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## On normal subgroups of $GL_2$ over rings with many units

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**Abstract.** For several classes of rings  $A$ , we prove the “sandwich theorem” for all normal subgroups of  $GL_2 A$ : for every normal subgroup  $H$  of  $GL_2 A$  there is a unique ideal  $B$  of  $A$  such that  $[E_2 A, E_2 B] \subset H \subset G_2(A, B)$ .

### Introduction

For any associative ring  $A$  with 1, let  $GL_1 A$  denote its multiplicative group, i.e., the group of its units, and  $GL_2 A$  the group of all invertible 2 by 2 matrices over  $A$ . For any ideal  $B$  of  $A$ , let  $E_2 B$  denote the subgroup of  $GL_2 A$  generated by all elementary matrices  $b^{1,2}$  and  $b^{2,1}$  where  $b \in B$ . We denote by  $E_2(A, B)$  the normal subgroup of  $E_2 A$  generated by  $E_2 B$ . Let  $GE_2 A$  be the subgroup of  $GL_2 A$  generated by all its diagonal and elementary matrices.

Let  $GL_2 B$  denote the principal congruence subgroup of  $GL_2 A$ , i.e., the kernel of the homomorphism  $GL_2 A \rightarrow GL_2(A/B)$ . Let  $G_2(A, B)$  be the inverse image of the center of  $GL_2(A/B)$  under this homomorphism. Clearly, the center of  $GL_2(A/B)$  consists of scalar matrices over the center of the ring  $A/B$ .

We will assume that

(1)  $\sum A(GL_1 A - 1) = \sum (GL_1 A - 1)A = A$ , i.e., no proper one-sided ideal of  $A$  contains all  $u - 1$ , where  $u \in GL_1 A$ .

The formulas  $[\text{diag}(u, 1), b^{1,2}] = (ub - b)^{1,2}$  and  $[\text{diag}(u, 1), b^{2,1}] = (bu^{-1} - b)^{2,1}$  give the following result.

**PROPOSITION 2.** Under condition (1),  $E_2(A, B) \subset [E_2 B, GE_2 A]$  for every ideal  $B$  of  $A$ .

To obtain the “sandwich theorem”, we will use one of the following conditions on  $A$ :

(3)  $\text{sr}(A) \leq 1$ , i.e.,  $A$  satisfies the first Bass stable range condition;

(4)  $A/\text{rad}(A)$  is von Neumann regular;

(5) for every element  $a \in A$  there is a finite sequence  $x_1, \dots, x_N$  in  $A$  such that

$$x_1 + \dots + x_N = 1 \quad \text{and} \quad 1 - ax_i \in GL_1 A \quad \text{for all } i.$$

See [3], [23], [24], [11] about condition (3). Under this condition,  $GE_2 A = GL_2 A$ .

In condition (4),  $\text{rad}(A)$  means the Jacobson radical of  $A$ . Recall that a ring  $A$  is called *von Neumann regular* if for every  $a \in A$  there is  $x \in A$  such that  $axa = a$ . Replacing, if necessary,  $x$  here by  $xax$ , we can have both  $axa = a$  and  $xax = x$ . If  $A/\text{rad}(A)$  is commutative, condition (4) implies (3).

Taking  $x_i = 1/N$  with a large  $N$  (depending on  $a$ ) in condition (5), we satisfy this condition for any Banach algebra  $A$ , as well as for the ring  $A$  of all bounded smooth functions on any smooth manifold. More generally, condition (5) holds for any connected topological ring  $A$  with 1 with  $GL_1 A$  open in  $A$ .

If  $A$  is a semilocal ring, i.e.,  $A/\text{rad}(A)$  is the product of matrix rings over division rings (e.g.,  $A$  is an Artinian ring), then both (3) and (4) hold. Condition (1) for a semilocal ring  $A$  is equivalent to condition (5) as well as to any of the following six conditions:

- $A$  has no factor rings of two elements;
- every element of  $A$  is the sum of two units;
- condition (5) holds with  $N = 2$ ;

$$E_2 A \subset [E_2 A, GL_2 A];$$

$$E_2 A = [E_2 A, GL_2 A];$$

- (6) every element of  $A$  is a sum of elements of the form  $u - v$ , where  $u, v \in GL_1 A$ .

Moreover, every semilocal ring  $A$  satisfies the following condition:

- (7) every element of  $A$  is a sum of units.

Note that condition (6) implies both (7) and (1) for an arbitrary  $A$ . Obviously, condition (5) implies (1) (because  $1 - ax_i \in GL_1 A$  implies  $1 - x_i a \in GL_1 A$ , see [19]).

Here is the main result of this paper.

**THEOREM 8.** *Assume conditions (1) and (7) and one of the conditions (3), (4), or (5). Then for every subgroup  $H$  of  $GL_2 A$  which is normalized by  $GE_2 A$  there is a unique ideal  $B$  of  $A$  such that  $[E_2 A, E_2 B] \subset H \subset G_2(A, B)$ .*

**EXAMPLE.** Let  $A$  be a semilocal ring. Then the conditions of Theorem 8 hold if and only if  $A$  has no factor rings of two elements. Now we give an example of a commutative local ring  $A$  with  $A/\text{rad}(A) = \mathbb{Z}/2\mathbb{Z}$  and a normal subgroup  $H$  of  $GL_2 A = GE_2 A$  such that the conclusion of Theorem 8 is false. Let  $A$  be the commutative algebra with 1 over the field  $\mathbb{Z}/2\mathbb{Z}$  with a single generator  $z$  subject to the relation  $z^2 = 0$ . Clearly,  $\text{rad}(A) = \{0, z\}$  and  $A/\text{rad}(A) = \mathbb{Z}/2\mathbb{Z}$ . Let  $H$  be the subgroup of  $GL_2 A$  consisting of the following 4 matrices:

$$1_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{diag}(1+z, 1+z) = \begin{bmatrix} 1+z & 0 \\ 0 & 1+z \end{bmatrix},$$

$$h = \begin{bmatrix} 1+z & z \\ z & 1 \end{bmatrix}, h' = \begin{bmatrix} 1 & z \\ z & 1+z \end{bmatrix}.$$

Then  $g^2 = 1_2$  for all  $g \in H$  and  $hh' = \text{diag}(1 + z, 1 + z)$ . Clearly,  $[E_2A, E_2B] \subset H$  for an ideal  $B$  of  $A$  only when  $B = 0$ . (Indeed, otherwise  $B = \text{rad}(A)$  or  $A$ , and the commutator subgroup  $[E_2A, E_2B]$  contains a matrix, namely  $[1^{1,2}, z^{2,1}] = \text{diag}(1 + z, 1 + z)z^{1,2}$ , outside of  $H$ .) On the other hand,  $H$  is normal in  $GE_2A = GL_2A$  and  $H$  is not contained in the center  $G_2(A, 0) = \{1_2, \text{diag}(1 + z, 1 + z)\}$  of  $GL_2A$ . Thus, condition (1) in Theorem 8 cannot be dropped.

The question arises naturally whether the converse to the conclusion of Theorem 8 is true. To be more cautious, one may ask the following question:

(9) suppose that  $E_2(A, B) \subset H \subset G_2(A, B)$  for a subgroup  $H$  of  $GL_2A$  and an ideal  $B$  of  $A$ ; is then  $H$  normalized by  $GE_2A$ ?

The positive answer for local rings  $A$  was “obtained” in [7]. However, Bass [3] pointed out a gap in [7], and [26] gives a counter-example (with a local ring  $A$ , i.e.,  $A/\text{rad}(A)$  is a division ring) to the results of [7]. Namely, let  $F$  be a field, and let  $A$  be the associative  $F$ -algebra with two generators  $x, y$  subject to the relations: all the 8 monomials of degree 3 in  $x, y$  are 0. Then  $\dim_F A = 7$  and  $\text{rad}(A)$  is the six-dimensional  $F$ -vector space with a basis  $x, y, yx, xy, x^2, y^2$ . Let  $B$  be the four-dimensional subspace with the basis  $yx, xy, x^2, y^2$ . Then  $B$  is an ideal of  $A$ ,  $F + B$  is the center of  $A$ , and the factor ring  $A/B$  is commutative. Then  $[G_2(A, B), E_2A]$  is not contained in  $E_2(A, B)$  which shows that the results of [7] are wrong.

Using this example, we will show that the answer to (7) is negative, in general, even for local rings  $A$ . Namely, if we define  $H$  to be the subgroup of  $G_2(A, B)$  generated by  $GL_2B$  and the diagonal matrix  $\text{diag}(1 + x, 1)$ , then  $E_2(A, B) \subset H \subset G_2(A, B)$ , but  $H$  is not normalized by  $\text{diag}(1 + y, 1) \in GE_2A$ .

However,  $H$  is normalized by  $[E_2A, E_2A]$  under the conditions of Theorem 8, by [3], [20], [22]:

PROPOSITION 10. *Suppose that one of the conditions (3), (4), or (5) holds. Then*

$$[G_2(A, B), [E_2A, E_2A]] \subset E_2(A, B)$$

for any ideal  $B$  of  $A$ . Therefore, if  $H$  is a subgroup of  $GL_2A$  such that  $E_2(A, B) \subset H \subset G_2(A, B)$  for an ideal  $B$  of  $A$ , then  $[H, [E_2A, E_2A]] \subset E_2(A, B)$ , hence  $H$  is normalized by  $[E_2A, E_2A]$ .

Indeed, by [3], [20], and [22],  $[GL_2B, E_2A] \subset E_2(A, B)$ . Take now an arbitrary  $h \in G_2(A, B)$  and consider the map  $\varphi: E_2A \rightarrow (GL_2B \cap E_2B)/E_2(A, B)$  given by  $\varphi(g) = [g, h]E_2(A, B)$  for any  $g \in E_2A$ . Then  $\varphi$  is a group homomorphism into an abelian group, so  $\varphi([E_2A, E_2A]) = 1_2$ . That is,  $[[E_2A, E_2A], h] \subset E_2(A, B)$ . So  $[G_2(A, B), [E_2A, E_2A]] \subset E_2(A, B)$ .

The question arises naturally whether

(11)  $[E_2A, E_2B] \subset H \subset G_2(A, B)$  for every subgroup  $H$  of  $GL_2A$  which is normalized by  $E_2A$ ?

This question is partially answered in [15]. By [21], a necessary condition for

(11) to be true is that every quasi-ideal of  $A$  is an ideal. Recall [21, Section 4] that a quasi-ideal  $Y$  of a ring  $A$  is defined as an additive subgroup of  $A$  such that  $aya, yay \in Y$  for all  $a \in A$  and  $y \in Y$ .

If every element of  $A$  has the form  $2c + d$ , where  $c$  belongs to the center of  $A$  and  $d$  belongs to the ideal of  $A$  generated by the additive commutators  $ab - ba$ , then every quasi-ideal of  $A$  is an ideal, see [21, Section 4]. In particular, this is the case when  $A$  is semilocal and either  $A$  is commutative with  $2A = A$  or  $A$  has no factor fields. Another example is any Banach algebra  $A$  with 1 which is either commutative or simple. Yet another example is any von Neumann regular ring  $A$ , see [15].

See [15] for results about subgroups  $H$  of  $GL_2 A$  which are normalized by  $E_2 A$ .

Theorem 8 will be proved in the next section. The uniqueness of the ideal  $B$  in Theorem 8 is easy to see. It follows, for example, from the following fact which is true for an arbitrary associative ring  $A$  with 1: if  $H$  is a subgroup of  $GL_2 A$  such that  $[E_2 A, E_2 B] \subset H \subset G_2(A, B)$  for an ideal  $B$  of  $A$ , then  $B$  is the ideal of  $A$  consisting of all non-diagonal entries of matrices in  $H$ . Also  $B$  coincides with the set  $\{x \in A: x^{1,2} x^{2,1} \in H\}$ .

Some partial positive results about normal subgroups of  $GL_2 A$  were known for: von Neumann regular rings  $A$  (see [20]), Banach algebras  $A$  (see [21]), rings  $A$  with  $sr(A) = 1$  (see [3]), including commutative rings  $A$  with  $sr(A) = 1$  (see [4], [13], [26]), commutative local  $A$  (see [1], [6], [9], [10], [12], [14], [17]), commutative semilocal rings (see [2] and [5]). If  $2A \neq A$ , then quasi-ideals appear even for commutative local rings [1]. In [8], Theorem 8 is proved for any semilocal ring satisfying the following two conditions: there is a unit  $\varepsilon$  in the center of  $A$  such that  $1 - \varepsilon$  is also a unit;  $A$  has neither factor rings which are division algebras with centers of  $\leq 5$  elements nor factor rings which are isomorphic to  $M_2 \mathbb{Z}/2\mathbb{Z}$ .

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### Proof of Theorem 8

LEMMA 12. *Let  $G$  be a subgroup of  $GL_1 A$ . Let  $H$  be a subgroup of  $GL_2 A$  which is normalized by  $E_2 A$  and all diagonal matrices  $\text{diag}(u, v)$  with  $u, v \in G$ . Set  $K =$*

$\{g \in GL_2 A : [g, E_2 A] \subset H\}$ . Let  $K$  contain a matrix

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with  $a \in GL_1 A$ . Then  $((1 - v)ca^{-1}(1 - u)(1 - w))^{2,1} \in K$  for all  $u, v, w \in G$ .

*Proof.* Note that  $K$  is a subgroup of  $GL_2 A$  containing  $H$  and normalized by  $E_2 A$  and all diagonal matrices  $\text{diag}(u, v)$  with  $u, v \in G$ . We write  $h = (c')^{2,1} \text{diag}(a, d')(b)^{1,2}$ , where  $c' = ca^{-1} \in A$ ,  $b' = a^{-1}b$ , and  $d' = d - c'b$ . We have

$$\begin{aligned} h_1 &= [h, \text{diag}(u^{-1}, 1)] = (c')^{2,1} \text{diag}(a, d')b^{1,2}(-u^{-1}b')^{1,2} \\ &\quad \times \text{diag}(u^{-1}a^{-1}u, (d')^{-1})(-c'u)^{2,1} \\ &= (c'')^{2,1} \text{diag}([a, u^{-1}], 1)(*)^{1,2}(-c'u)^{2,1} \in K, \end{aligned}$$

hence  $h_2 = (-c'u)^{2,1} h_1 (c'u)^{2,1} = (c' - c'u')^{2,1} \text{diag}([a, u^{-1}], 1)(*) \in K$ .

So  $h_3 = [h_2, \text{diag}(1, v)] = (c' - c'u')^{2,1} \text{diag}([a, u^{-1}], 1)(*)^{1,2} (v(c' - c'u'))^{2,1} \in K$ , hence

$$h_4 = (v(c' - c'u'))^{2,1} h_3 (-v(c' - c'u'))^{2,1} = ((1 - v)(c' - c'u'))^{2,1} (*)^{1,2} \in K.$$

Since  $h_4 = ((1 - v)(c' - c'u'))^{2,1} (*)^{1,2} \in K$ ,  $((1 - v)(c' - c'u'))^{2,1} \in K$ . So

$$[((1 - v)(c' - c'u'))^{2,1}, \text{diag}(w, 1)] = ((1 - v)(c' - c'u')(1 - w))^{2,1} \in K.$$

**COROLLARY 13.** *Under the conditions of Lemma 12, assume that  $\Sigma A(G - 1) = \Sigma(G - 1)A = A$  and that every element of  $A$  is a sum of units from  $G$ . Then  $(AcA + AbA)^{2,1} \subset K$ .*

*Proof.* Set  $B = \{x \in A : x^{2,1} \in K\}$ . We have to prove that  $AcA + AbA \subset B$ . Since  $K$  is normalized by all diagonal matrices  $\text{diag}(u, v)$  with  $u, v \in G$ , the condition  $\Sigma G = A$  gives that  $B$  is an ideal of  $A$ . By Lemma 12,  $(1 - v)c'(1 - u)(1 - w) \in B$  for arbitrary units  $u, v$  and  $w$  of  $A$ , where  $c' = ca^{-1} \in B$ . Using the condition  $\Sigma A(G - 1) = \Sigma(G - 1)A = A$ , we obtain that  $Ac'A = AcA \subset B$ .

Now we apply this to the element  $tht^{-1}$  of  $K$  instead of  $h$ , where  $t = 1^{1,2}(-1)^{2,1} 1^{1,2} \in E_2 A$ . This gives that  $AbA \subset B$ .

**LEMMA 14.** *Let*

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2 A$$

*commute with  $E_2 A$  modulo the center  $G_2(A, 0)$  of  $GL_2 A$ . Then  $a \in GL_1 A$ .*

*Proof.* Case 1.  $A$  is commutative. Then the second entry in the first column of the matrix  $[1^{1,2}, h] \in G_2(A, 0)$  is  $c^2/\det(h) = 0$ , hence  $c^2 = 0$ . So  $aA = A$ .

*General case.* Since  $1^{1,2}h = h 1^{1,2} \text{diag}(\mu, \mu)$  with  $\mu$  in the center of  $A$ , we conclude that  $a + b = a\mu$ , hence  $ab = ba$ . Similarly, every off-diagonal entry of  $h$  commutes with every diagonal entry of  $h$ .

Since  $tht^{-1} \in G_2(A, 0)$ , where  $t = -t^{-1} = 1^{1,2}(-1)^{2,1} 1^{1,2} \in E_2A$ , we conclude that  $ad = da$  and  $bc = cb$ . So the subring  $A_0$  of  $A$  generated by  $1, a, b, c, d$  is commutative. Consider now the dominion of  $A_0$  in  $A$  [16]. This is a commutative subring of  $A$ , and it contains all the entries of  $h^{-1}$  [16, Propositions 1.1 and 1.3]. So we are reduced to Case 1.

LEMMA 15. *Let  $G, H$ , and  $K$  be as in Lemma 12. Suppose that  $\Sigma A(G - 1) = \Sigma(G - 1)A = A$ , i.e., no proper one-sided ideal of  $A$  contains all  $1 - u$  with  $u \in G$ . Assume one of the conditions (3), (4), (5). Assume also that  $K$  is not contained in the center of  $GL_2A$ . Then there is a non-zero  $y \in A$  such that  $y^{1,2} \in K$ .*

*Proof.* Pick a matrix  $h$  in  $K$  outside the center  $G_2(A, 0)$  of  $G_2(A, A) = GL_2A$ . Write

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H. \text{ Let } h^{-1} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

*Case 1.*  $a \in GL_1A$ . By Lemma 12 and the condition on  $G$ , we can assume that  $b = c = 0$ . Since  $h$  is outside the center, there is  $x \in A$  such that  $ax \neq xc$ . Then  $[h, x^{1,2}] = y^{1,2} \in K$  with  $y = axd^{-1} - x \neq 0$ . Note that we did not use any of the conditions (3), (4), (5).

*General case.* Under the condition (3), we argue as follows. By (3), there is  $e \in A$  such that  $a + xc$  is a unit of  $A$ . Replacing  $h$  by  $e^{1,2}h(-e)^{1,2}$ , we can assume that  $a$  is a unit. So we are reduced to Case 1.

Under condition (4), we argue as follows. If  $b, c \in \text{rad}(A)$ , then  $a \in GL_1A$ , so we are done by Case 1. Otherwise, we can assume that  $c \notin \text{rad}(A)$  (the case  $b \notin \text{rad}(A)$  is similar).

We pick  $z \in A$  such that  $czc \equiv c \pmod{\text{rad}(A)}$  and set  $e = zc \in A$ . Note that  $e^2 \equiv e$  and  $ce \equiv c \pmod{\text{rad}(A)}$ . Replacing  $h$  by  $(-az)^{1,2}h(az)^{1,2} \in K$ , we change  $a$  by  $a - azc = a(1 - e)$ , and keep the same  $c$ . So we can assume that  $a = a(1 - e)$ .

Then  $d'c \equiv d'ce \equiv (d'c + c'a)e = 0 \pmod{\text{rad}(A)}$  and the first row of the matrix  $[h, e^{1,2}]$  (modulo  $\text{rad}(A)$ ) is  $(1, -e)$ . Since  $ce \equiv c \neq 0 \pmod{\text{rad}(A)}$ ,  $e \neq 0 \pmod{\text{rad}(A)}$ . So  $[h, e^{1,2}] \in H$  is outside the center of  $GL_2A$ , and we are reduced to Case 1.

Under condition (5), having Case 1 done, we can assume, by Lemma 14, that there is an elementary matrix  $g$  which does not commute with  $h$  modulo the center. Say,  $g = z^{1,2}$ . By condition (5), we find  $x_i \in A$  such that  $1 + c'azzx_i \in GL_1A$  and

$GL_1 A$  and  $\sum x_i = 1$ . Since  $g = z^{1,2} = \prod(zx_i)^{1,2}$ , we can pick some  $i$  such that  $(zx_i)^{1,2}$  does not commute with  $h$  modulo the center. The first entry of the first row of the matrix  $h' = [h, (zx_i)^{1,2}] \in H$  is  $1 + azx_i c' \in GL_1 A$ . So we are reduced to Case 1, applied to  $h'$  instead of  $h$ .

Now we can complete our proof of Theorem 8. We define

$$K = \{g \in GL_2 A : [g, E_2 A] \subset H\}$$

as in Lemma 12 above. We want to prove that  $E_2(A, B) \subset K \subset G_2(A, B)$  for an ideal  $B$  of  $A$ . Set  $B = \{x \in A : x^{2,1} \in K\}$ . Since  $K$  is normal in  $GL_2 A$ , condition (7) gives that  $B$  is an ideal of  $A$ . Since  $t = -t^{-1} = 1^{1,2}(-1)^{2,1}1^{1,2} \in E_2 A$  and  $tB^{2,1}t^{-1} = B^{2,1}$ , we obtain that  $E_2(A, B) \subset H$ . We have to prove that  $K \subset G_2(A, B)$ , i.e.  $f(H)$  is central in  $GL_2(A/B)$ , where  $f: GL_2(A/B)$  is the canonical homomorphism.

Set  $H' = f(H)$  and  $K' = \{g \in GL_2(A/B) : [g, E_2(A/B)] \subset H'\}$ . Note that both  $H'$  and  $K'$  are normalized by  $f(GL_2 A)$ . Clearly,  $f(K) \subset K'$ . So it suffices to show that  $K' \subset G_2(A/B, 0)$ , the center of  $GL_2(A/B)$ .

Note that the conditions of Theorem 8 on  $A$  imply the corresponding conditions on  $A/B$ . We assume now that  $K'$  is not central, and will obtain a contradiction.

Applying Lemma 15 (with  $G$  being the image of  $GL_1 A$  in  $GL_1(A/B)$ ) to  $H'$  and  $K'$  instead of  $H$  and  $K$ , we obtain that  $K'$  contains a non-trivial elementary matrix. Say,  $c^{1,2} \in K'$  with  $0 \neq c = a + B \in A/B$ , where  $a \in A \setminus B$ . Then  $[c^{1,2}, t] = c^{1,2}c^{2,1} \in H' = f(H)$ . That is,  $H$  contains a matrix of the form  $hg$ , where  $h \in GL_2 B$  and  $g = a^{1,2}a^{2,1}$ . Now we use that  $[GL_2 B, E_2 A] \subset E_2(A, B)$  (see [3], [20], and [22]). In particular,  $[1^{2,1}, h] \in E_2(A, B) \subset H$ . So  $[1^{2,1}, g] = [1^{2,1}, a^{1,2}] \in H$ . Therefore  $(-1)^{1,2}[1^{2,1}, a^{1,2}]1^{1,2} = (-a)^{2,1}(-a)^{1,2} \in H \subset K$ , hence  $(-a)^{2,1} \in K$  by the definition of  $K$  and  $a \in B$  by the definition of  $B$ . A contradiction!

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