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Quasi-periodic functions and Drinfeld modular forms

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1. Introduction

Let $E = \mathbb{C}/\Lambda$ be a complex elliptic curve with period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and let $\wp(z)$, $\zeta(z)$ be the associated Weierstrass functions. Then $\wp(z)$ and $\zeta(z)$ are meromorphic on $\mathbb{C}$ and $\wp(z) = -\zeta'(z)$ is Λ-periodic, whereas $\zeta(z)$ satisfies

$$\zeta(z + \omega_i) = \zeta(z) + \eta_i \quad (i = 1, 2) \tag{1.1}$$

with certain $\eta_i \in \mathbb{C}$. Between the periods of first kind ω_i and that of second kind η_i, *Legendre's relation*

$$\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i \tag{1.2}$$

holds (suppose $\mathrm{im}(\omega_1/\omega_2) > 0$). Further, the η_i, considered as functions on the complex upper half-plane, are closely related to the "false Eisenstein series of weight two," i.e., the logarithmic derivative of the discriminant function $\Delta(\omega)$ (See [8], App. 1 for a precise statement).

(1.3) Let now K be a function field in one variable having the field $\mathbb{F}_q$ with q elements as exact constant field (q = some power of the prime p). Fix a place ∞ of K of degree δ, say, and put

$$A = \{f \in K \mid f \text{ has poles at most at } \infty\}.$$

On A, we have the degree function $\deg a = \delta$ times (pole order of a at ∞). Further, let K_∞ be the completion of K at ∞ with its normalized absolute value $|?|$, and C the completion of an algebraic closure $\bar{K}_\infty$ with respect to the unique extension, also named $|?|$, of $|?|$ to $\bar{K}_\infty$. Note that C is again algebraically closed, i.e., is the minimal complete algebraically closed field containing K_∞. The basic example (in fact, the only one considered later on) is given by

$$\begin{aligned} K &= \mathbb{F}_q(T) \text{ the field of rational functions,} \\ \text{“}\infty\text{”} &= \text{the place at infinity, so} \\ A &= \mathbb{F}_q[T] \text{ is the polynomial ring, and} \\ K_\infty &= \mathbb{F}_q((T^{-1})) \text{ the field of finite-tailed Laurent series in } T^{-1}. \end{aligned} \tag{1.4}$$

(1.5) It is well known that there is a deep analogy between the arithmetic of $\mathbb{Q}$ and that of K, where A (resp. K_∞, C) replaces the ring $\mathbb{Z}$ of integers (resp. $\mathbb{R}$, $\mathbb{C}$). Roughly, the role of complex analysis in number theory is played by rigid analysis over the "complex" field C. (See [5] and the references given there.)

As explained below, the classical theory of elliptic modular forms corresponds to the theory of rank two Drinfeld modules. By following ideas of Pierre Deligne, as elaborated by Greg Anderson and Jing Yu, we will define quasi-periodic functions, periods of second kind, and de Rham cohomology for a Drinfeld module. Specializing to the case $A = \mathbb{F}_q[T]$ and rank $=$ two, where the theory of modular forms is sufficiently well developed, we obtain a non-vanishing result (Cor. 6.3) for the associated determinant which replaces (1.2). For weight reasons, the latter will not be a constant as in (1.2), but a meromorphic modular form of negative weight. Applying the results of [6], we get a very precise description of that form, in particular its expansion at ∞. As in the classical case, the maps $\omega \mapsto \eta_i(\omega)$ $(i = 1, 2)$ which to each ω associate its periods of second kind are related to a "false Eisenstein series" (Thm. 7.10). Finally, we calculate the Gauss-Manin connection ∇ on $H^*_{\mathrm{DR}}(\phi)$, where the Drinfeld module ϕ varies over the "upper half-plane" Ω (Thm. 8.8). After a suitable normalization, ∇ is defined for the Tate–Drinfeld module, i.e., by a matrix whose entries are power series over A.

Certainly, some of our methods and results may be widely generalized. (For the "de Rham cohomology" introduced in section 3, see the forthcoming articles [2] and [9].)

For example, the local system over the moduli scheme defined by H^*_{DR}, which is rather simple in our case, will be of high interest if A is general. Nevertheless, the relation with modular forms presented here seems to be a feature special to the polynomial case $A = \mathbb{F}_q[T]$ where one disposes of a "natural" basis for H^*_{DR}.

2. Review of Drinfeld modules

(See [3] or [5] for proofs.)

Let Λ be a rank r lattice in C, i.e., a projective, discrete A-submodule of C of A-rank $r \in \mathbb{N}$. Put

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda}' (1 - z/\lambda), \tag{2.1}$$

where as usual, Π' will denote the product over the non-zero elements of a lattice. The product is easily seen to converge for $z \in C$; it defines an entire surjective mapping e_Λ: $C \to C$ which is additive and $\mathbb{F}_q$-linear. Further, e_Λ is Λ-periodic with kernal Λ.

Let $C\{\tau\}$ be the non-commutative ring of polynomials over C of the form $\Sigma\, l_i \tau^i$, where τ corresponds to X^q, and the multiplication $f \circ g$ is defined by inserting $g(X)$ into $f(X)$. Elements of $C\{\tau\}$ will be referred to as *q-additive polynomials*. Note that $\tau^0 = X$ is the identity of $C\{\tau\}$. We may consider

$$e_\Lambda(X) = \sum \alpha_i X^{q^i} = \sum \alpha_i \tau^i$$

as an element of the ring $C\{\{\tau\}\}$ of "power series" in τ with certain growth conditions on the coefficients α_i.

Next, for $a \in A$, let $\phi_a^\Lambda \in C\{\tau\}$ be defined by the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C & \xrightarrow{e_\Lambda} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a & & \downarrow \phi_a^\Lambda & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C & \xrightarrow{e_\Lambda} & C & \longrightarrow & 0. \end{array} \tag{2.2}$$

A closer look shows ϕ_a^Λ to be of the form

$$\phi_a^\Lambda = l_0 \tau^0 + \cdots + l_{rd} \tau^{rd}, \tag{2.3}$$

where $l_i \in C$, d $=$ deg a, $l_0 = a$, $l_{rd} \neq 0$. Further, $a \mapsto \phi_a$ defines a ring homomorphism ϕ^Λ: $A \to C\{\tau\}$.

(2.4) A ring homomorphism ϕ: $A \to C\{\tau\}$, $a \mapsto \phi_a$ satisfying the conditions of (2.3) will be called a *Drinfeld module* over C of rank r. Given ϕ, one may recover e_Λ and Λ (see, e.g., [5]). This sets up a bijection $\Lambda \mapsto \phi^\Lambda$ between the set of rank r lattices in C and the set of rank r Drinfeld modules over C. By means of ϕ, the additive group $G_a | C$ gets another structure as an A-module which differs from the canonical one. Two lattices, Λ, Λ', define isomorphic Drinfeld modules if and only if they are similar, i.e., there exists $c \in C^*$ such that $\Lambda = c\Lambda$.

(2.5) From now on, let us restrict to lattices Λ which are free over A. Let Γ be the group $GL(r, A)$, considered as a discrete subgroup of the Lie group $GL(r, K_\infty)$ that operates on the projective space $\mathbb{P}^{r-1}(C)$ by fractional linear transformations. We define

$$\Omega^r = \{(z_1: \ldots :z_r) \in \mathbb{P}^{r-1}(C) \mid z_1, \ldots, z_r \ K_\infty\text{-linearly independent}\}.$$

Then Ω^r carries the structure of an r-1-dimensional rigid analytic manifold over C, and the induced action of Γ on Ω^r is analytic. We have the bijection

$$\Gamma\backslash\Omega^r \xrightarrow{\cong} \{\text{Similarity classes of free lattices of rank } r\}$$
$$\mathbf{z} = (z_1: \ldots :z_r) \mapsto \text{class of } \Lambda_{\mathbf{z}} = Az_1 + \cdots + Az_r$$

which in fact gives an analytic description of the moduli scheme for Drinfeld modules of rank r.

2.6 Example: Let $A = \mathbb{F}_q[T]$. Any rank r Drinfeld module is given by

$$\phi_T = T\tau^0 + \ldots l_r\tau^r = TX + \ldots l_rX^{q^r},$$

where $l_r \neq 0$. If further $r = 2$, then $\Omega = \Omega^2 = \mathbb{P}^1(C)\backslash\mathbb{P}^1(K_\infty) = C\backslash K_\infty$, and $\Gamma = GL(2, A)$ acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = (az + b)/(cz + d)$. Any rank two ϕ is given by $\phi_T = T\tau^0 + g\tau + \Delta\tau^2$, where $g, \Delta \in C$, $\Delta \neq 0$. Letting vary ϕ, g and Δ are modular forms for Γ, see (4.9).

3. Quasi-periodic functions

(This section is developed entirely from the ideas of Greg Anderson, Pierre Deligne, and Jing Yu.)

Let Λ be a rank r lattice in C with exponential function $e = e_\Lambda$ and Drinfeld module $\phi = \phi^\Lambda$. Let $a \in A$ be non-constant and $\partial \in \tau C\{\tau\}$ an element of $C\{\tau\}$ without constant term. The difference equation

$$\partial(e(X)) = F(aX) - aF(X) \tag{3.1}$$

has a solution $F \in C\{\{\tau\}\}$ which is well-defined up to adding a multiple of $\tau^0 = X$. Thus, assuming the τ^0-coefficient to be zero (*which we will always assume in what follows*), there exists a unique solution $F \in C\{\{\tau\}\}$. Comparing coefficients, one easily verifies F to define an entire function $C \to C$, also called F.

(3.2) Let $C_\phi\{\tau\}$ be the A-bimodule $C\{\tau\}$, where the left action is as usual, and the right action of $a \in A$ is right multiplication with ϕ_a.

3.3. LEMMA: (i) *Let* $\partial^{(0)}: A \to C\{\tau\}$ $a \mapsto \partial_a^{(0)}$ *be defined by* $\partial_a^{(0)} = \phi_a - a\tau^0$. *Then* $\partial_{ab}^{(0)} = a\partial_b^{(0)} + \partial_a^{(0)} \circ \phi_b$, *i.e.,* $\partial^{(0)}$ *is a derivation* $A \to C_\phi\{\tau\}$.

(ii) *Let* $\partial: a \mapsto \partial_a$ *be any* $\mathbb{F}_q$*-linear derivation of* A *into* $C_\phi\{\tau\}$ *with values in* $\tau C\{\tau\}$. *Then the unique solution* $F_\partial \in \tau C\{\{\tau\}\}$ *of*

$$\partial_a(e(X)) = F(aX) - aF(X) \qquad (*)$$

does not depend on a, provided that a is non-constant. (For $a =$ *constant,* $\partial_a = 0 = F$.*)*

Proof: (i) is trivial: $\partial_{ab}^{(0)} = \phi_{ab} - ab\tau^0 = a(\phi_b - b\tau^0) + (\phi_a - a\tau^0) \circ \phi_b$ since $\phi_{ab} = \phi_a \circ \phi_b$. (ii) Let F_a be the unique solution (without τ^0-term) of $(*)$. From the derivation relations, it is straightforward to show
(1) $F_{ca} = F_a$ $(c \in \mathbb{F}_q^*)$;
(2) $F_a = F_b \Rightarrow F_{a+b} = F_a = F_b$;
(3) $F_a = F_b \Rightarrow F_{ab} = F_a = F_b$.
Clearly, (1), (2), (3) imply the assertion for A a polynomial ring $A = \mathbb{F}_q[a]$. Since any A is an integral extension of a polynomial ring, (1), (2), (3) give the result.

(3.4) In the following, derivations will always assumed to be $\mathbb{F}_q$-linear. Let $D(\phi)$, $D_i(\phi)$, $D_{si}(\phi)$ respectively be the A-bimodule of derivations, inner derivations, strictly inner derivations respectively from A to $C_\phi\{\tau\}$ with values in $\tau C\{\tau\}$. (Some $\partial \in D(\phi)$ is called an *inner* (*strictly inner*) *derivation* if there exists $m \in C\{\tau\}$ ($m \in \tau C\{\tau\}$) such that for $a \in A$, $\partial_a = m \circ \phi_a - am$ holds.) The function F_∂ associated with some $\partial \in D(\phi)$ will be called quasi-periodic for Λ.

3.5. *Remarks:* (i) F: $C \to C$ to be quasi-periodic implies $F(z + \lambda) = F(z) + F(\lambda)$ $(\lambda \in \Lambda)$, where F restricted to Λ is A-linear. This is analogous with (1.1).

(ii) If $\partial = \partial^{(0)}$ is the inner derivation defined in (3.3i), then $F_\partial(z) = e(z) - z$ which thus is quasi-periodic.

(iii) Let ∂ be the strictly inner derivation associated with $m \in \tau C\{\tau\}$. Then $F_\partial(z) = m(e(z))$ which in fact is periodic and vanishes on Λ.

3.6. EXAMPLE: Let $A = \mathbb{F}_q[T]$ as in (1.4). Some $\partial \in D(\phi)$ is given by $\partial_T \in \tau C\{\tau\}$ which may be arbitrarily prescribed. The corresponding F_∂ is the solution of

$$\partial_T(e(z)) = F(Tz) - TF(z).$$

Since $\phi_T = T\tau^0 + \cdots + l_r\tau^r$, $l_r \neq 0$, it is easy to see that $H^*_{\mathrm{DR}}(\phi) = D(\phi)/D_{\mathrm{si}}(\phi)$, considered as a left C-vector space, has dimension r. A basis is given by the r derivations $\partial^{(i)}$ defined by $\partial^{(i)}_T = \tau^i$, $0 < i \leqslant r$, or rather $\{\partial^{(i)} | 0 \leqslant i < r\}$, where $\partial^{(0)}_T = \phi_T - T\tau^0$. A generalization of this fact for arbitrary A, as well as an interpretation of $H^*_{\mathrm{DR}}(\phi)$ as *de Rham cohomology group* of the Drinfeld module ϕ, will be given in [2], see also [1]. In that context, the map $\Lambda \times H^*_{\mathrm{DR}}(\phi) \to C$: $(\omega, \partial) \mapsto F_\partial(\omega)$ will appear as some sort of "path integration."

4. Relations with modular forms

From now on, we assume the situation of (1.4), (2.6), (3.6), i.e., $A = \mathbb{F}_q[T]$. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in C^r$, the ω_i linearly independent over K_∞, and $\Lambda_{\boldsymbol{\omega}}$ the lattice $A\omega_1 + \cdots + A\omega_r$. Let $e = e_\Lambda$, and $\phi = \phi^\Lambda$ the associated Drinfeld module. We further put for $0 \leqslant i < r$ $F^{(i)} = F_{\partial^{(i)}}$. Thus

$$F^{(0)}(z) = e(z) - z, \quad \text{and for } 0 < i < r,$$

$$e^{q^i}(z) = F^{(i)}(Tz) - TF^{(i)}(z). \tag{4.1}$$

We use a subscript $\boldsymbol{\omega}$ to indicate the dependence of all the data introduced from $\boldsymbol{\omega}$, thus $\mathrm{e} = \mathrm{e}_{\boldsymbol{\omega}}$, $\phi = \phi_{\boldsymbol{\omega}}$, $F^{(i)} = F^{(i)}_{\boldsymbol{\omega}}$. Now for any lattice Λ, z and $c \in C$, $c \neq 0$, we have

$$e_{c\Lambda}(cz) = ce_\Lambda(z).$$

As an easy consequence, we deduce

$$F^{(i)}_{c\boldsymbol{\omega}}(cz) = c^{q^i} F^{(i)}_{\boldsymbol{\omega}}(z). \tag{4.2}$$

Note that for $i > 0$, $F^{(i)}_{\boldsymbol{\omega}}(\omega_j)$ is the analogue of a *period of second kind* on an elliptic curve.

We are interested in the *Legendre determinant*

$$\mathrm{L}(\boldsymbol{\omega}) = \det\,(F^{(i)}_{\boldsymbol{\omega}}(\omega_j))\ (0 \leqslant i < r, 0 < j \leqslant r).$$

By the above,

$$L(c\boldsymbol{\omega}) = c^k L(\boldsymbol{\omega}) \quad \text{with} \quad k = 1 + q + \cdots + q^{r-1}. \tag{4.3}$$

4.4. LEMMA: *For $\gamma \in GL(r, A)$, we have*

$$L(\gamma\boldsymbol{\omega}) = (\det \gamma) L(\boldsymbol{\omega}).$$

Proof: $F_{\boldsymbol{\omega}}^{(i)}$ depends only on the lattice $\Lambda_{\boldsymbol{\omega}} = \Lambda_{\gamma\boldsymbol{\omega}}$. Thus, $L(\gamma\boldsymbol{\omega}) = \det(F_{\boldsymbol{\omega}}^{(i)}((\gamma\boldsymbol{\omega})_j)) = (\det\gamma)\det(F_{\boldsymbol{\omega}}^{(i)}(\omega_j)) = (\det\gamma)L(\boldsymbol{\omega})$ since $F^{(i)}|\Lambda$ is A-linear.

We now finally restrict to the case $r = 2$ where we will give a description of the map $\boldsymbol{\omega} \mapsto L(\boldsymbol{\omega})$. By (4.3), it suffices to consider $\boldsymbol{\omega}$ of the form $\boldsymbol{\omega} = (\omega, 1)$ with $\omega \in \Omega = C\setminus K_\infty$. The function $L(\omega) = L((\omega, 1))$ now satisfies

(4.5) (i) $\mathrm{L}(\gamma\omega) = (\det\gamma)(c\omega + d)^{-q-1}\mathrm{L}(\omega)$ ($\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma = GL(2, A)$ acts by $\gamma\omega = (a\omega + b)/(c\omega + d)$), and (ii) L is holomorphic on Ω in the rigid analytic sense.

Here, (i) is just a repetition of (4.3) and (4.4), whereas (ii) had to be shown. Since we did not give a description of the analytic structure on Ω (which may be found e.g., in [7]), we omit the simple proof. But compare (5.3).

(4.6) In order to make a meromorphic modular form out of $\mathrm{L}(\omega)$, we have to introduce cusp conditions. Let ϱ the rank one Drinfeld module (sometimes called the *Carlitz module*) defined by $\varrho_T = T\tau^0 + \tau = TX + X^q$. Via (2.4), it corresponds to the rank one lattice $L = \bar{\pi}A$ with some number $\bar{\pi} \in C$. Hopefully, the double occurrence of the symbol L will cause no confusion. *Note*: $\bar{\pi}$ is only defined up to a $(q - 1)$-st root of unity. We choose one $\bar{\pi}$ and fix it for what follows. It is similar to the period $2\pi i$ of the exponential function. Several additive and multiplicative expressions are known for $\bar{\pi}$ ([6], (4.9)–(4.11)). In particular, its absolute value equals $q^{q/(q-1)}$. Let e_L be the lattice function associated with L, and $t(z) = 1/e_L(\bar{\pi}z)$. Obviously, $t(z)$ is A-periodic. In our framework, it plays the part of $q(z) = e^{2\pi i z}$ as a uniformizer at ∞.

(4.7) For $z \in C$, we define the "imaginary part" $|z|_i = \inf|z - x|$, where x runs through K_∞. Also, let $\Omega_s = \{z \in \Omega \mid |z|_i \geqslant s\}$, where $s \in \mathbb{R}$. Then we have:

4.8. LEMMA ([6], LEMMA 5.5): For $|z|_i > 1$, $|e_L(\bar{\pi}z)|$ is a monotonically increasing function of $|z|_i$. Namely, if $|z| = |z|_i = q^{d-\varepsilon}$ with some $d \in \mathbb{N}$, $0 \leqslant \varepsilon < 1$, then $\log_q|e_L(\bar{\pi}z)| = q^d(q/(q - 1) - \varepsilon)$. There exists a constant

$c_0 > 1$ such that $|z|_i \leqslant \log_q|e_L(\bar{\pi}z)| \leqslant c_0|z|_i$. Further, $t(z)$ defines an analytic isomorphism of $A\backslash\Omega_s$ $(s > 1)$ with some pointed ball $\{z \in C \,|\, |z| \leqslant r\} - \{0\}$.

4.9. DEFINITION: A function $f\colon \Omega \to C$ is a holomorphic *modular form* of weight $k \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)$ if the following conditions hold:

(i) $f(\gamma z) = (\det \gamma)^{-m}(cz + d)^k f(z)$ $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$;

(ii) f is holomorphic;

(iii) f is holomorphic at ∞.

The last point signifies that for $|z|_i \gg 0$, $f(z)$ has a series expansion $f(z) = \Sigma_{i\geqslant 0} a_i t^i(z)$, which, due to (4.8), corresponds to the cusp condition in the elliptic modular case. Replacing "holomorphic" by "meromorphic" and admitting Laurent series around ∞ with a finite number of terms of negative order, we may define meromorphic modular forms. For a further discussion, see [5] or [6]. In the next sections, we will calculate the expansion of $L(\omega)$ around ∞, thereby verifying it as meromorphic modular of weight $-q-1$ and type -1. For that purpose, we will need the following series of polynomials: for $0 \neq a \in A$ of degree d, let $\varrho_a(X) = aX + \ldots \in A[X]$ be the polynomial derived from the Carlitz module. We have $\deg(\varrho_a(X)) = q^d = |a|$, and its leading coefficient agrees with that of a as a polynomial in T. Put $f_a(X) = \varrho_a(X^{-1})X^{|a|}$. Then for example, $f_1(X) = 1$, $f_T(X) = 1 + TX^{q-1}$, $f_{T^2}(X) = 1 + (T + T^q)X^{q^2-q} + T^2X^{q^2-1}$.

5. Behavior of L at infinity

In all that follows, $A = \mathbb{F}_q[T]$, and we are considering the rank two case only.

For $\omega \in \Omega$ and $i = 0, 1$ let $F^{(i)}_\omega(z) = F^{(i)}_{(\omega,1)}(z)$ as defined in (4.1). Also, let e_ω, $\Lambda_\omega \ldots$ be the objects associated with $\boldsymbol{\omega} = (\omega, 1)$. We are going to investigate the function

(5.1) $\omega \mapsto L(\omega) = F^{(1)}_\omega(\omega) - \omega F^{(1)}_\omega(1)$ on Ω which, in view of $F^{(0)}(z) = e(z) - z$, is the inhomogenous expression of $L(\boldsymbol{\omega}) = L((\omega, 1))$. Actually, we will show

5.2. THEOREM: $\lim\limits_{|\omega|_i \to \infty} t(\omega)L(\omega) = \bar{\pi}^{-q}$,

the proof of which will occupy this section. Let us first treat the function $F_\omega(z) = F^{(1)}_\omega(z)$ (there is no further need for the superscript (1)). From (4.1) we get

$$F_\omega(z) = e^q_\omega(z/T) + TF_\omega(z/T)$$
$$= e^q_\omega(z/T) + Te^q_\omega(z/T^2) + T^2F_\omega(z/T^2)$$
$$\vdots$$

Looking at the definition of e_ω, we see $|e_\omega(z)| = |z|$ whenever $|z| < \inf_{(0,0) \neq (a,b) \in A^2} |a\omega + b| = \inf(|\omega|_i, 1)$. Thus

$$|T^i e^q_\omega(z/T^{i+1})| = |z|^q q^{i-q(i+1)} \quad \text{for} \quad i \gg 0$$

which shows the convergence (pointwise, in fact locally uniformly in z and in ω) of the series

$$F_\omega(z) = \sum_{i \geqslant 0} T^i e^q_\omega(z/T^{i+1}). \tag{5.3}$$

We are interested in the behavior of $F_\omega(\omega)$ and $F_\omega(1)$ for $|\omega|_i \gg 0$. The above shows:

$$F_\omega(1) \text{ is bounded on } \Omega_1 = \{\omega \in \Omega \,|\, |\omega|_i \geqslant 1\}. \tag{5.4}$$

In order to handle $F_\omega(\omega)$, we have to control the different terms in (5.3).

(5.5) Let $u = (u_1, u_2) \in K^2 \setminus A^2$, where for $i = 1, 2$, $u_i = s_i/n$, the s_i and n in A, $\deg s_i < \deg n$, and n monic. Let further $\Gamma(n)$ be the congruence subgroup $\{\gamma \in \Gamma \,|\, \gamma \equiv 1 \bmod n\}$ of Γ. In [4], we studied the functions

$$e_u(\omega) = e_\omega(u_1\omega + u_2)$$

on Ω which are meromorphic modular forms of weight -1 for $\Gamma(n)$. Their behavior at ∞ is described using the uniformizer $t_{1/n}(\omega) = t(\omega/n) = 1/e_L(\bar{\pi}\omega/n)$. It is related with $t(\omega)$ by

$$t = t_{1/n}^{|n|}/f_n(t_{1/n}). \tag{5.6}$$

(In loc. cit., $t_{1/n}$ had been labelled by t_n which, unfortunately, is inconsistent with notations in [6] and in Sections 6–8.) Now, e_u has a convergent product expansion given by (*loc. cit.* (2.1)):

$$e_u(\omega) = \bar{\pi}^{-1}\frac{f_{s_1}(t_{1/n}) + \zeta}{t_{1/n}^{|s_1|}} \prod_{c\in A}' \frac{f_{nc-s_1}(t_{1/n}) - \zeta t_{1/n}^{|nc|}}{f_{nc}(t_{1/n})}. \tag{5.7}$$

Here $\zeta = \zeta_{u_2} = e_L(\bar{\pi}u_2)$ is an n-division point (= root of $\varrho_n(X)$) of the Carlitz module ϱ. The factor corresponding to $0 \neq c \in A$ in (5.7) equals $e_L(\bar{\pi}((c - u_1)\omega - u_2))/e_L(\bar{\pi}c\omega)$. Since $|(c - u_1)\omega - u_2|_i = |c||\omega|_i$, (4.8) implies its absolute value to be 1, provided that $|\omega|_i > 1$. Due to (5.3), we may write

$$F_\omega(\omega) = \sum_{i\geqslant 0} T^i e^q_{u(i)}(\omega) \quad \text{with } u(i) = (T^{-i-1}, 0). \tag{5.8}$$

5.9. Lemma: *For* $i > 0$, $\lim t(\omega)T^i e^q_{u(i)}(\omega) = 0$ *uniformly in i, where the* lim *is over those* $\omega \in \Omega$ *that satisfy* $|\omega| = |\omega|_i$, *and* $|\omega| \to \infty$.

Proof: Put ad hoc $t_i = t_{1/n}$ where $n = T^i$, and let $|\omega|_i = |\omega| > 1$. Then by (5.7), $e_{u(i)}(\omega) = \bar{\pi}^{-1}t^{-1}_{i+1}(\omega) \times$ unit since $u(i) = (T^{-i-1}, 0)$, $s_1 = 1$, and $u_2 = 0 = \zeta$. Therefore,

$$\begin{aligned} t(\omega)T^i e^q_{u(i)}(\omega) &= t(\omega)T^i\bar{\pi}^{-q}t^{-q}_{i+1}(\omega) \times \text{unit} \\ &= \bar{\pi}^{-q}T^i e^q_L(\bar{\pi}\omega/T^{i+1})/e_L(\bar{\pi}\omega) \times \text{unit}. \end{aligned}$$

Using (4.8), one easily sees this tends to zero with $|\omega| \to \infty$. In order to get uniformity in i, we have to use (4.8) in full detail. Let $|\omega| = |\omega|_i = q^{d-\varepsilon}$ where $d \in \mathbb{N}$ and $0 \leqslant \varepsilon < 1$. We compute the q-logarithm of

$$(*) = |T^i e^q_L(\bar{\pi}\omega/T^{i+1})/e_L(\bar{\pi}\omega)|.$$

By (4.8) $\log_q|e_L(\bar{\pi}\omega)| = q^d(q/(q - 1) - \varepsilon)$.
For the numerator, we distinguish the cases
(a) $|\omega/T^{i+1}| > 1$, i.e., $i + 1 < d$
(b) $|\omega/T^{i+1}| \leqslant 1$, i.e., $i + 1 \geqslant d$.
In case (a), $\omega' = \omega/T^{i+1}$ satisfies $|\omega'| = |\omega'|_i = q^{d-i-1-\varepsilon}$, thus

$$\log_q|e_L(\bar{\pi}\omega')| = q^{d-i-1}(q/(q - 1) - \varepsilon).$$

We get the estimate

$$\log_q(*) = q^{d-i}(q/q - 1) - \varepsilon) + i - q^d(q/(q - 1) - \varepsilon)$$

$$< d - 1 - q^{d-1} =: h_1(d)$$

since $i < d - 1$ and $\varepsilon < 1$.

In case (b), $|e_L(\bar{\pi}\omega/T^{i+1})| = |\bar{\pi}\omega/T^{i+1}|$ as an immediate consequence of the definition of e_L. Thus

$$\log_q(*) = q \log_q|\bar{\pi}| + q(d - i - 1 - \varepsilon) + i - q^d(q/(q - 1) - \varepsilon).$$

Since $\log_q|\bar{\pi}| = q/(q - 1)$, this becomes

$$q(q/(q - 1) - 1 - \varepsilon) + qd + (1 - q)i - q^d(q/(q - 1) - \varepsilon)$$

$$\leqslant q/(q - 1) + qd - q^d/(q - 1) =: h_2(d).$$

Clearly, sup $(h_1(d), h_2(d))$ tends to $-\infty$ if d increases, which establishes the result.

Now we are able to prove Theorem 5.2: since $L(\omega)$ is A-periodic, we may restrict the limit $\lim_{|\omega|_i \to \infty}$ (simply denoted by "lim") to those ω satisfying $|\omega| = |\omega|_i$. Then

$$\lim t(\omega)L(\omega) = \lim t(\omega) \sum_{i\geqslant 0} T^i e^q_{u(i)}(\omega) - \lim t(\omega)\omega F_\omega(1).$$

The second limit vanishes by virtue of (4.8), (5.4), and the assumption $|\omega| = |\omega|_i$. As to the first, since the limits in (5.9) are uniform, we may interchange lim and Σ. Hence

$$\lim t(\omega)L(\omega) = \lim t(\omega)e^q_{u(0)}(\omega)$$

$$= \bar{\pi}^{-q} \lim e^q_L(\bar{\pi}\omega/T)/e_L(\bar{\pi}\omega)$$

since the limit of the factors of $\Pi'_{c\in A}(\ldots)$ in (5.7) gives 1. Finally,

$$e_L(\bar{\pi}\omega) = \varrho_T(e_L(\bar{\pi}\omega/T)) = Te_L(\bar{\pi}\omega/T) + e^q_L(\bar{\pi}\omega/T),$$

thus

$$\lim e^q_L(\bar{\pi}\omega/T)/e_L(\bar{\pi}\omega) = 1 - \lim Te_L(\bar{\pi}\omega/T)/e_L(\bar{\pi}\omega)$$

$$= 1 \text{ by (4.8).} \qquad \text{QED.}$$

6. L as a modular form

Consider the inverse $1/L(\omega)$ as a function on Ω. It is meromorphic on Ω, of weight $q + 1$ and type 1 for $\Gamma = GL(2, A)$, and has a simple zero at ∞ (see (4.9)). More precisely, its t-expansion begins $1/L(\omega) = \bar{\pi}^q t(\omega) + \ldots$, due to Theorem 5.2. If $0 \neq f(\omega)$ is any meromorphic modular form for Γ of weight k and type m, we have (see [6] for explanation):

$$\sum v_x(f) + v_0(f)/(q + 1) + v_\infty(f)/(q - 1) = k/(q^2 - 1), \tag{6.1}$$

where $v_x(f)$ is the order of zero of f at x (negative if f has a pole at x), the sum on the left hand side is over the non-elliptic points of $\Gamma\backslash\Omega$, and v_0, v_∞ is the order at the elliptic point, at ∞, respectively.

Since L is holomorphic on Ω (so $1/L$ has no zeroes on Ω), (6.1) shows $1/L$ to have neither zeroes nor poles on Ω. Thus $1/L$ is a holomorphic modular form of weight $q + 1$ and type 1. But the space of those forms is one-dimensional, generated by the form $h(\omega)$ discussed in [6], Section 9. Using theorem 9.1 of that paper, we have

6.2. THEOREM: $L(\omega) = -\bar{\pi}^{-q}h^{-1}(\omega)$, *where the modular form h of weight $q + 1$ and type 1 has the product expansion around* ∞

$$h(\omega) = -t \prod_{a \in A \text{ monic}} f_a^{q^2-1}(t).$$

Note that h has several a priori different characterizations, e.g., as a Poincaré series, or as the "Serre derivation" of the normalized Eisenstein series of weight $q - 1$ (which explains the minus sign).

6.3. COROLLARY: *In the notation of* (4.3), $L(\boldsymbol{\omega}) = L(\omega_1, \omega_2)$ *is always non-zero.*

We may even compute that value. The Drinfeld module ϕ associated with $A\omega_1 + A\omega_2$ is of the form

$$\phi_T = T\tau^0 + g\tau + \Delta\tau^2,$$

where $\Delta(\boldsymbol{\omega})$, considered as a modular form, equals $-\bar{\pi}^{q^2-1}h^{q-1}(\omega)$. Replacing $\boldsymbol{\omega} = (\omega_1, \omega_2)$ by $c\boldsymbol{\omega}$ replaces (g, Δ) by $(c^{1-q}g, c^{1-q^2}\Delta)$, so we may assume $\Delta(\boldsymbol{\omega}) = -1$. From the above, we get

6.4. COROLLARY: *If $\boldsymbol{\omega}$ is normalized such that $\Delta(\boldsymbol{\omega}) = -1$ then* $L^{q-1}(\boldsymbol{\omega}) = \bar{\pi}^{q-1}$.

Clearly, this is a nice analogue of Legendre's relation (1.2). The occurrence of $(q - 1)$-st powers reflect the fact that $\bar{\pi}$ is not intrinsically defined, but $\bar{\pi}^{q-1}$ is. Similarly, the definition of h (but not that of $\bar{\pi}h$ or $\bar{\pi}^q h$) depends on the choice of $\bar{\pi}$.

(6.5) For $v \in \Lambda_\omega = A\omega_1 + A\omega_2$ and $\partial \in D(\phi)$ we formally write

$$\int_v \partial = F_\partial(v)$$

which by remark 3.5(i) vanishes if $\partial \in D_{si}(\phi)$. We thus obtain a pairing of the two-dimensional C-vector spaces $\Lambda_\omega \otimes C$ and $H^*_{\mathrm{DR}}(\phi_\omega)$ which by (6.3) is non-singular. For a generalization of this fact, see [9].

7. The periods of second kind

For the last time, we change notation and set for $\omega \in \Omega$

$$\eta_1(\omega) = F^{(1)}_\omega(\omega), \quad \eta_2(\omega) = F^{(1)}_\omega(1), \tag{7.1}$$

the *periods of second kind* of the Drinfeld module associated with $A\omega + A$. They are related by

$$\eta_1(\omega) = \omega^q \eta_2(\omega^{-1}), \tag{7.2}$$

which follows immediately from (4.2). Further,

$$\eta_2(\gamma\omega) = \eta_2(\omega) \tag{7.3}$$

if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with c $= 0$. Therefore, $\eta_2(\omega)$ has a t-expansion that we will calculate. By means of section 5, we may write down two kinds of series representations for η_2. First, (5.3) and (5.5) give

$$\eta_2(\omega) = \sum_{i \geqslant 0} T^i e^q_{v(i)}(\omega), \quad v(i) = (0, T^{-i-1}).$$

Applying (5.7) and noting $f_{nc}(t_{1/n}) = f_c(t)$ yields

$$\eta_2(\omega) = \bar{\pi}^{-q} \sum_{i \geqslant 0} T^i \zeta^q(i) \prod_{c \in A}{}' (f_c(t) - \zeta(i) t^{|c|})/f_c(t) \tag{7.4}$$

with $\zeta(i) = e_L(\bar{\pi}/T^{i+1})$.

Next, putting $e_\omega(z) = \Sigma_{i\geqslant 0}\alpha_i(\omega)z^{q^i}$, the $\alpha_i(\omega)$ are modular forms of weight $q^i - 1$ and type 0. Their t-expansions may be calculated with the methods of [6]. Let [i] be short for the element $T^{q^i} - T$ of A. Comparing coefficients in the difference equation that defines $F_\omega^{(1)}$, we see

$$F_\omega^{(1)}(z) = \sum_{i\geqslant 0} \alpha_i^q(\omega)z^{q^i}/[i+1],$$

in particular

$$\eta_2(\omega) = \sum_{i\geqslant 0} \alpha_i^q(\omega)/[i+1]. \tag{7.5}$$

Note that (7.4) is an infinite sum of meromorphic modular forms of weight $-q$ of different levels, whereas (7.5) is a sum of holomorphic modular forms of level one, but with different weights. Both sums converge in a neighborhood of ∞, but fail to give us rationality properties of the t-coefficients since for each term, an infinite number of summands contribute. At least we may read off the constant term:

7.6. LEMMA: $\eta_2(\infty) = -\bar{\pi}^{1-q}$.

Proof: As in (5.3), we have

$$F(z) = \sum_{i\geqslant 0} T^i e_L^q(z/T^{i+1})$$

for the unique solution $F(z)$ without z-term of the difference equation

$$e_L^q(z) = F(Tz) - TF(z).$$

But $e_L(Tz) = Te_L(z) + e_L^q(z)$, thus $F(z) = e_L(z) - z$ and

$$-\bar{\pi} = e_L(\bar{\pi}) - \bar{\pi} = \sum_{i\geqslant 0} T^i e_L^q(\bar{\pi}/T^{i+1}) = \sum_{i\geqslant 0} T^i \zeta^q(i).$$

Since $f_c(0) =$ leading coefficient of c as a polynomial in T, the right hand factors of (7.4) evaluate to 1 at $t = 0$.

Now (6.2) combined with (7.2) gives the transformation law for η_2:

$$\omega^q\eta_2(\omega^{-1}) = \omega\eta_2(\omega) - \bar{\pi}^{-q}h^{-1}(\omega). \tag{7.7}$$

But $h(\omega^{-1}) = -\omega^{q+1}h(\omega)$, thus $\tilde{\eta}_2(\omega) = \bar{\pi}^{q-1}h(\omega)\eta_2(\omega)$ satisfies

$$\tilde{\eta}_2(\omega^{-1}) = -\omega^2\tilde{\eta}_2(\omega) + \bar{\pi}^{-1}\omega. \tag{7.8}$$

Let θ be the differential operator $\bar{\pi}^{-1}\mathrm{d}/\mathrm{d}\omega$, given in terms of t by $-t^2\mathrm{d}/\mathrm{d}t$. (For this and the following, see [6], Section 8.) Let $E = \theta(\Delta)/\Delta$ be the logarithmic derivative of the discriminant function $\Delta(\omega)$. Then E satisfies the *same* transformation rule (7.8) as $\tilde{\eta}_2$. $E(\omega)$ is an analogue of the "false Eisenstein series of weight 2" which up to a constant gives the periods of second kind of a complex elliptic curve ([8], p. 166). Let us note the t-expansion of $E(\omega)$:

$$E(\omega) = \sum_{a \in A \text{ monic}}{}' at(a\omega) = \sum{}' at^{|a|}(\omega)/f_a(t) \tag{7.9}$$

which in particular has coefficients in A.

The function $\omega \mapsto \tilde{\eta}_2(\omega) - E(\omega)$ has the following properties (which result from corresponding properties of η_2, h, and E):

(i) It is of weight 2 and type 1 for $\Gamma = GL(2, A)$;
(ii) it is holomorphic on Ω;
(iii) it has at least a double zero at ∞.

The last fact comes from Lemma 7.6, $h(\omega) = -t + \ldots$, and $E(\omega) = t + \ldots$. Thus by (6.1), it has to vanish identically, and we have shown:

7.10. THEOREM: $\eta_2(\omega) = \bar{\pi}^{1-q}E(\omega)/h(\omega)$.

7.11. COROLLARY: *Up to the factor $\bar{\pi}^{1-q}$, $\eta_2(\omega)$ has its t-coefficients in A.*

7.12. COROLLARY: *Up to the factor $\bar{\pi}^{1-q}$, the t-series given by (7.4) and (7.5) have coefficients in A.*

8. The Gauss–Manin connection

Let $\phi = \phi_\omega$ be the generic rank two Drinfeld module associated with $\Lambda_\omega = A\omega + A$, ω varying over the upper half-plane Ω, and $\omega_1 = \omega$, $\omega_2 = 1$. Following Katz (see [8], A 1.3), we define the *Gauss-Manin connection* $\nabla = \nabla_\theta$ of the differential operator $\theta = \bar{\pi}^{-1}\mathrm{d}/\mathrm{d}\omega$ as the unique endomorphism $\nabla\colon H^*_{\mathrm{DR}}(\phi) \to H^*_{\mathrm{DR}}(\phi)$ satisfying

$$\int_{\omega_i} \nabla(\partial) = \bar{\pi}^{-1}\frac{\mathrm{d}}{\mathrm{d}\omega}\int_{\omega_i}\partial \quad (i = 1, 2,\ \partial \in H^*_{\mathrm{DR}}(\phi)). \tag{8.1}$$

∇ is well-defined in view of (6.5). Clearly, for f holomorphic on Ω and $\partial \in H^*_{\mathrm{DR}}(\phi)$ a "differential form,"

$$\nabla(f\partial) = \theta(f)\partial + f\nabla(\partial) \tag{8.2}$$

holds. Now $H^*_{\mathrm{DR}}(\phi)$ is spanned by $\partial^{(i)}$ ($i = 0, 1$), and $F_{\partial^{(i)}}(\omega_j)$ (where $i = 0, 1$ and $j = 1, 2$) is given by the matrix

$$\begin{pmatrix} -\omega & -1 \\ \eta_1(\omega) & \eta_2(\omega) \end{pmatrix}.$$

Trivially, $(\mathrm{d}/\mathrm{d}\omega)\eta_i(\omega) = 0$ as results from (5.8) and (7.5). Hence

$$\left(\frac{\mathrm{d}}{\mathrm{d}\omega} F_{\partial^{(i)}}(\omega_j)\right) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and solving (8.1) gives

$$\nabla\begin{pmatrix} \partial^{(0)} \\ \partial^{(1)} \end{pmatrix} = \bar{\pi}^{q-1}h(\omega)\begin{pmatrix} \eta_2(\omega) & 1 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} \partial^{(0)} \\ \partial^{(1)} \end{bmatrix}. \tag{8.3}$$

From [6], (8.5) we know that $\theta h + Eh$ is a modular form of weight $q + 3$ and type 2. Comparing t-expansions, we see it has a zero of order $\geqslant 3$ at ∞. By (6.1), it vanishes identically, hence

$$\theta(h) = -Eh. \tag{8.4}$$

Similarly ([6], Theorem 9.1, note the different normalization used there):

$$\theta(g) = Eg + \bar{\pi}^{q-1}h. \tag{8.5}$$

Let now TD be the Tate–Drinfeld module over the power series ring $R = A[[t]]$, defined by

$$(\mathrm{TD})_T = T\tau^0 + \tilde{g}\tau + \tilde{\Delta}\tau^2, \tag{8.6}$$

where for $\tilde{g}$ (resp. $\tilde{\Delta}$), we insert the t-expansion of $\bar{\pi}^{1-q}g$ (resp. $\bar{\pi}^{1-q^2}\Delta$) which in fact has coefficients in A. In order to have everything defined over R, we replace the basis $\{\partial^{(0)}, \partial^{(1)}\}$ of $H^*_{\mathrm{DR}}(\phi)$ by $\{\tilde{\partial}^{(0)}, \tilde{\partial}^{(1)}\}$, where

$$\begin{aligned} \tilde{\partial}^{(0)} &= \bar{\pi}^{1-q^2}\partial^{(0)} + (\bar{\pi}^{1-q} - \bar{\pi}^{1-q^2})g(\omega)\partial^{(1)} \\ \tilde{\partial}^{(1)} &= h(\omega)\partial^{(1)}. \end{aligned} \tag{8.7}$$

Note that $\mathfrak{d}_T^{(0)} = \tilde{g}\tau + \tilde{\Delta}\tau^2$, i.e., $\mathfrak{d}^{(0)}$ is the inner derivation canonically associated with TD. Using (8.2)–(8.7) and some calculation, the following matrix representation for ∇ is yielded:

8.8. THEOREM: $$\nabla \begin{pmatrix} \mathfrak{d}^{(0)} \\ \mathfrak{d}^{(1)} \end{pmatrix} = \begin{pmatrix} \mathrm{E}(\omega) & 1 \\ 0 & -\mathrm{E}(\omega) \end{pmatrix} \begin{pmatrix} \mathfrak{d}^{(0)} \\ \mathfrak{d}^{(1)} \end{pmatrix}.$$

By (7.9), the entries may be considered as elements of R. Stressing further the analogy with the elliptic modular case, the nilpotency of the matrix for ∇ evaluated at $\omega = \infty$, i.e., $t = 0$ should indicate $\{\mathfrak{d}^{(0)}, \mathfrak{d}^{(1)}\}$ to be a basis for the canonical extension of $(H_{\mathrm{DR}}^*(\mathrm{TD}), \nabla)$ to ∞.

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