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On normal subgroups of GL_2 over rings with many units

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Abstract. For several classes of rings A, we prove the "sandwich theorem" for all normal subgroups of $\mathrm{GL}_2 A$: for every normal subgroup H of $\mathrm{GL}_2 A$ there is a unique ideal B of A such that $[E_2 A, E_2 B] \subset H \subset G_2(A, B)$.

Introduction

For any associative ring A with 1, let $\mathrm{GL}_1 A$ denote its multiplicative group, i.e., the group of its units, and $\mathrm{GL}_2 A$ the group of all invertible 2 by 2 matrices over A. For any ideal B of A, let $\mathrm{E}_2 B$ denote the subgroup of $\mathrm{GL}_2 A$ generated by all elementary matrices $b^{1,2}$ and $b^{2,1}$ where $b \in B$. We denote by $\mathrm{E}_2(A, B)$ the normal subgroup of $\mathrm{E}_2 A$ generated by $E_2 B$. Let $\mathrm{GE}_2 A$ be the subgroup of $\mathrm{GL}_2 A$ generated by all its diagonal and elementary matrices.

Let $\mathrm{GL}_2 B$ denote the principal congruence subgroup of $\mathrm{GL}_2 A$, i.e., the kernel of the homomorphism $\mathrm{GL}_2 A \to \mathrm{GL}_2(A/B)$. Let $\mathrm{G}_2(A, B)$ be the inverse image of the center of $\mathrm{GL}_2(A/B)$ under this homomorphism. Clearly, the center of $\mathrm{GL}_2(A/B)$ consists of scalar matrices over the center of the ring A/B.

We will assume that

(1) $\Sigma A(\mathrm{GL}_1 A - 1) = \Sigma(\mathrm{GL}_1 A - 1)A = A$, i.e., no proper one-sided ideal of A contains all $u - 1$, where $u \in \mathrm{GL}_1 A$.

The formulas $[\mathrm{diag}(u, 1), b^{1,2}] = (ub - b)^{1,2}$ and $[\mathrm{diag}(u, 1), b^{2,1}] = (bu^{-1} - b)^{2,1}$ give the following result.

PROPOSITION 2. *Under condition* (1), $\mathrm{E}_2(A, B) \subset [\mathrm{E}_2 B, \mathrm{GE}_2 A]$ *for every ideal* B *of* A.

To obtain the "sandwich theorem", we will use one of the following conditions on A:

(3) $\mathrm{sr}(A) \leqslant 1$, i.e., A satisfies the first Bass stable range condition;

(4) $A/\mathrm{rad}(A)$ is von Neumann regular;

(5) for every element $a \in A$ there is a finite sequence $x_1, \ldots, x_N$ in A such that

$$x_1 + \cdots + x_N = 1 \quad \text{and} \quad 1 - ax_i \in \mathrm{GL}_1 A \quad \text{for all } i.$$

See [3], [23], [24], [11] about condition (3). Under this condition, $\mathrm{GE}_2 A = \mathrm{GL}_2 A$.

In condition (4), rad(A) means the Jacobson radical of A. Recall that a ring A is called *von Neumann regular* if for every $a \in A$ there is $x \in A$ such that $axa = a$. Replacing, if necessary, x here by xax, we can have both $axa = a$ and $xax = x$. If $A/\mathrm{rad}(A)$ is commutative, condition (4) implies (3).

Taking $x_i = 1/N$ with a large N (depending on a) in condition (5), we satisfy this condition for any Banach algebra A, as well as for the ring A of all bounded smooth functions on any smooth manifold. More generally, condition (5) holds for any connected topological ring A with 1 with $\mathrm{GL}_1 A$ open in A.

If A is a semilocal ring, i.e., $A/\mathrm{rad}(A)$ is the product of matrix rings over division rings (e.g., A is an Artinian ring), then both (3) and (4) hold. Condition (1) for a semilocal ring A is equivalent to condition (5) as well as to any of the following six conditions:

A has no factor rings of two elements;

every element of A is the sum of two units;

condition (5) holds with $N = 2$;

$\mathrm{E}_2 A \subset [\mathrm{E}_2 A, \mathrm{GL}_2 A]$;

$\mathrm{E}_2 A = [\mathrm{E}_2 A, \mathrm{GL}_2 A]$;

(6) every element of A is a sum of elements of the form $u - v$, where $u, v \in \mathrm{GL}_1 A$.

Moreover, every semilocal ring A satisfies the following condition:

(7) every element of A is a sum of units.

Note that condition (6) implies both (7) and (1) for an arbitrary A. Obviously, condition (5) implies (1) (because $1 - ax_i \in \mathrm{GL}_1 A$ implies $1 - x_i a \in \mathrm{GL}_1 A$, see [19]).

Here is the main result of this paper.

THEOREM 8. *Assume conditions* (1) *and* (7) *and one of the conditions* (3), (4), *or* (5). *Then for every subgroup H of* $\mathrm{GL}_2 A$ *which is normalized by* $\mathrm{GE}_2 A$ *there is a unique ideal B of A such that* $[\mathrm{E}_2 A, \mathrm{E}_2 B] \subset H \subset \mathrm{G}_2(A, B)$.

EXAMPLE. Let A be a semilocal ring. Then the conditions of Theorem 8 hold if and only if A has no factor rings of two elements. Now we given an example of a commutative local ring A with $A/\mathrm{rad}(A) = \mathbb{Z}/2\mathbb{Z}$ and a normal subgroup H of $\mathrm{GL}_2 A = \mathrm{GE}_2 A$ such that the conclusion of Theorem 8 is false. Let A be the commutative algebra with 1 over the field $\mathbb{Z}/2\mathbb{Z}$ with a single generator z subject to the relation $z^2 = 0$. Clearly, $\mathrm{rad}(A) = \{0, z\}$ and $A/\mathrm{rad}(A) = \mathbb{Z}/2\mathbb{Z}$. Let H be the subgroup of $\mathrm{GL}_2 A$ consisting of the following 4 matrices:

$$1_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathrm{diag}(1 + z, 1 + z) = \begin{bmatrix} 1 + z & 0 \\ 0 & 1 + z \end{bmatrix},$$

$$h = \begin{bmatrix} 1 + z & z \\ z & 1 \end{bmatrix}, h' = \begin{bmatrix} 1 & z \\ z & 1 + z \end{bmatrix}.$$

Then $g^2 = 1_2$ for all $g \in H$ and $hh' = \mathrm{diag}(1 + z, 1 + z)$. Clearly, $[\mathrm{E}_2 A, \mathrm{E}_2 B] \subset H$ for an ideal B of A only when $B = 0$. (Indeed, otherwise $B = \mathrm{rad}(A)$ or A, and the commutator subgroup $[\mathrm{E}_2 A, \mathrm{E}_2 B]$ contains a matrix, namely $[1^{1,2}, z^{2,1}] = \mathrm{diag}(1 + z, 1 + z)z^{1,2}$, outside of H.) On the other hand, H is normal in $\mathrm{GE}_2 A = \mathrm{GL}_2 A$ and H is not contained in the center $G_2(A, 0) = \{1_2, \mathrm{diag}\,(1 + z, 1 + z)\}$ of $\mathrm{GL}_2 A$. Thus, condition (1) in Theorem 8 cannot be dropped.

The question arises naturally whether the converse to the conclusion of Theorem 8 is true. To be more cautious, one may ask the following question:

(9) suppose that $\mathrm{E}_2(A, B) \subset H \subset \mathrm{G}_2(A, B)$ for a subgroup H of $\mathrm{GL}_2 A$ and an ideal B of A; is then H normalized by $\mathrm{GE}_2 A$?

The positive answer for local rings A was "obtained" in [7]. However, Bass [3] pointed out a gap in [7], and [26] gives a counter-example (with a local ring A, i.e., $A/\mathrm{rad}(A)$ is a division ring) to the results of [7]. Namely, let F be a field, and let A be the associative F-algebra with two generators x, y subject to the relations: all the 8 monomials of degree 3 in x, y are 0. Then $\dim_F A = 7$ and $\mathrm{rad}(A)$ is the six-dimensional F-vector space with a basis x, y, yx, xy, x^2, y^2. Let B be the four-dimensional subspace with the basis yx, xy, x^2, y^2. Then B is an ideal of A, $F + B$ is the center of A, and the factor ring A/B is commutative. Then $[G_2(A, B), \mathrm{E}_2 A]$ is not contained in $\mathrm{E}_2(A, B)$ which shows that the results of [7] are wrong.

Using this example, we will show that the answer to (7) is negative, in general, even for local rings A. Namely, if we define H to be the subgroup of $\mathrm{G}_2(A, B)$ generated by $\mathrm{GL}_2 B$ and the diagonal matrix $\mathrm{diag}(1 + x, 1)$, then $\mathrm{E}_2(A, B) \subset H \subset \mathrm{G}_2(A, B)$, but H is not normalized by $\mathrm{diag}(1 + y, 1) \in \mathrm{GE}_2 A$.

However, H is normalized by $[\mathrm{E}_2 A, \mathrm{E}_2 A]$ under the conditions of Theorem 8, by [3], [20], [22]:

PROPOSITION 10. *Suppose that one of the conditions* (3), (4), *or* (5) *holds. Then*

$$[\mathrm{G}_2(A, B), [\mathrm{E}_2 A, \mathrm{E}_2 A]] \subset \mathrm{E}_2(A, B)$$

for any ideal B of A. Therefore, if H is a subgroup of $\mathrm{GL}_2 A$ *such that* $\mathrm{E}_2(A, B) \subset H \subset \mathrm{G}_2(A, B)$ *for an ideal B of A, then* $[H, [\mathrm{E}_2 A, \mathrm{E}_2 A]] \subset \mathrm{E}_2(A, B)$, *hence H is normalized by* $[\mathrm{E}_2 A, \mathrm{E}_2 A]$.

Indeed, by [3], [20], and [22], $[\mathrm{GL}_2 B, \mathrm{E}_2 A] \subset \mathrm{E}_2(A, B)$. Take now an arbitrary $h \in \mathrm{G}_2(A, B)$ and consider the map $\varphi\colon \mathrm{E}_2 A \to (\mathrm{GL}_2 B \cap \mathrm{E}_2 B)/\mathrm{E}_2(A, B)$ given by $\varphi(g) = [g, h]\mathrm{E}_2(A, B)$ for any $g \in \mathrm{E}_2 A$. Then φ is a group homomorphism into an abelian group, so $\varphi([\mathrm{E}_2 A, \mathrm{E}_2 A]) = 1_2$. That is, $[[\mathrm{E}_2 A, \mathrm{E}_2 A], h] \subset \mathrm{E}_2(A, B)$. So $[\mathrm{G}_2(A, B), [\mathrm{E}_2 A, \mathrm{E}_2 A]] \subset \mathrm{E}_2(A, B)$.

The question arises naturally whether

(11) $[\mathrm{E}_2 A, \mathrm{E}_2 B] \subset H \subset \mathrm{G}_2(A, B)$ for every subgroup H of $\mathrm{GL}_2 A$ which is normalized by $\mathrm{E}_2 A$?

This question is partially answered in [15]. By [21], a necessary condition for

(11) to be true is that every quasi-ideal of A is an ideal. Recall [21, Section 4] that a quasi-ideal Y of a ring A is defined as an additive subgroup of A such that $aya, yay \in Y$ for all $a \in A$ and $y \in Y$.

If every element of A has the form $2c + d$, where c belongs to the center of A and d belongs to the ideal of A generated by the additive commutators $ab - ba$, then every quasi-ideal of A is an ideal, see [21, Section 4]. In particular, this is the case when A is semilocal and either A is commutative with $2A = A$ or A has no factor fields. Another example is any Banach algebra A with 1 which is either commutative or simple. Yet another example is any von Neumann regular ring A, see [15].

See [15] for results about subgroups H of $\mathrm{GL}_2 A$ which are normalized by $\mathrm{E}_2 A$.

Theorem 8 will be proved in the next section. The uniqueness of the ideal B in Theorem 8 is easy to see. It follows, for example, from the following fact which is true for an arbitrary associative ring A with 1: if H is a subgroup of $\mathrm{GL}_2 A$ such that $[\mathrm{E}_2 A, \mathrm{E}_2 B] \subset H \subset G_2(A, B)$ for an ideal B of A, then B is the ideal of A consisting of all non-diagonal entries of matrices in H. Also B coincides with the set $\{x \in A : x^{1,2} x^{2,1} \in H\}$.

Some partial positive results about normal subgroups of $\mathrm{GL}_2 A$ were known for: von Neumann regular rings A (see [20]), Banach algebras A (see [21]), rings A with $\mathrm{sr}(A) = 1$ (see [3]), including commutative rings A with $\mathrm{sr}(A) = 1$ (see [4], [13], [26]), commutative local A (see [1], [6], [9], [10], [12], [14], [17]), commutative semilocal rings (see [2] and [5]). If $2A \neq A$, then quasi-ideals appear even for commutative local rings [1]. In [8], Theorem 8 is proved for any semilocal ring satisfying the following two conditions: there is a unit ε in the center of A such that $1 - \varepsilon$ is also a unit; A has neither factor rings which are division algebras with centers of $\leqslant 5$ elements nor factor rings which are isomorphic to $M_2 \mathbb{Z}/2\mathbb{Z}$.

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Proof of Theorem 8

LEMMA 12. *Let G be a subgroup of* $\mathrm{GL}_1 A$. *Let H be a subgroup of* $\mathrm{GL}_2 A$ *which is normalized by* $\mathrm{E}_2 A$ *and all diagonal matrices* $\mathrm{diag}(u, v)$ *with* $u, v \in G$. *Set* $K =$

$\{g \in \mathrm{GL}_2 A : [g, E_2 A] \subset H\}$. *Let K contain a matrix*

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a \in \mathrm{GL}_1 A$. Then $((1-v)ca^{-1}(1-u)(1-w))^{2,1} \in K$ for all $u, v, w \in G$.

Proof. Note that K is a subgroup of $\mathrm{GL}_2 A$ containing H and normalized by $E_2 A$ and all diagonal matrices $\mathrm{diag}(u, v)$ with $u, v \in G$. We write $h = (c')^{2,1}$ $\mathrm{diag}(a, d')(b)^{1,2}$, where $c' = ca^{-1} \in A$, $b' = a^{-1}b$, and $d' = d - c'b$. We have

$$\begin{aligned} h_1 = [h, \mathrm{diag}(u^{-1}, 1)] &= (c')^{2,1}\,\mathrm{diag}(a, d')b^{1,2}(-u^{-1}b')^{1,2} \\ &\quad\times \mathrm{diag}(u^{-1}a^{-1}u, (d')^{-1})(-c'u)^{2,1} \\ &= (c'')^{2,1}\,\mathrm{diag}([a, u^{-1}], 1)(*)^{1,2}(-c'u)^{2,1} \in K, \end{aligned}$$

hence $h_2 = (-c'u)^{2,1}\, h_1 (c'u)^{2,1} = (c' - c'u')^{2,1}\ \mathrm{diag}([a, u^{-1}], 1)(*) \in K$.

So $h_3 = [h_2, \mathrm{diag}(1, v)] = (c' - c'u')^{2,1}\ \mathrm{diag}([a, u^{-1}], 1)(*)^{1,2}\ (v(c' - c'u'))^{2,1} \in K$, hence

$$h_4 = (v(c' - c'u'))^{2,1}\, h_3(-v(c' - c'u'))^{2,1} = ((1-v)(c' - c'u'))^{2,1}(*)^{1,2} \in K.$$

Since $h_4 = ((1-v)(c' - c'u'))^{2,1}(*)^{1,2} \in K$, $((1-v)(c' - c'u'))^{2,1} \in K$. So

$$[((1-v)(c' - c'u'))^{2,1}, \mathrm{diag}(w, 1)] = ((1-v)(c' - c'u')(1-w))^{2,1} \in K.$$

COROLLARY 13. *Under the conditions of Lemma* 12, *assume that* $\Sigma A(G-1) = \Sigma(G-1)A = A$ *and that every element of A is a sum of units from G. Then* $(AcA + AbA)^{2,1} \subset K$.

Proof. Set $B = \{x \in A : x^{2,1} \in K\}$. We have to prove that $AcA + AbA \subset B$. Since K is normalized by *all diagonal matrices* $\mathrm{diag}(u, v)$ *with* $u, v \in G$, the condition $\Sigma G = A$ gives that B is an ideal of A. By Lemma 12, $(1-v)c'(1-u)(1-w) \in B$ for arbitrary units u, v and w of A, where $c' = ca^{-1} \in B$. Using the condition $\Sigma A(G-1) = \Sigma(G-1)A = A$), we obtain that $Ac'A = AcA \subset B$.

Now we apply this to the element tht^{-1} of K instead of h, where $t = 1^{1,2}(-1)^{2,1}\,1^{1,2} \in \mathrm{E}_2 A$. This gives that $AbA \subset B$.

LEMMA 14. *Let*

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2 A$$

commute with $\mathrm{E}_2 A$ *modulo the center* $\mathrm{G}_2(A, 0)$ *of* $\mathrm{GL}_2 A$. *Then* $a \in \mathrm{GL}_1 A$.

Proof. Case 1. A is commutative. Then the second entry in the first column of the matrix $[1^{1,2}, h] \in G_2(A, 0)$ is $c^2/\det(h) = 0$, hence $c^2 = 0$. So $aA = A$.

General case. Since $1^{1,2}h = h\ 1^{1,2}\operatorname{diag}(\mu, \mu)$ with μ in the center of A, we conclude that $a + b = a\mu$, hence $ab = ba$. Similarly, every off-diagonal entry of h commutes with every diagonal entry of h.

Since $tht^{-1} \in G_2(A, 0)$, where $t = -t^{-1} = 1^{1,2}(-1)^{2,1}\ 1^{1,2} \in E_2A$, we conclude that $ad = da$ and $bc = cb$. So the subring A_0 of A generated by 1, a, b, c, d is commutative. Consider now the dominion of A_0 in A [16]. This is a commutative subring of A, and it contains all the entries of h^{-1} [16, Propositions 1.1 and 1.3]. So we are reduced to Case 1.

LEMMA 15. *Let G, H, and K be as in Lemma 12. Suppose that* $\Sigma A(G - 1) = \Sigma(G - 1)A = A$, *i.e., no proper one-sided ideal of A contains all* $1 - u$ *with* $u \in G$. *Assume one of the conditions* (3), (4), (5). *Assume also that K is not contained in the center of* GL_2A. *Then there is a non-zero* $y \in A$ *such that* $y^{1,2} \in K$.

Proof. Pick a matrix h in K outside the center $G_2(A, 0)$ of $G_2(A, A) = GL_2A$. Write

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H. \text{ Let } h^{-1} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Case 1. $a \in GL_1A$. By Lemma 12 and the condition on G, we can assume that $b = c = 0$. Since h is outside the center, there is $x \in A$ such that $ax \neq xc$. Then $[h, x^{1,2}] = y^{1,2} \in K$ with $y = axd^{-1} - x \neq 0$. Note that we did not use any of the conditions (3), (4), (5).

General case. Under the condition (3), we argue as follows. By (3), there is $e \in A$ such that $a + xc$ is a unit of A. Replacing h by $e^{1,2}h(-e)^{1,2}$, we can assume that a is a unit. So we are reduced to Case 1.

Under condition (4), we argue as follows. If $b, c \in \operatorname{rad}(A)$, then $a \in GL_1A$, so we are done by Case 1. Otherwise, we can assume that $c \notin \operatorname{rad}(A)$ (the case $b \notin \operatorname{rad}(A)$ is similar).

We pick $z \in A$ such that $czc \equiv c (\operatorname{mod} \operatorname{rad}(A))$ and set $e = zc \in A$. Note that $e^2 \equiv e$ and $ce \equiv c$ (mod rad(A)). Replacing h by $(-az)^{1,2}h(az)^{1,2} \in K$, we change a by $a - azc = a(1 - e)$, and keep the same c. So we can assume that $a = a(1 - e)$.

Then $d'c \equiv d'ce \equiv (d'c + c'a)e = 0 (\operatorname{mod} \operatorname{rad}(A))$ and the first row of the matrix $[h, e^{1,2}]$ (modulo rad(A)) is $(1, -e)$. Since $ce \equiv c \neq 0$ (mod rad(A)), $e \neq 0$ (mod rad(A)). So $[h, e^{1,2}] \in H$ is outside the center of GL_2A, and we are reduced to Case 1.

Under condition (5), having Case 1 done, we can assume, by Lemma 14, that there is an elementary matrix g which does not commute with h modulo the center. Say, $g = z^{1,2}$. By condition (5), we find $x_i \in A$ such that $1 + c'azx_i \in GL_1A$ and

$GL_1 A$ and $\Sigma x_i = 1$. Since $g = z^{1,2} = \Pi(zx_i)^{1,2}$, we can pick some i such that $(zx_i)^{1,2}$ does not commute with h modulo the center. The first entry of the first row of the matrix $h' = [h, (zx_i)^{1,2}] \in H$ is $1 + azx_ic' \in GL_1 A$. So we are reduced to Case 1, applied to h' instead of h.

Now we can complete our proof of Theorem 8. We define

$$K = \{g \in GL_2 A: [g, E_2 A] \subset H\}$$

as in Lemma 12 above. We want to prove that $E_2(A, B) \subset K \subset G_2(A, B)$ for an ideal B of A. Set $B = \{x \in A: x^{2,1} \in K\}$. Since K is normal in $GL_2 A$, condition (7) gives that B is an ideal of A. Since $t = -t^{-1} = 1^{1,2}(-1)^{2,1}1^{1,2} \in E_2 A$ and $tB^{2,1}t^{-1} = B^{2,1}$, we obtain that $E_2(A, B) \subset H$. We have to prove that $K \subset G_2(A, B)$, i.e, $f(H)$ is central in $GL_2(A/B)$, where $f: GL_2(A/B)$ is the canonical homomorphism.

Set $H' = f(H)$ and $K' = \{g \in GL_2(A/B): [g, E_2(A/B)] \subset H'\}$. Note that both H' and K' are normalized by $f(GL_2 A)$. Clearly, $f(K) \subset K'$. So it suffices to show that $K' \subset G_2(A/B, 0)$, the center of $GL_2(A/B)$.

Note that the conditions of Theorem 8 on A imply the corresponding conditions on A/B. We assume now that K' is not central, and will obtain a contradiction.

Applying Lemma 15 (with G being the image of $GL_1 A$ in $GL_1(A/B)$) to H' and K' instead of H and K, we obtain that K' contains a non-trivial elementary matrix. Say, $c^{1,2} \in K'$ with $0 \neq c = a + B \in A/B$, where $a \in A \backslash B$. Then $[c^{1,2}, t] = c^{1,2}c^{2,1} \in H' = f(H)$. That is, H contains a matrix of the form hg, where $h \in GL_2 B$ and $g = a^{1,2}a^{2,1}$. Now we use that $[GL_2 B, E_2 A) \subset E_2(A, B)$ (see [3], [20], and [22]). In particular, $[1^{2,1}, h] \in E_2(A, B) \subset H$. So $[1^{2,1}, g] = [1^{2,1}, a^{1,2}] \in H$. Therefore $(-1)^{1,2}[1^{2,1}, a^{1,2}]\, 1^{1,2} = (-a)^{2,1}(-a)^{1,2} \in H \subset K$, hence $(-a)^{2,1} \in K$ by the definition of K and $a \in B$ by the definition of B. A contradiction!

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