linear homomorphism from E_1 to \mathbb{G}_a can be extended to a \mathbb{F}_q -linear homomorphism from E to \mathbb{G}_a (cf. [11], Lemma 5.2). □

The exponential maps associated to t -modules also give E_q -functions:

LEMMA 2.2. *Let (E, ϕ) be a t -module of dimension n defined over \bar{k} . Let $e_E(z) = (e^{(1)}(z), \dots, e^{(n)}(z))$ be the associated exponential map with respect to a fixed coordinate system. Let $V \in \bar{k}^n$, and let $f_i(y) = e^{(i)}(yV)$, for $i = 1, \dots, n$ and $y \in \bar{k}_\infty$. Then the functions $f_i: \bar{k}_\infty \rightarrow \bar{k}_\infty, i = 1, \dots, n$, are E_q -functions relative to some finite extension field over k .*

Proof. Let s be an integer such that $p^s \geq n$. Then $\phi(t^{p^s})_*$ is the scalar multiplication T^{p^s} on $\text{Lie}(E)$. Hence one has a functional equation for the exponential which is of the form

$$e_E(T^{p^s}) = \sum_{j=0}^l G_j \begin{pmatrix} e^{(1)}(z)^{q^j} \\ \vdots \\ e^{(n)}(z)^{q^j} \end{pmatrix},$$

where $G_0 = T^{p^s}I, G_1, \dots, G_l$ are $n \times n$ matrices with entries in a suitable finite extension field K/k .

Write

$$e_E(yV) = \sum_{h=0}^{\infty} y^{q^h} b_h, \quad \text{with } b_h \in \bar{k}^n.$$

We can solve the vector Taylor coefficients b_h recursively from the formula

$$[(T^{p^s})^{q^h} - T^{p^s}]b_h = \sum_{j=1}^{\inf(h,l)} G_j b_{h-j}^{q^j},$$

where $b_{h-j}^{q^j}$ denotes the column vector obtained from b_{h-j} by raising all coordinate components to its q^j -th power. From this recursive formula, it is rather easy to see that the functions $f_i(y), i = 1, \dots, n$ are E_q -functions with respect to K . □

Now we come to the main point. Let $E = (E, \phi)$ be an abelian t -module defined over \bar{k} . Let V be any period vector of E . We contend that at least one coordinate component of V is transcendental. This is special case of the following

THEOREM 2.3. *Let E be an abelian t -module of dimension n defined over \bar{k} . Let $e_E(z)$ be the associated exponential map with respect to a fixed coordinate system. Let $V \in \bar{k}_\infty^n$ such that $V \neq 0$ and $e_E(V) \in \bar{k}^n$. Then at least one coordinate component of V is transcendental.*

Proof. We first verify that the one-parameter map $y \mapsto e_E(yV)$ is not a polynomial map. Let s be an integer such that $\phi(t^{p^s})_*$ acts as scalar multiplication on $\text{Lie } E$.

Suppose $y \mapsto e_E(yV)$ is polynomial. Let Z be the image of \bar{k}_∞^n under this map in $\bar{k}_\infty^n \simeq E(\bar{k}_\infty)$. Then the connected component of the Zariski closure of Z is an one-dimensional algebraic subgroup E_1 . By Lemma 2.1 $(E_1, \phi|_{\mathbb{F}_q[t^{p^s}]})$ is an abelian t^{p^s} -module over \bar{k}_∞ .

Identify $\text{Lie } E_1(\bar{k}_\infty)$ inside $\text{Lie } E(\bar{k}_\infty)$, and regard \exp_{E_1} as a restriction of \exp_E . Under our chosen coordinate system, $\text{Lie } E_1(\bar{k}_\infty)$ coincides with $\bar{k}_\infty V$, because of the inverse mapping theorem. This implies that the abelian t^{p^s} -module E_1 has a polynomial exponential, which is impossible.

We may then write $e_E(yV) = (f_1(y), \dots, f_n(y))$ and assume $f_1(y)$ is not a polynomial in y . Suppose $V \in \bar{k}_n^n$. Then the functions $f_i(y), i = 1, \dots, n$ are E_q -functions with respect to some finite extension K/k . We apply Theorem 1.1 to the two E_q -functions, $f_1(y)$ and $f(y) = y$. By Artin's theorem we then have non-trivial additive relation of the form

$$\sum_{i=0}^{m_1} \alpha_i (f_1(y))^{p^i} + \sum_{j=0}^{m_2} \beta_j y^{p^j} \equiv 0.$$

Since $f_1(y)$ is entire but not polynomial, it has an infinite number of zeros. Hence all β_j are zero. This is clearly impossible. Therefore, we have $V \notin \bar{k}^n$. □

Finally, we note that we have proved the stronger result in [11] for those abelian t -modules over \bar{k} which admit sufficiently many "real" endomorphisms (i.e.

Hilbert-Blumenthal abelian t -modules). In that case, if $V \neq 0$ and $\exp_E(V) \in E(\bar{k})$, then all coordinate components of V with respect to suitably normalized coordinate system are transcendental.

3. Quasi-periodic functions and transcendence

To introduce quasi-periodic functions into Drinfeld’s theory, we first recall some facts from classical function theory.

- (I) Let $E_1 = \mathbb{G}_m$. The exponential function e^z gives complex analytic isomorphism $\mathbb{C}/2\pi i\mathbb{Z} \simeq E_1(\mathbb{C})$, where e^z is a solution of the algebraic differential equation $f'(z) = f(z)$, and $2\pi i\mathbb{Z}$ is the period lattice.
- (II) Let L be a rank two lattice in \mathbb{C} . The periodic Weierstrass function $\wp_L(z)$ leads to complex analytic isomorphism from \mathbb{C}/L onto $E_2(\mathbb{C})$, where E_2 is the elliptic curve associated to L . In this connection, one also has quasi-periodic Weierstrass function $\zeta_L(z)$. Both $\wp_L(z)$ and $\zeta_L(z)$ are solutions of suitable algebraic differential equations.

$$[\wp'_L(z)]^2 = 4\wp_L(z)^3 - g_2(L)\wp_L(z) - g_3(L)$$

$$\zeta'_L(z) = -\wp_L(z).$$

Write $L = \langle \omega_1, \omega_2 \rangle$, with $\text{Im}(\omega_1/\omega_2) > 0$. Let $\eta_i = 2\zeta_L(\frac{1}{2}\omega_i)$, for $i = 1, 2$. Then one has the Legendre’s relation connecting (I) and (II),

$$\begin{vmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{vmatrix} = 2\pi i.$$

We regard the entries ω_i, η_i as elliptic integrals of the first and second kind respectively, then the non-vanishing of this determinant gives the de Rham isomorphism theorem for the elliptic curve E_2 .

In Drinfeld’s theory, one starts with lattices $L \subset \bar{k}_\infty$ (i.e. finitely generated discrete A -submodules). One can associate to given lattice L a Drinfeld A -module $\phi = \phi_L: A \rightarrow \text{End}_{\mathbb{F}_q} \mathbb{G}_a$. We let $E_\phi = \mathbb{G}_a$, equipped with the A -action given by ϕ . Drinfeld’s exponential function $e_\phi(z)$ then gives an analytic A -module isomorphism $\bar{k}_\infty/L \simeq E_\phi(\bar{k}_\infty)$. Fix any non-constant a in A , the function $e_\phi(z)$ is a solution of the “algebraic differential equation” below

$$e_\phi(az) = \phi(a)(e_\phi(z)).$$

If the lattice L has rank $r > 1$, one also has interesting quasi-periodic functions associated to L , as first noticed by P. Deligne in the case $A = \mathbb{F}_q[T]$.

To get these quasi-periodic functions, we consider $\bar{k}_\infty\{\tau\}$ as A -bimodule, with right multiplication by $\phi(a)$ and left multiplication by scalars $a, a \in A$. By a biderivation from A into $\bar{k}_\infty\{\tau\}\tau$, we mean a \mathbb{F}_q -linear map $\delta: A \rightarrow \bar{k}_\infty\{\tau\}\tau$ satisfying

$$\delta(ab) = a\delta(b) + \delta(a)\phi(b), \quad \text{for all } a, b \in A.$$

Given such a biderivation, and given non-constant a in A , we can always solve the unique entire \mathbb{F}_q -linear solution $F(z)$ of the following ‘‘algebraic differential equation’’

$$F(az) - aF(z) = \delta(a)(e_\phi(z)),$$

$$F(z) \equiv 0 \pmod{z^q}.$$

This solution is independent of a , and is henceforth denoted by $F_\delta(z)$. It is quasi-periodic with respect to the lattice L , in the sense that the following properties always hold

- (i) $F_\delta(z + \omega) = F_\delta(z) + F_\delta(\omega)$, for $z \in \bar{k}_\infty$ and $\omega \in L$,
- (ii) $F_\delta(\omega)$ is A -linear in $\omega \in L$.

We shall call the values $F_\delta(\omega), \omega \in L$, the quasi-periods of F_δ , and following G. Anderson [2], we shall adopt the integral notation

$$\int_\omega \delta \stackrel{\text{def}}{=} -F_\delta(\omega).$$

We call biderivations $\delta: A \rightarrow \bar{k}_\infty\{\tau\}\tau$ differentials of second kind on the Drinfeld A -module ϕ . The set of all such biderivations will be denoted by $BD(\phi)$.

The Drinfeld A -module ϕ itself gives rise to a biderivation satisfying

$$\delta_\phi(a) = \phi(a) - a\tau^0, \quad \text{for all } a \in A.$$

The solution of the corresponding equation is $F_{\delta_\phi}(z) = e_\phi(z) - z$. Thus, one has $\int_\omega \delta_\phi = \omega$ for all $\omega \in L$. We call scalar multiples of δ_ϕ differentials of the first kind.

One can also form inner biderivations $\delta_\phi^{(P)}$ from any $P \in \bar{k}_\infty\{\tau\}\tau$, i.e.

$$\delta_\phi^{(P)}(a) = P\phi(a) - aP, \quad \text{for all } a \in A.$$

These are also called exact differentials, since $\int_\omega \delta_\phi^{(P)} = -P(e_\phi(\omega)) \equiv 0$. All quasi-periodic functions obtained from exact differentials are actually periodic.

The set of all $\delta_\phi^{(P)}, P \in \bar{k}_\infty\{\tau\}$, will be denoted by $IBD(\phi)$. The vector space $BD(\phi)/IBD(\phi)$ is therefore called the de Rham cohomology of the Drinfeld A -module ϕ , and is denoted by $H_{DR}^*(\phi)$.

Just as in the classical theory, one is able to write down genuine quasi-periodic functions only if the lattice L has rank $r > 1$. In fact, as observed by P. Deligne and G. Anderson, one has

$$\dim_{\bar{k}_\infty} H_{DR}^*(\phi) = \text{rank } \phi = r.$$

An illuminating way to get this dimension is through the so-called de Rham isomorphism: $H_{DR}^*(\phi) \simeq \text{Hom}_A(L, \bar{k}_\infty)$ via the mapping induced by $\delta \mapsto (\omega \mapsto \int_\omega \delta)$. We refer to E.-U. Gekeler [6] for a proof of this theorem.

Since our purpose here is to derive transcendence properties of the quasi-periodic functions, we will not go into the deeper part of Anderson’s theory, which culminates in an analogue of the Legendre’s relation for Drinfeld A -modules.

We now restrict ourselves to Drinfeld A -module ϕ defined over \bar{k} . For these ϕ , it is natural to consider biderivations δ defined over \bar{k} , i.e. satisfying $\delta(A) \subset \bar{k}\{\tau\}$. The set of all such biderivations is denoted by $BD(\phi/\bar{k})$. The set of all $\delta_\phi^{(P)}, P \in \bar{k}\{\tau\}$, is denoted by $IBD(\phi, \bar{k})$. Putting $H_{DR}^*(\phi/\bar{k})$ to be the quotient of $BD(\phi/\bar{k})$ by $IBD(\phi/\bar{k})$, then one has

$$H_{DR}^*(\phi) = H_{DR}^*(\phi/\bar{k}) \otimes_{\bar{k}} \bar{k}_\infty.$$

Thus, if $\delta \in BD(\phi/\bar{k})$ and $\int_\omega \delta = 0$ for all periods ω , the de Rham isomorphism implies $\delta \in IBD(\phi/\bar{k})$.

The fundamental theorem we want to prove is

THEOREM 3.1. *Let ϕ be a Drinfeld A -module defined over \bar{k} , with corresponding exponential $e_\phi(z)$. Let $\delta \in BD(\phi/\bar{k}) - IBD(\phi/\bar{k})$, with corresponding quasi-periodic function $F_\delta(z)$. Let $u \in \bar{k}_\infty$ such that $u \neq 0$ and $e_\phi(u) \in \bar{k}$. Then $F_\delta(u)$ is transcendental. In particular, $\int_\omega \delta$ is transcendental for all periods ω of ϕ .*

Proof. We let A act on \mathbb{G}_a^2 according to the following recipe

$$\Phi(a)(X_1, X_2) = (\phi(a)(X_1), aX_2 + \delta(a)(X_1)).$$

Then (\mathbb{G}_a^2, Φ) becomes an A -module, a fortiori a t -module for any choice of non-constant T in A . The exponential map for this module is easily seen to be the following map

$$(z_1, z_2) \mapsto (e_\phi(z_1), F_\delta(z_1) + z_2).$$

Since (\mathbb{G}_a^2, Φ) is defined over \bar{k} , we may apply Lemma 2.2 with $V = (1, 0)$. It follows that $e_\phi(z_1)$ and $F_\delta(z_1)$ are E_q -functions with respect to some finite extension field K/k . Suppose $F_\delta(u) \in \bar{k}$. Then $F_\delta(au) \in \bar{k}$ for all $a \in A$, since $e_\phi(u) \in \bar{k}$ by assumption. Thus, applying our Theorem 1.1, we know that $e_\phi(z_1)$ and $F_\delta(z_1)$ are algebraically dependent functions. By Artin's theorem, we then have algebraic dependence relations of the form

$$\sum_{i=0}^{m_1} \alpha_i e_\phi(z_1)^{p^i} + \sum_{j=0}^{m_2} \alpha'_j F_\delta(z_1)^{p^j} \equiv 0.$$

Let L be the period lattice of ϕ , and let $\omega_1 \in L$ be a period. The dependence relation above implies that the values $F_\delta(a\omega_1)$, $a \in A$, must be among the finitely many roots of the additive polynomial $\sum \alpha'_j X^{p^j}$. Thus we can find $a_1 \neq 0$ in A such that $F_\delta(a_1\omega_1) = 0$. Since L is finitely generated, we conclude that $F_\delta(z_1)$ vanishes on a sublattice of L of finite index. This implies $\int_\omega \delta = 0$ for all $\omega \in L$ which contradicts the de Rham isomorphism theorem. □

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