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Pseudoconvex domains on complex spaces with singularities

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Section 1. Introduction

This short note deals with pseudoconvex domains (and more generally locally hyperconvex domains) on complex spaces with singularities. For strongly pseudoconvex domains the results are well known [13]: any strongly pseudoconvex domain $D \Subset X$ is a proper modification of a Stein space at a finite set, in particular it is holomorphically convex.

On the other hand an example of Grauert [12] shows that the pseudoconvexity of D is not sufficient to guarantee its holomorphic convexity. For complex manifolds the most general positive result in this direction seems to be the following theorem of Elencwajg [4]: Let X be a complex manifold and $D \Subset X$ a locally Stein open subset. Assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of $\bar{D}$. Then D is Stein.

The main purpose of this note is to generalize Elencwajg's theorem for complex spaces with singularities. We are able to prove only the following partial result:

THEOREM 1: *Let X be a complex space, $D \Subset X$ a relatively compact open subset which is locally hyperconvex and assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of $\bar{D}$. Then D is Stein.*

As a direct consequence we obtain:

COROLLARY 1: *Let X be a K-complete space and $D \Subset X$ a relatively compact open subset which is locally hyperconvex. Then D is Stein. In particular any pseudoconvex domain $D \Subset X$ is Stein.*

When X is a Stein space the above corollary can be strengthened as follows:

THEOREM 2. *Let X be a Stein space and $D \subset X$ a locally Stein open subset. Assume that D is locally hyperconvex at $\partial D \cap \mathrm{Sing}(X)$. Then D is a Stein space.*

REMARK 1.

(a) Corollary 1 for pseudoconvex domains was proved in ([1], Theorem 2) under the additional assumption that D has a globally defined boundary.

(b) A weaker result than Theorem 2 is proved in ([1], Corollary 2). Namely it is assumed that D is strongly pseudoconvex at $\partial D \cap \mathrm{Sing}(X)$.

Section 2. Preliminaries

All complex space are supposed reduced and countable at infinity.

A Stein space is called hyperconvex [18] if there exists a continuous plurisubharmonic exhaustion function $\varphi: X \to (-\infty, 0)$ (the empty set is considered hyperconvex).

Examples of hyperconvex spaces.

Let $D \subset \mathbb{C}^n$ be a Stein open set. Each of the following conditions are sufficient for the hyperconvexity of D:

(a) D is bounded and convex [18]
(b) D is bounded and has C^2 boundary [2] or C^1 boundary [11]
(c) D is a bounded Reinhardt domain containing the origin [5]
(d) D is a tube whose base $\mathrm{Re}(D) \subset \mathbb{R}^n$ is bounded and convex [5].

Other examples can be found in ([5], [8]).

To get examples of hyperconvex spaces in the singular case one may take subspaces or finite morphisms into the nonsingular ones given above. In particular any relatively compact analytic polyhedron in a Stein space is hyperconvex and any Stein space can be exhausted with hyperconvex open sets.

DEFINITION 1. Let X be a complex space, $D \subset X$ an open subset and $A \subset \partial D$ any subset. We say that D is *locally hyperconvex* at A if for any $x_0 \in A$ there exists an open neightbourhood U of x_0 such that $U \cap D$ is hyperconvex. When $A = \partial D$ D is called locally hyperconvex.

DEFINITION 2 ([1], [12], [13]). Let X be a complex space and $D \Subset X$ a relatively compact open subset. D is called *pseudoconvex* if for any $x_0 \in \partial D$ there exists an open neighbourhood U of x_0 and a continuous plurisubharmonic function $\varphi: U \to \mathbb{R}$ such that $U \cap D = \{x \in U \mid \varphi(x) < 0\}$.

It is clear from the above definitions that any pseudoconvex domain is locally hyperconvex.

The proof of Theorem 1 relies on a patching technique which allows us to produce a continuous strongly plurisubharmonic exhaustion function $\varphi: D \to \mathbb{R}$. To obtain the Steiness of D we invoke the following result of Narasimham [13]:

THEOREM 3. *Let D be a complex space and assume that there exists a continuous strongly plurisubharmonic exhaustion function $\varphi: D \to \mathbb{R}$. Then D is a Stein space.*

For the proof of Theorem 2 we shall need the following two results:

THEOREM 4 ([1], Theorem 4). *Let X be a Stein space and $D \subset X$ a locally Stein open subset. Assume that there is an open neighbourhood U of $\partial D \cap \mathrm{Sing}(X)$ such that $D \cap U$ is a Stein space. Then D itself is a Stein space.*

THEOREM 5 ([14], Theorem 2). *Let X be a Stein space, $A \subset X$ a closed analytic subset and V an open neighbourhood of A. Then there exists a continuous plurisubharmonic function $p: X \to \mathbb{R}$ such that $A \subset \{p < 0\} \subset V$.*

Let us recall also the following:

DEFINITION 3. A complex space X is called K-complete if for any $x_0 \in X$ there is a holomorphic map $f: X \to \mathbb{C}^p, p = p(x_0)$ such that x_0 is an isolated point of $f^{-1}(f(x_0))$.

It is known [9] that a complex space X of pure dimension n is K-complete iff X can be realised as a remified domain over $\mathbb{C}^n$, but we shall not need this result.

In ([1], Lemma 5) it was proved:

THEOREM 6. *Every relatively compact open subset of a K- complete space carries a C^∞ strongly plurisubharmonic function.*

Section 3. Proof of the main results

In the proof of Theorem 1 the existence of some special convex increasing functions on $(-\infty, 0)$ will play an important role. So we state:

LEMMA 1. *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of negative real numbers such that $a_n \to 0$. Then there exists a function $\tau: (-\infty, 0) \to \mathbb{R}$ with the following properties:*

(1) *τ is continuous, increasing and convex*
(2) $\tau \geqslant 0$
(3) $\lim_{x\to 0} \tau(x) = \infty$
(4) $\tau(a_{n+1}) - \tau(a_n) < 1$ for every $n \in \mathbb{N}$

Proof. We define τ to be linear on each interval $[a_n, a_{n+1}]$ and to vanish identically near $-\infty$. The precise definition is as follows:

$$\tau(x) = \begin{cases} n - \left(\dfrac{a_2}{a_1} + \cdots + \dfrac{a_n}{a_{n-1}}\right) - \dfrac{x}{a_n} & \text{if } a_n \leqslant x \leqslant a_{n+1} \\ 0 & \text{if } x \leqslant a_1 \end{cases}$$

Properties (1), (2) and (4) follow easily from the definition of τ so it remains to verify (3). Since τ is increasing it suffices to show that $\tau(a_n) \to \infty$. Now

$$\tau(a_{n+p}) - \tau(a_n) = \frac{a_n - a_{n+1}}{a_n} + \cdots + \frac{a_{n+p-1} - a_{n+p}}{a_{n+p-1}} \geqslant \frac{a_n - a_{n+p}}{a_n},$$

hence for a given n $\tau(a_{n+p}) - \tau(a_n) \geqslant \frac{1}{2}$ if p is sufficiently large (depending on n). It follows that $\tau(a_n) \to \infty$ which proves the lemma.

LEMMA 2. *Let* $f_1, \ldots, f_n: (-\infty, 0) \to (-\infty, 0)$ *be increasing functions such that for any* $i \in \{1, \ldots, n\}$ $\lim_{x \to 0} f_i(x) = 0$. *Then there exists a continuous increasing convex function* $\tau: (-\infty, 0) \to \mathbb{R}$ *such that:*

(a) $\lim_{x \to 0} \tau(x) = \infty$
(b) $\tau \circ f_i - \tau \circ f_j$ is bounded for any $i, j \in \{1, \ldots, n\}$

Proof. From the assumption "$\lim_{x \to 0} f_i(x) = 0$ for any $i \in \{1, \ldots, n\}$" it follows that there exists an increasing sequence $\{\alpha_\nu\}_{\nu \in \mathbb{N}}$ of negative real numbers, $\alpha_\nu \to 0$ such that:

$$\max\{f_1(\alpha_\nu), \ldots, f_n(\alpha_\nu)\} < \min\{f_1(\alpha_{\nu+1}), \ldots, f_n(\alpha_{\nu+1})\} \quad \text{for any } \nu \in \mathbb{N}. \tag{$*$}$$

If we set $a_\nu = \min\{f_1(\alpha_\nu), \ldots, f_n(\alpha_\nu)\}$ for odd ν and $a_\nu = \max\{f_1(\alpha_\nu), \ldots, f_n(\alpha_\nu)\}$ for even ν then $a_1 < \ldots a_\nu < a_{\nu+1} < \cdots < 0$ and $a_\nu \to 0$.

By Lemma 1 there is a continuous convex increasing function

$$\tau: (-\infty, 0) \to \mathbb{R}, \tau \geqslant 0, \lim_{x \to 0} \tau(x) = \infty$$

and

$$\tau(a_{\nu+1}) - \tau(a_\nu) < 1 \quad \text{for any } \nu \in \mathbb{N}.$$

To prove Lemma 2 it remains to verify that $\tau \circ f_i - \tau \circ f_j$ is bounded. Since τ is bounded below ($\tau \geqslant 0$) it suffices to check that $\tau(f_i(x)) - \tau(f_j(x))$ is bounded for $x < 0$ sufficiently close to 0. If $\alpha_{2\nu} \leqslant x \leqslant \alpha_{2\nu+2}$ then

$$a_{2\nu-1} \leqslant \min\{f_i(x), f_j(x)\} \leqslant \max\{f_i(x), f_j(x)\} \leqslant a_{2\nu+2},$$

hence $\tau(f_i(x)) - \tau(f_j(x)) < 3$, which proves Lemma 2.

LEMMA 3. *Let* Y *be a complex space which carries a continuous strongly plurisubharmonic function and let* $D \Subset Y$ *be a relatively compact open subset. Assume that there exists open subsets of* $Y A_i \Subset B_i \Subset C_i i \in \{1, \ldots, k\}, D \subset \bigcup_{i=1}^{K} A_i$ *and continuous plurisubharmonic exhaustion functions* $\varphi_i: C_i \cap D \to \mathbb{R}$ *such that*

$\varphi_i|_{B_i\cap B_j\cap D} - \varphi_j|_{B_i\cap B_j\cap D}$ *is bounded for any* $i, j \in \{1, \ldots, k\}$. *Then* D *is a Stein space.*

Proof. The proof is obtained by a slight modification of the arguments given by M. Peternell in ([16], Lemma 10). For the sake of completeness we shall indicate the modifications to be done.

Take $p_i' \in C_0^\infty(Y)$ with $p_i' \geqslant 0$, supp $p_i' \subset B_i$ and $p_i'|_{A_i} = 1$. We define the functions $p_i \in C_0^\infty(Y)$ in the following way: for each i the functions $\varphi_j - \varphi_i$ $j \in \{1, \ldots, k\}$ are bounded on $\partial B_j \cap A_i \cap D$ so we can choose a sufficiently large constant $\lambda_i > 0$ with $\lambda_i p_i' > \varphi_j - \varphi_i$ on $\partial B_j \cap A_i \cap D$. We set $p_i = \lambda_i p_i'$. Since $p_j = 0$ on ∂B_j we have:

$$p_i + \varphi_i > p_j + \varphi_j \quad \text{on } \partial B_j \cap A_i \cap D \tag{$*$}$$

Let now φ be a continuous strongly plurisubharmonic function on Y and let $A > 0$ be a sufficiently large constant such that $A\varphi + p_i$ is strongly plurisubharmonic for any $i \in \{1, \ldots, k\}$. We set $I = \{1, \ldots, k\}$ and for $x \in D$ we define $I(x) \subset I$ by $I(x) = \{i \in I \mid x \in B_i\}$. If $x \in D$ we set $u(x) = \max_{i \in I(x)}\{p_i(x) + \varphi_i(x)\}$. We show that $\psi = A\varphi + u$ is a continuous strongly plurisubharmonic exhaustion function on D. It is clear that ψ is an exhaustion function because φ_i are exhaustion functions on $C_i \cap D$, hence it remains to verify that ψ is a continuous strongly plurisubharmonic function on D. Let $x_0 \in D$ and set $I'(x_0) = \{i \in I \mid x_0 \in \partial B_i\}$. Choose a neighbourhood $D_{x_0} \subset D$ of x_0 such that $D_{x_0} \cap B_i = \emptyset$ if $i \notin I(x_0) \cup I'(x_0)$ and let $i_0 \in I(x_0)$ with $x_0 \in A_{i_0}$. For each $j \in I'(x_0)$ it follows from ($*$) that $p_{i_0} + \varphi_{i_0} > p_j + \varphi_j$ on D_{x_0} if $D_{x_0} \subset A_{i_0}$ is chosen small enough. We get $u|_{D_{x_0}} = \max_{i \in I(x_0)}\{p_i + \varphi_i\}$ hence $\psi|_{D_{x_0}} = \max_{i \in I(x_0)}\{A\varphi + p_i + \varphi_i\}$ which shows that ψ is a continuous strongly plurisubharmonic function. By Theorem 3 D is Stein and the proof of Lemma 3 is complete.

THEOREM 1. *Let* X *be a complex space and* $D \Subset X$ *a relatively compact open subset which is locally hyperconvex and assume that there exists a continuous strongly plurisubharmonic function in a neighbourhood of* $\bar{D}$. *Then* D *is Stein.*

Proof. Let Y be a neighbourhood of $\bar{D}$ and φ a continuous strongly plurisubharmonic function on Y. Choose open subsets $A_i \Subset B_i \Subset C_i \subset Y$ $i \in \{1, \ldots, k\}$ such that:

(1) $D \subset \bigcup_{i=1}^k A_i$

(2) for any $i \in \{1, \ldots, k\}$ there exists a continuous plurisubharmonic exhaustion function v_i: $C_i \cap D \to (-\infty, 0)$.

For every $i, j \in \{1, \ldots, k\}$ such that $B_i \cap B_j \cap D \neq \emptyset$ we define the function E_{ij}: $(-\infty, 0) \to (-\infty, 0)$ by $E_{ij}(x) = \inf\{v_j(z) \mid z \in B_i \cap B_j \cap D \ v_i(z) \geqslant x\}$. E_{ij} are increasing functions and $\lim_{x\to 0} E_{ij}(x) = 0$ because v_i are exhaustion functions. Let h: $(-\infty, 0) \to (-\infty, 0)$ be the identity map. Now we use Lemma 2 for the finite set of functions $\{E_{ij}, h\}$ and we get a continuous increasing convex function

$\tau : (-\infty, 0) \to \mathbb{R}$ such that:

(1) $\lim_{x \to 0} \tau(x) = \infty$
(2) $\tau - \tau \circ E_{ij}$ is bounded for any $i, j \in \{1, \ldots, k\}$ with $B_i \cap B_j \cap D \neq \emptyset$.

Setting $\varphi_i = \tau \circ v_i$ we get continuous plurisubharmonic exhaustion functions on $C_i \cap D$. Moreover, if $z \in B_i \cap B_j \cap D$ $E_{ij}(v_i(z)) \leqslant v_j(z)$, therefore $\varphi_i(z) - \varphi_j(z) \leqslant (\tau - \tau \circ E_{ij})(v_i(z))$. From Lemma 3 D is Stein and the proof of Theorem 1 is complete.

We give now some immediate consequences of Theorem 1. By Theorem 6 we know that any relatively compact open subset of a K-complete space carries a C^∞ strongly plurisubharmonic function. Therefore we obtain:

COROLLARY 1. *Let X be a K-complete space and $D \Subset X$ a relatively compact open subset which is locally hyperconvex. Then D is Stein. In particular any pseudoconvex domain $D \Subset X$ is Stein.*

Corollary 1 is a particular case of the following open problem (see [8]):

LEVI PROBLEM: *Let X be a K-complete space and $D \Subset X$ a locally Stein open set. Is D itself a Stein space?*

We show that this is the case at least when X is a Stein space and D is locally hyperconvex at $\partial D \cap \operatorname{Sing}(X)$, namely we prove:

THEOREM 2. *Let X be a Stein space and $D \subset X$ a locally Stein open subset. Assume that D is locally hyperconvex at $\partial D \cap \operatorname{Sing}(X)$. Then D is a Stein space.*

Proof. For each $x \in \operatorname{Sing}(X)$ we choose a hyperconvex neighbourhood $V_x \Subset X$ of x such that $V_x \cap D$ is hyperconvex. Then $V = \bigcup_{x \in \operatorname{Sing}(X)} V_x$ is an open neighbourhood of $\operatorname{Sing}(X)$ and by Theorem 5 there is a continuous plurisubharmonic function p on X such that $B = \{p < 0\}$ contains $\operatorname{Sing}(X)$ and $\bar{B} \subset V$. We show that $B \cap D$ is locally hyperconvex. Indeed, for any $x_0 \in \overline{B \cap D} \subset \bar{B} \subset V$ there exists $x \in \operatorname{Sing}(X)$ with $x_0 \in V_x$. On the other hand $V_x \cap B \cap D = (V_x \cap B) \cap (V_x \cap D)$ which is hyperconvex as an intersection of two hyperconvex open subsets. Therefore $B \cap D$ is locally hyperconvex and by Corollary 1 and an exhaustion argument it follows that B $\cap$ D is Stein. In view of Theorem 4 D itself is a Stein space and the proof is complete.

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