COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 69, nº 3 (1989), p. 241-276

<http://www.numdam.org/item?id=CM_1989__69_3_241_0>

Compositio Mathematica 69: 241–276, 1989

Unipotent representations and Dixmier algebras

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Received 17 November 1987; accepted 27 June 1988

Abstract. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$. David Vogan has studied algebras A with 1 having also the (compatible) structure of a finitely-generated Harish–Chandra $U(\mathfrak{g})$-bimodule with G-action. Here we study such A satisfying also (1) the kernel of the natural map $U(\mathfrak{g}) \to A$ is a maximal unipotent ideal and (2) A is completely reducible as a bimodule. We define a property called strong primality lying between primality and complete primality and give necessary and sufficient conditions under mild hypotheses for A to be strongly prime. Under stronger hypotheses, all strongly prime A are actually completely prime. This fact enables us to disprove a conjecture of Vogan that completely prime A are in bijection to ramified covers of orbit closures in $\mathfrak{g}^*$; in a wide family of examples, there are too many algebras to correspond to orbit covers. Finally, we prove some results concerning filtrations of A and investigate what happens when the mild hypotheses alluded to above are not satisfied.

1. Introduction

Let G be a complex semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The aim of this paper is to present some results concerning finitely-generated admissible Harish–Chandra bimodules over $U(\mathfrak{g})$, endowed also with the structure of an associative algebra A with 1. I studied these in my thesis [McGovern, 1987], calling them Dixmier algebras. They had previously been studied by David Vogan [Vogan, 1986], who was attempting to rescue and generalize a false conjecture suggested by the famous orbit method of Kostant and Kirillov, namely that completely prime primitive quotients of $U(\mathfrak{g})$ should be in bijection to coadjoint G-orbits. Vogan conjectured that completely prime Dixmier algebras should be in bijection to irreducible varieties X equipped with a finite map $X \to \mathfrak{g}^*$ and a rational action of G on X compatible with this map. This formulation represents in part an attempt to extend the methods of algebraic geometry to an important, (slightly) noncommutative case; it is an analogue of the equivariant Nullstellensatz.

Research supported in part by NSF Grant DMS-8120790.

We restrict attention in the first five sections to Dixmier algebras A that are completely reducible as $U(\mathfrak{g})$-bimodules and that contain a quotient $U(\mathfrak{g})/I$ with I maximal unipotent, calling them unipotent also. The term "unipotent" is defined in several papers of Barbasch and Vogan most notably [Barbasch–Vogan, Definition 5.23]; the definition goes as follows. Given a nilpotent orbit $\mathfrak{O}$ of $\mathfrak{g}^*$, assume that it is special in the sense of [Lusztig, 1979] or equivalently [Barbasch–Vogan, 1985, Definition 1.10]. Attach to $\mathfrak{O}$ an infinitesimal character $\lambda_{\mathfrak{O}}$ by the recipe of [Barbasch–Vogan, 1985, 5.4], assuming as they do that the dual nilpotent orbit ${}^L\mathfrak{O}$ is even. Then a maximal ideal I of $U(\mathfrak{g})$ is attached to $\mathfrak{O}$ if its infinitesimal character is $\lambda_{\mathfrak{O}}$; I is unipotent if it is attached to some special $\mathfrak{O}$ with ${}^L\mathfrak{O}$ even in this sense. The point of this restriction is that unipotent Dixmier algebras are nicely built out of irreducible bimodules and these pieces behave in a nice way.

THEOREM 1.1 (THEOREM III OF [Barbasch–Vogan, 1985]): *Let $\mathfrak{O}$ be special and ${}^L\mathfrak{O}$ even (as in the definition of maximal unipotent ideal). Let $e \in \mathfrak{O}$ and suppose I is attached to $\mathfrak{O}$. Then the set $HC_{\mathfrak{O}}$ of irreducibnle Harish-Chandra bimodules with right and left annihilator equal to I and a compatible locally finite G-action is parametrized by irreducible representations of $\overline{A(\mathfrak{O})}$, Lusztig's canonical quotient of the G-equivariant fundamental group $\pi_1^G(\mathfrak{O})$ of $\mathfrak{O}$. Here $\pi_1^G(\mathfrak{O})$ is isomorphic to the component group G^e/G_0^e of G^e, and Lusztig's canonical quotient is defined* [Lusztig, *1984, 13.1.2*] *and* [Barbasch–Vogan, *1985, 4.4c*]. *Moreover, if μ, $\nu \in \overline{A(\mathfrak{O})}^\wedge$ and the bimodules corresponding to them are denoted by V_μ, V_ν and if*

$$\mu \otimes_{\mathbb{C}} \nu \cong \oplus_{i=1}^n X_i \tag{a}$$

then

$$V_\mu \otimes_{V_{\mathrm{triv}}} V_\nu \cong \oplus_{i=1}^n V_{X_i}. \tag{b}$$

In other words, the character theories of $HC_{\mathfrak{O}}$ and $\overline{A(\mathfrak{O})}$ coincide.

We start in Section 3 by considering the case where $\overline{A(\mathfrak{O})}$ is abelian, so that the right sides of (a) and (b) above have only one term. This puts a very strong constraint on possible multiplication tables of Dixmier algebras. In particular, we prove the following result.

THEOREM 1.2: *There is a natural finite-to-one correspondence from the set of all strongly prime unipotent Dixmier algebras corresponding to $\overline{A(\mathfrak{O})}$ to the set of subgroups S of $\overline{A(\mathfrak{O})}$. The fiber over S has cardinality* 0 *or* $|H^2(S, \mathbb{C}^\times)|$, *the* *order of the Schur multiplier of S, and consists of algebras that are isomorphic as bimodules.*

The notion of strongly primality is defined in Section 2; it is weaker than complete primality but stronger than primality. We also give a criterion for the surjectivity of the correspondence. In Section 3 we show that under certain circumstances these concepts coincide (in a setting slightly different from that of Theorem 1.1) and discuss a counterexample to Vogan's conjecture in detail.

In Section 4, we investigate what happens if $\overline{A(\mathfrak{O})}$ is not abelian. Here we are forced to make an existence assumption which was needed only for the surjectivity of the correspondence in Section 3. The most interesting non-abelian groups that arise in practice are the symmetric groups S_3, S_4, and S_5. We give a rather complete analysis of the first of these and sketchy remarks about the other two.

The foregoing results seem to indicate that we must consider sheaves of noncommutative algebras on nilpotent orbits in order to arrive at a class of geometric objects capable of parametrizing even unipotent Dixmier algebras. Unfortunately, it is not at all clear just what kind of sheaves ought to be used. In Section 5, we analyze (possibly noncommutative) completely prime algebras of finite type over $S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$. We will see that this class of geometric objects is an unsatisfactory one for parametrizing unipotent Dixmier algebras, although it possibly represents a step in the right direction.

Since one often studies noncommutative algebras by filtering them, we conclude by considering filtrations of (not necessarily prime or unipotent) Dixmier algebras A in Section 6. We show that any A has a filtration making the associated graded algebra gr A commutative. If A is completely prime, it has another filtration making gr A completely prime. It is not always possible to filter A so as to make gr A both commutative and completely prime, but we mention some results of Moeglin indicating what happens if this is possible.

2. Notation and definitions

We will denote Lie groups by upper case roman letters and their Lie algebras by the corresponding gothic letters. All groups and algebras will be complex; groups will be assumed to be connected unless otherwise stated. Given G and $\mathfrak{g}$ semisimple, $\mathfrak{h}$ will denote a Cartan subalgebra of $\mathfrak{g}$ with dual $\mathfrak{h}^*$,

$\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots, $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ a choice of positive roots, W the Weyl group, R the root lattice, and $P = P(G)$ the weight lattice with standard partial order $\prec$ induced by Δ^+. It will be convenient to make P a totally ordered abelian group by an ordering $<$ which we may assume extends $\prec$. We also use $<$ to order finite-dimensional holomorphic G-representations according to their highest weights.

Denote the universal enveloping algebra of $\mathfrak{g}$ by $U(\mathfrak{g})$ and the center of $U(\mathfrak{g})$ by $Z(\mathfrak{g})$. Recall that the maximal spectrum of $Z(\mathfrak{g})$ identifies with $\mathfrak{h}^*/W$ by the Harish–Chandra isomorphism; we call elements of this latter set infinitesimal characters and denote the maximal ideal corresponding to λ by $Z(\lambda)$. A (proper two-sided) ideal I of $U(\mathfrak{g})$ is said to have infinitesimal character λ if I contains the minimal primitive ideal $U(\mathfrak{g})Z(\lambda)$. Recall that maximal ideals correspond $1 - 1$ to their infinitesimal characters.

By an (admissible) Harish–Chandra $U(\mathfrak{g})$-bimodule we will mean a finitely-generated $U(\mathfrak{g})$-bimodule V satisfying

(1) V is a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules E when regarded as a $\mathfrak{g}$-module under the adjoint action (which we assume lifts to G) and each E occurs with finite multiplicity $[V:E]$ in V. If $[V:E] \neq 0$, we call E a K-type of V.

(2) *L Ann* V and *R Ann* V, the left and right annihilators of V, coincide.

Harish–Chandra bimodules then automatically have finite length.

As stated in the introduction, a Dixmier algebra is a Harish–Chandra bimodule which is also an associative algebra with 1, all structures being compatible. This terminology was chosen because Vogan and I came to consider such algebras while trying to generalize Dixmier's bijection between primitive ideals and coadjoint orbits from the solvable to the semisimple case. A is unipotent if it is completely reducible as a bimodule and $\mathrm{Ker}(U(\mathfrak{g}) \to A)$ is maximal unipotent. It may be that the second of these conditions implies the first.

Finally, we recall for completeness a few definitions from algebra and introduce one of our own. An algebra A is called prime (resp. completely prime) if the product of two nonzero ideals (resp. nonzero elements) is nonzero. If A is also a bimodule over a ring R, then we will call A strongly prime if the product of two nonzero R-bisubmodules is nonzero. Clearly strong primality lies between primality and complete primality. A is said to be (left) primitive if it has an irreducible (left) module V with *Ann* $V = 0$. Finally, an ideal I of A is prime (resp. completely prime, . . .) if the quotient ring A/I is prime (resp. completely prime, . . .).

3. Unipotent Dixmier algebras parametrized by abelian groups

3.1. Classification of strongly prime algebras

Unraveling the definitions, we see that a unipotent Dixmier algebra A parametrized by an abelian $\overline{A(\mathfrak{O})}$ is graded by a finite abelian group $F \cong \overline{A(\mathfrak{O})}$, so that $A = \oplus_{g \in F} A_g$, and each homogeneous component A_g is isotypic as a $U(\mathfrak{g})$-bimodule. In particular, A_1 is isomorphic to a nonzero direct sum of copies of $U(\mathfrak{g})/I$, I a fixed maximal unipotent ideal.

LEMMA 3.1.1: *If A is strongly prime, then $A_1 \cong U(\mathfrak{g})/I$ (i.e., A_1 is irreducible).*

Proof: Put $A_1 = \oplus_{i=1}^{n} B_i$, where each B_i is isomorphic to $U(\mathfrak{g})/I$. Then B_i^G, the set of G-fixed vectors in B_i, is isomorphic to $\mathbb{C}$ as a G-module, and the sum A_1^G of all the B_i^G is a finite-dimensional $\mathbb{C}$-algebra. If $\dim_{\mathbb{C}} A_1^G > 1$, then it is well known from the theory of such algebras that A_1^G is not completely prime. But if A_1^G is not completely prime, then compatibility of the product and $U(\mathfrak{g})$-action on A_1 force the product of two copies of $U(\mathfrak{g})/I$ inside A_1 to be 0, a contradiction to strong primality. Hence $A_1^G \cong \mathbb{C}$ and $A_1 \cong U(\mathfrak{g})/I$.

LEMMA 3.1.2: *Under the same hypotheses, all homogeneous components A_g are irreducible.*

Proof: It is clear from strong primality that $\{g \in F | A_g \neq 0\}$ is a subgroup of F. Suppose now that $B_1, B_2 \subset A_g$, $B_2 \subset A_{g-1}$ with all B_i irreducible as bimodules and $B_1 \neq B_2$. Fix isomorphisms i: $B_2 \otimes_{A_1} B_3 \to A_1, j$: $B_1 \to B_2$. Then multiplication of B_1 and B_2 by B_3 must be given by nonzero scalar multiples of i and ji, respectively. Whatever these multiples are, it is clear that we can find $c \in \mathbb{C}$ such that the product of the bimodules $\{x - cj(x) | x \in B_1\}$ and B_2 is 0, a contradiction. The conclusion follows.

Proof of Theorem 1.2: Associate to each Dixmier algebra A attached to $\overline{A(\mathfrak{O})}$ the subgroup $S = \{g \in \overline{A(\mathfrak{O})} | A_g \neq 0\}$ of $\overline{A(\mathfrak{O})}$; this gives the correspondence of the theorem. If a subgroup S occurs in this correspondence, we get an isomorphism τ_{st}: $V_s \otimes_{V_1} V_t \to V_{st}$ induced from the multiplication of A_s and A_t in A and arbitrarily chosen bimodule isomorphisms $V_i \to A_i$ ($i = s, t, st$), for each $s, t \in S$. Associativity of multiplication in A then forces the following diagram to commute.

$$\begin{array}{ccc} V_s \otimes V_t \otimes V_u & \xrightarrow{\tau_{st} \otimes id} & V_{st} \otimes V_u \\ {\scriptstyle id \otimes \tau_{tu}} \downarrow & & \downarrow {\scriptstyle \tau_{st,u}} \\ V_s \otimes V_{tu} & \xrightarrow[\tau_{s\,tu}]{} & V_{stu} \end{array} \qquad (3.1.3)$$

In particular, if S occurs in the correspondence it is possible to choose isomorphisms τ_{st} making (3.1.3) commute. The reasoning is reversible; given the τ_{st}, it is obvious how to place a ring structure on $\oplus_{t \in S} V_t$ making it a strongly prime algebra corresponding to S. We now assume that S satisfies this condition and ask how many algebras are associated to it by the correspondence. Any such algebra gives rise to a new set of isomorphisms τ'_{st} which must be complex scalar multiples of the old ones: $\tau'_{st} = n(s, t)\tau_{st}$. In order that (3.1.3) commute with τ_{st} replaced by τ'_{st}, it is necessary and sufficient that

$$n(s, t)n(st, u) = n(t, u)n(s, tu). \qquad (3.1.4)$$

Once again all steps in this argument are reversible; given $n(s, t) \neq 0$ satisfying (3.1.4), we recover the τ'_{st} and the attached algebra. We now ask when two algebras attached to the same S are isomorphic. Since any isomorphism must respect the bimodule structures, it must reduce to a scalar on each homogeneous component, by (3.1.2). Consequently, the $n(s, t)$ and $m(s, t)$ attached to the algebras must satisfy

$$n(s, t) = \lambda(s)\lambda(t)\lambda(st)^{-1}m(s, t) \qquad (3.1.5)$$

for some $\lambda \colon S \to \mathbb{C}^\times$. But (3.1.4) and (3.1.5) are exactly the 2-cocylce and 1-coboundary conditions defining $H^2(S, \mathbb{C}^\times)$, where S acts trivially on $\mathbb{C}^\times$. This completes the proof.

We degress briefly to calculate $H^2(F, \mathbb{C}^\times)$, F an arbitrary finite abelian group acting trivially on $\mathbb{C}^\times$, as we need to know in the next section that this group is not always trivial. Write $F = F_1 \times \cdots \times F_n$, with each F_i cyclic of prime-power order. It is well known that $H^2(F, \mathbb{C}^\times)$ may be calculated as the group of extensions of $\mathbb{C}^\times$ by F; i.e., as the set of exact sequences of groups

$$L \to \mathbb{C}^\times \to G \to F \to 1$$

with the image of $\mathbb{C}^\times$ central in G, modulo equivalence, defined by the commutativity of an obvious diagram. F has a presentation $\langle a_1, \ldots, a_n | a_i^{|F_i|} = 1, a_i a_j = a_j a_i \rangle$. Given the short exact sequence we may pull back the generators a_i of F to elements $\overline{a_i}$ of G satisfying $\overline{a_i}^{|F_i|} = 1, \overline{a_i a_j} = b_{ij}\overline{a_j a_i}$ for some $b_{ij} \in \mathbb{C}^\times$, since $\mathbb{C}^\times$ is divisible. It is not difficult to decide that the b_{ij} are uniquely determined, that each is an nth root of unity for $n = g.c.d.(|F_i|, |F_j|)$, and that they are otherwise unrestricted. We deduce the following result.

PROPOSITION 3.1.5: *$H^2(F, \mathbb{C}^\times) \cong \oplus_{i<j} C_{ij}$, where C_{ij} is cyclic of order $g.c.d.(|F_i|, |F_j|)$ (in the above notation).*

In particular $H^2(F, \mathbb{C}^\times)$ is trivial if and only if F is cyclic, but has order $2^{\binom{n}{2}}$ if $F \cong (\mathbb{Z}/2)^n$.

3.2. Criterion for complete primality and a disproof of Vogan's conjecture

In this section all unipotent Dixmier algebras A will be assumed to satisfy

$$\text{Ker}(U(\mathfrak{g}) \to A) \text{ is completely prime} \tag{3.2.1}$$

unless otherwise stated. Barbasch and Vogan have conjectured that every unipotent A has this property; at any rate, it is known that it holds whenever the kernel is the annihilator of some irreducible unitary representation [Vogan, 1986, 7.12]. Letting $A = \oplus_{g\in F} A_g$ as in the last section, we now localize A by the non-zero elements A_1^* in A_1. By Goldie's theorem [Herstein, 1968], these form an Ore set in A_1, and A_1 becomes a division ring upon localization by this set.

LEMMA 3.2.2: *A_1^* is right and left Ore in A, so that the localization $B = (A_1^*)^{-1}$ exists. In fact, for any prime Dixmier algebra, not necessarily unipotent, one may localize A by the regular elements of A_1. The resulting ring coincides with the Goldie right or left quotient ring of A.*

Proof: Goldie's theorem applies also to A (A being Noetherian), so A has a quotient ring. If $x \in A$ is regular, consider the sum $A_1 + A_1 x + A_1 x^2 + \cdots$ of left A_1-submodules of A. Since A is Noetherian as a left A_1-module, this sum cannot be direct, so we have $a_0 + a_1 x + \cdots + a_n x^n = 0$ for some a_i not all 0. As x is regular, we may cancel a power of x if necessary to conclude that some left multiple of x lies in A_1^*. Hence localization amounts to no more than localizing by some element of A_1^*. The conclusion follows.

Observe that B is still F-graded, but now B_1 is a division ring and other pieces B_g are isomorphic as left or right B_1-modules to B_1 itself. This follows from (3.2.2): the left or right dimension of B_g over B_1 must be 1 since $a_g b_g^{-1}$ lands in B_1 for any $a_g, b_g \in B_g - \{0\}$. In particular B is prime right Artinian, hence isomorphic to $D_n = M_n(D)$, the right of $n \times n$ matrices over some division ring D. (Note also that G no longer acts finitely on B.)

LEMMA 3.2.3: *Assume that $|F|$ is prime. Then A is completely prime if and only if the only homogeneous roots of unity in B lie in $\mathbb{C} \subset B_1$; that is, if and only if no $b \in B_g$ is a root of unity for $g \neq 1$.*

Proof: Obviously A is completely prime if and only if B is. It is clear that this condition is necessary. Conversely, if it satisfied, then I first claim that any sum $b_g + b_h$ of two nonzero homogeneous elements of B is invertible. Premultiplying by b_g^{-1} (which exists), we may assume that $b_g = 1$. If $1 + b_h$ is not invertible, it is a zero-divisor. Expanding out $(1 + b_h)(\Sigma_{g \in F} b_g) = 0$, we see that this forces b_h to be a root of unity, a contradiction. Now let $p = |F|$, denote the elements of F by $0, 1, \ldots, p - 1$, and let $c_0 + \cdots + c_{p-1}$ generate a minimal right ideal of B. Applying various automorphisms of B to this element, we get $p - 1$ elements $c_0 + \epsilon^j c_1 + \cdots + \epsilon^{(p-1)j} c_{p-1}$ $(j = 0, \ldots, p - 2; \epsilon = e^{2\pi i/p})$ of B, each generating a minimal right ideal. But from linear algebra the $\mathbb{C}$-span of these elements contains an invertible sum of two homogeneous elements. Hence B is a sum of at most $p - 1$ minimal right ideals and has Goldie rank at most $p - 1$. On the other hand, all minimal right ideals of B have the same right dimension over B_1, while B itself has right dimension p over B_1, so the Goldie rank of B divides p. Hence the Goldie rank of B is 1, and B is a division ring.

In general, if $|F|$ is not necessarily prime, F has a composition series $F = F_0 > F_1 > \cdots > F_n = 0$ in which each subquotient F_i/F_{i+1} is cyclic of prime order p_i. Setting $C_i = \oplus_{g \in F_i} B_g$, we see that C_i is $\mathbb{Z}/p_i$-graded with 0-graded piece C_{i+1}. Then we have the following criterion for complete primality.

THEOREM 3.2.4: *With the above notations, A is completely prime if and only if the only homogeneous roots of unity of each C_i (with respect to its $\mathbb{Z}/p_i$-grading) lie in $\mathbb{C} \subset C_{i+1}$.*

Proof: Again the necessity of the condition is clear. If it is satisfied, we argue as in (3.2.3) to show each C_i is a division ring by downward induction on i.

The criterion of (3.2.4) is admittedly difficult to apply in general. In certain favorable circumstances, however, it is easy to verify. Barbasch and

Vogan have extended the character formulas of (1.1) to certain non-integral infinitesimal characters $\tilde{\lambda}_{\mathfrak{O}}$ attached to certain nilpotent orbits $\mathfrak{O}$.

In these cases there is a fixed primitive ideal I of infinitesimal character $\tilde{\lambda}_{\mathfrak{O}}$ such that irreducible Harish–Chandra bimodules with left and right annihilator equal to I are parametrized by irreducible representations of $\pi_1(\mathfrak{O})$ (rather than its quotient $\overline{A(\mathfrak{O})}$.) Moreover, the character formulas of (1.1) carry over with $\pi_1(\mathfrak{O})$ replacing $\overline{A(\mathfrak{O})}$. If $Z(G)$ surjects onto $\pi_1(\mathfrak{O})$ (as always happens in type A), then it is known that the K-types occurring in A_g, A_h have highest weights that are noncongruent modulo the root lattice R for $g \neq h$. (This work of Barbasch and Vogan has not been published, but we will give an example in the next section for which these properties are easily verified.) It is clear that all of the above results apply to Dixmier algebras with left and right annihilator equal to I, provided that $\pi_1(\mathfrak{O})$ is abelian. These considerations motivate the next result.

THEOREM 3.2.5: *With the above notation, assume that the K-types of A_g, A_h have noncongruent highest weights modulo R for $g \neq h$. Then the criterion of (3.2.4) is satisifed, so that A is completely prime.*

Proof: It is clear that the K-type condition carries over to the C_i occurring in (3.2.4), so it suffices to assume that $|F|$ is prime. Suppose contrarily that $b^n = 1$, $b = q_1^{-1}a_g$, $a_1 \in A_1$, $a_g \in A_g \neq A_1$. For definiteness assume that $n = 3$; the general case is similar. By definition of $(A_1^*)^{-1}A_1$, we get equations

$$a_g a_1^{-1} = b_1^{-1} b_g$$

$$b_g b_1^{-1} = c_1^{-1} c_g$$

$$c_1 b_1 a_1 = c_g b_g a_g$$

for some b_1, $c_1 \in A_1$, b_g, $c_g \in A_g$. Rearranging we get

$$b_1 a_g = b_g a_1$$

$$c_1 g_b = c_g b_1$$

$$c_1 b_1 a_1 = c_g b_g a_g$$

Now let $<$ totally order the weight lattice P, as in section 2. Write all a_i, b_i, c_i as sums of h-weight vectors, and let α_1, β_1, γ_1, α_g, β_g, γ_g be the $<$-highest weights appearing in the sums. We know that B has no homogeneous zero-divisors. It follows that the $<$-highest weight appearing in the sum for $a_1 b_1$ is $\alpha_1 + \beta_1$, and similarly for the other products. Accordingly, we get equations

$$\beta_1 + \alpha_g = \beta_g + \alpha_1$$

$$\gamma_1 + \beta_g = \gamma_g + \beta_1$$

$$\gamma_1 + \beta_1 + \alpha_1 = \gamma_g + \beta_g + \alpha_g$$

If $\alpha_1 < \alpha_g$ (resp. $\alpha_1 > \alpha_g$), then the first two equations force $\beta_1 < \beta_g$, $\gamma_1 < \gamma_g$ (resp. $\beta_1 > \beta_g$, $\gamma_1 > \gamma_g$), contradicting the last equation. Hence $\alpha_1 = \alpha_g$. But this last equality contradicts the hypothesis, since the weights of any K-type are congruent to its highest weight modulo R.

Another sufficient condition for complete primality will be given in section 6, when we discuss filtrations. The proof of (3.2.5) actually yields the following slightly stronger result.

PROPOSITION 3.2.6: *If $|F|$ is prime and the criterion of (3.2.4) fails, then the set of $a_1 \in A_1^*$ such that $a_1^{-1} a_g$ is a root of unity is closed under left and right multiplication by A_1^*, the G-action, and the following two operations: replacing a sum of weight vectors by the $<$-highest vector occurring in it, and replacing a sum of weight vectors belonging to various K-types by a highest weight vector for the $<$-highest K-type. In particular, if $(A_1^*)^{-1} A_g$ has any roots of unity, then it is infinitely many that are quotients of highest weight vectors of the same highest weight.*

Proof: We saw above that if $a_1^{-1} a_g$ is a root of unity, then it is still a root of unity if a_1, a_g are replaced by the $<$-highest weight vectors occurring in their sums; also, it is clear that $[g \cdot b_1 a_1 c_1]^{-1} [g \cdot b_1 a_g c_1]$ is a root of unity for any $g \in G$, $b_1, c_1 \in A_1^*$. The last assertions follow from an observation in section 6; any sum of vectors belonging to the K-types $K_1, \ldots, K_n$ is G-conjugate to a sum of weight vectors including highest weight vectors for all K_i.

This proposition shows that if there are any non-completely-prime unipotent Dixmier algebras A parametrized by abelian groups $\overline{A(\mathfrak{O})}$ then the various graded pieces A_g must look very similar. We have already seen above that their Goldie ranks must divide $|\overline{A(\mathfrak{O})}|$ (even if $|\overline{A(\mathfrak{O})}|$ is not prime). Moreover, if $|\overline{A(\mathfrak{O})}|$ is prime, then we have the following result.

PROPOSITION 3.2.7: *Let $B = \bigoplus_{g \in F} B_g$ be a localized Dixmier algebra graded by an F of prime order. If B is not completely prime, then its Goldie field embeds in B_1 as the fixed point set of some automorphism.*

Proof: We have seen that the hypotheses imply that the Goldie rank of B is $|F| = p$. Let $b = \Sigma_{g \in F} b_g$ generate a minimal right ideal of B with the $b_g \in B_g$. Premultiplying by b_1^{-1}, we may assume that $b_1 = 1$. Choose g with $b_g \neq 0, 1$. If $b(1 - b_g)$ is not 0 then it generates the same minimal right ideal as b does, while its B_g-component is 0. Arguing as in the last part of the proof of (3.2.3), we deduce that the Goldie rank of B is less than p, a contradiction. Hence $b(1 - b_g) = 0$; it easily follows that $b_g^p = 1$, $b = 1 + b_g + b_g^2 + \cdots + b_g^{p-1}$. Since the Goldie rank is p, the right dimension of bB over B_1 is 1.. Now conjugation by b_g is an automorphism of B_1 or order p, whence the dimension of B_1 over its division subring $B_1^{b_g}$ on either side is p. We can recover D from bB as the commuting ring of the right B-action on bB. But this commuting ring contains all left translations by elements of $B_1^{b_g}$, which actually fill it up, by a dimension count. Hence $D \cong B_1^{b_g}$.

This line of argument shows more generally that, for arbitrary F, a large piece of D is isomorphic to a large piece of B_1. We will, however, see examples below in which the existence of a gradation with "nice" pieces is not sufficient to guarantee complete primality (for non-abelian $\overline{A(\mathfrak{O})}$).

We now give a criterion for the surjectivity of the correspondence in (1.2). Drop the hypothesis (3.2.1). Given $S < \overline{A(\mathfrak{O})}$, we try to construct an algebra corresponding of S by first constructing its localization. Filtering S as in the discussion before (3.2.4) and making use of repeated $\mathbb{Z}/m$-graded extensions of $(A_1^*)^{-1}A_1$, we see that it suffices to assume that S is cyclic, say of order n. (We will not need to assume that n is prime, nor even a prime power.) We first make a general observation about rings. If A is any (associative) ring with 1 and $y \in A$ is invertible, suppose that some automorphism φ of A fixes y and satisfies $\varphi^n = c_y$, conjugation by y, for some n. Then we may construct an extension B or A which as an additive group is just $A \oplus A_x \oplus \cdots \oplus A_x^{n-1}$ (x a new element), and which as a multiplicative semigroup has the presentation $\langle A, x/x^n = y, xax^{-1} = \varphi(a)$ for $a \in A\rangle$. We use this observation to study the desired extension of $(A_1^*)^{-1}A_1$, which is isomorphic to $M_m(\Delta)$, Δ some division ring. We need another lemma.

LEMMA 3.2.8: *The localized modules $(A_1^*)^{-1}A_g$ all exist and have the structure of $M_m(\Delta)$-bimodules.*

Proof: It is clear that $(A_1^*)^{-1}A_g$ exists and has the structure of an $M_m(\Delta) - A_1$ bimodule. As a left $M_m(\Delta)$-module, it is a direct sum of (say) r copies of Δ^m, so that its commuting ring C is isomorphic to $M_r(\Delta)$. Now A_1 embeds into C but not into any $M_s(\Delta)$ with $s < m$, so $r \geqslant m$. If we had $r > m$, then there would be $z \in C - \mathbb{C} \cdot 1$ whose action on $(A_1^*)^{-1}A_g$ commuted with the left and right actions of A_1, contradicting the irreducibility of $(A_1^*)^{-1}A_g$ as an A_1-bimodule. Hence $r = m$, and the conclusion follows.

As a by-product of the proof, we get an $a_1^{-1}a_g \in (A_1^*)^{-1}A_g$ whose left annihilator in A_1 is trivial. We construct $a_1^{-1}a_g$ by taking a sum of independent vectors, one from each copy of Δ^m inside $(A_1^*)^{-1}A_g$. This element will turn out to be invertible in the desired extension B over $B_1 = (A_1^*)^{-1}A_1$, if B exists. Now we can state our criterion. Choose isomorphisms τ_{st} as in (3.13) arbitrarily and use them to make $\Sigma_{h\in S} A_h$ into a ring (not necessarily associative).

THEOREM 3.2.9: *The correspondence in (1.2) is surjective if and only if $a_g^n \in A_1$ commutes with a_g in this ring (under our hypothesis that S is cyclic of order n).*

Proof: Note first that this condition does not depend on the choice of the τ_{st} or on how we insert parentheses to define a_g^n; any two definitions agree up to a scalar. The condition is then obviously necessary. If it is satisfied, observe first that the τ_{st} extend to isomorphisms $B_s \otimes_{B_1} B_t \to B_{st}$ (B_s = localization of A_s), making $\Sigma_{h\in S} B_h$ into a ring. The associative law $(b_g b_h)b_k = b_g(b_h b_k)$ holds up to a scalar depending only on g, h, k ($b_i \in B_i$). The right inverse a_g^{-1} of a_g exists; consider the map $B_1 \to B_1$ defined by $x \to (a_g x)a_g^{-1}$. In order to check that this is a homomorphism, it suffices to test it on a single pair of elements from B_1, as we just saw that homogeneous elements associate up to a fixed scalar. The pair (1,1) does the trick; one similarly shows that it is an automorphism whose nth power is conjugation by a_g^n. Since the condition implies that the automorphism fixes a_g^n, we can apply our general observation to make $\Sigma\ B_h$ an associative ring by redefining the τ_{st}. Then (3.1.3) will commute with the new choice of τ_{st}.

Hence the obstruction to constructing the extension is measured precisely by the scalar α satisfying $a_h^n a_h = \alpha a_h a_h^n$ for all $a_h \in A_g$. As the nth power of conjugation by a_g is conjugation by a_g^n, we see that α must be an nth root of unity. In case n is prime and the modules A_h have analogues in characteristic n satisfying the same character identities as above, we can construct the extension in characteristic n. A more satisfactory sufficient condition for α to be 1 is given below. Put $J = \mathrm{Ker}(U(\mathfrak{g}) \to A)$.

THEOREM 3.2.10: *Assume that*

(1) *there exists $a_h \in A_g$ such that the set of elements in A_{g-1} commuting with a_h^n is not $\{0\}$, and*

(2) *either A_1 admits a filtration making gr A_1 commutative and completely prime, or J is completely prime and satisfies the Kirillov conjecture, so that Fract$(U(\mathfrak{g})/I)$ is isomorphic to a Weyl field D_n.*

Then $\alpha = 1$, so an extension exists.

Proof: Pick a_h satisfying (1) and $a_k \in A_{g-1}$ such that $[a_h^n, a_k] = 0$. Either case of (2) implies that A_1 is completely prime, so $a_h a_k \in A_1^*$. Now we have $a_h^n(a_h a_k) = \alpha(a_h a_k)a_h^n \neq 0$. The first case of (2) implies that $\alpha = 1$ by passing to the images $\overline{a_h^n}$, $\overline{a_h a_k}$ of a_h^n, $a_h a_k$ in an appropriate level of gr A_1. In the second case of (2), first observe that $xy = \alpha yx \neq yx$ cannot hold for x, y in a Weyl algebra A_n, since A_n satisfies the first condition of (2). An easy calculation shows that if $xy = \alpha yx$ holds for some $x, y \in D_n$, it holds for a different $x, y \in A_n$, a contradiction.

Barbasch and Vogan have conjectured that every maximal unipotent J satisfies the first case of (2) with the standard filtration; [Joseph, 1980] shows that every completely prime J satisfies the second case of (2). Condition (1) is satisfied if, for example, one can find a_h^n contained in the image of $U(\mathfrak{n})$, $\mathfrak{n}$ the nilradical of some Borel subalgebra of $\mathfrak{g}$. I do not believe that α ever deserves to be anything but 1.

As was observed in the last section, $H^2(F, \mathbb{C}^\times)$ is not always trivial, so the correspondence in (1.2) is not in general 1 – 1. (We will see an example in the next section.) Since the geometric objects corresponding to unipotent Dixmier algebras attached to $\mathfrak{O}$ should be just covers of $\mathfrak{O}$, parametrized by conjugacy classes of subgroups of $\pi_1(\mathfrak{O})$, we see that Vogan's conjecture fails in general. In any event, the existence of nonisomorphic algebras with the same bimodule structure precludes a natural construction attaching distinct geometric objects to these algebras, if these geometric objects are to be determined by the G-module structure.

3.3. An example

We warm up to the main example with a smaller one. Let $G = SL(2, \mathbb{C})$, $\mathfrak{O}$ = principal nilpotent orbit in $\mathfrak{g}^*$, $\lambda_{\mathfrak{O}}$ = infinitesimal character of the lowest weight $\mathfrak{g}$-module with lowest weight $\frac{1}{2}$. In this case $\pi_1^G(\mathfrak{O}) = \pi_1(\mathfrak{O}) \cong Z(SL(2, \mathbb{C})) \cong \mathbb{Z}/2$. It has two subgroups, both having trivial Schur multiplier, by (3.1.5). So there are two Dixmier algebras attached to $\mathfrak{O}$. These turn out to be $U(\mathfrak{g})/U(\mathfrak{g})Z(\lambda_{\mathfrak{O}})$ and A_1, the first Weyl algebra [Vogan, 1986, p. 290]. The first embeds in the second as the even operators. Now take $G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, $\mathfrak{O}$ = principal nilpotent orbit, $\lambda_{\mathfrak{O}} = (\lambda, \lambda)$, where λ is the $\lambda_{\mathfrak{O}}$ of the last example. Then $\pi_1^G(\mathfrak{O}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the Klein 4-group. By (3.1.5), $H^2((\mathbb{Z}/2)^2, \mathbb{C}^\times) \cong \mathbb{Z}/2$, so there are two Dixmier algebras attached to $\pi_1^G(\mathfrak{O})$ itself. One of these is familiar: $A_2 = A_1 \times A_1$, the second Weyl algebra. The other has a presentation $\langle p_1, q_1, p_2, q_2 | [p_1 q_1] = [p_2 q_2] = 1, x_1 y_2 = -y_2 x_1 (x, y = p, q)\rangle$. In other words, it is "A_2 with anticommutativity"; call it A_2'. Anticipating the last section, let us consider filtrations of these algebras. The first is well known

known to have a standard filtration by degree with associated graded algebra $[p_1, q_1, p_2, q_2]$, a polynomial algebra on four generators. If we filter A_2' in the same way, the associated graded algebra gr A_2' becomes

$$\mathbb{C}\langle p_1, q_1, p_2, q_2\rangle / x_1y_2 = -y_2x_1, x_1y_1 = y_1x_1, x_2y_2 = y_2x_2 (x, y = p, q).$$

It is still completely prime, but no longer commutative. Indeed, the existence of anticommuting elements in A_2' precludes its having any filtration with gr A_2' commutative and completely prime. We will see in the last section that A_2' does have another filtration with gr A_2' commutative; here gr A_2' turns out to be

$$\mathbb{C}[p_1^2, p_1q_1, q_1^2, p_2^2, p_2q_2, q_2^2, n_1, \ldots, n_8]/n_in_j = 0, \ldots,$$

the dots denoting additional relations given in the last section. In other words, the new gr A_2' is the ring of regular functions on $\mathfrak{O}$ with nilpotent elements adjoined. This ought to be a common phenomenon.

Thus the Dixmier algebras attached to the same group, though isomorphic as bimodules, are far from indistinguishable. This fact probably runs contrary to the reader's impressions of the abstract constructions of the preceding two subsections, in which no one Dixmier algebra attached to any group could be regarded as canonical. Another important difference between A_2 and A_2' is given below.

Both A_2 and A_2' have a natural module, namely the polynomial ring $\mathbb{C}[x, y]$ on two generators. The action of A_2 on this ring is by differential operators; one puts $p_1 = M_x$, multiplication by x, $q_1 = \partial/\partial x$; $p_2 = M_y$; $q_2 = \partial/\partial y$. The action of A_2' is more complicated; p_1 and q_1 act on a monomial x^iy^j exactly as for A_2, but the action of p_2 and q_2 on this monomial is $(-1)^i$ times the A_2-action. There is no reasonable way to redefine multiplication in $\mathbb{C}[x, y]$ so as to make $p_2, q_2 \in A_2'$ act by derivations. Not all Dixmier algebras can be realized as rings of differential operators (or at least not on line bundles). Actually, it is better to think of the A_2'-module as polynomials in two anticommuting generators, for the following reason. If we put a $\mathbb{Z}/2$-gradation on gr A_2' by giving x, $\partial/\partial x$ grade 1, y and $\partial/\partial y$ grade 0, and extending canonically, the resulting algebra becomes supercommutative. If we then similarly put a $\mathbb{Z}/2$ gradation on polynomials in two anticommuting generators x, y by giving x grade 1, y grade 0, and extending canonically, then it too becomes supercommutative and A_2' acts on it by superdifferential operators. One may hope that by suitably generalizing the notion of supercommutativity one could filter many Dixmier algebras so as to make their associated graded algebras supercommutative and completely prime. Section 6 gives an inkling of the benefits to be gained from such filtrations.

Note finally that the images of $U(\mathfrak{g})$ in A_2 and A_2', generated by the even operators in one variable, act in exactly the same way on $\mathbb{C}[x, y]$. It is not difficult to show that no $a \in A_2$ acts on $\mathbb{C}[x, y]$ as $q_2 \in A_2'$, and similarly with A_2 and A_2' interchanged. We conclude that neither A_2 nor A_2' realizes the ring of $\mathfrak{g}$-finite endomorphisms of $\mathbb{C}[x, y]$.

4. Unipotent Dixmier algebras parametrized by non-abelian groups

4.1. General theory

It will be noted that the assumption that $\overline{A(\mathfrak{O})}$ is abelian was crucial to the arguments of the last section. It is much harder to construct Dixmier algebras attached to non-abelian $\overline{A(\mathfrak{O})}$ and to test them for complete primality; in particular, they are not obtainable as a sequence of nth root extensions of division rings. While one can easily write down the analogue of (3.1.3) its commutativity turns out to be neither necessary nor sufficient for the existence of corresponding associative Dixmier algebras. Fortunately, the following hypothesis will circumvent many of these difficulties.

ASSUMPTION (4.1.1) [Barbasch–Vogan, 1983]: There exists a completely prime Dixmier algebra A on which $\overline{A(\mathfrak{O})}$ acts by bimodule automorphisms. We have an isomorphism $i: A \to \Sigma_{\pi \in \overline{A(\mathfrak{O})}} \wedge (V_\pi \otimes_{\mathbb{C}} F_\pi)$ as a module for $U(\mathfrak{g} \times \mathfrak{g})$ and $\overline{A(\mathfrak{O})}$, where F_π is the $\overline{A(\mathfrak{O})}$-module corresponding to π.

For maximum flexibility, we do not insist that A be completely prime. (We will later give an example in which (4.1.1) is satisfied.) Note that in case $\overline{A(\mathfrak{O})}$ is abelian, (4.1.1) merely amounts to assuming that the fiber over $\overline{A(\mathfrak{O})}$ is nonempty, in the language of Theorem 1.2; note also that the action of $\overline{A(\mathfrak{O})}$ on an algebra in this fiber was used at several critical points in the last section.

In order to get the sharpest possible results we will need to strengthen (4.1.1) slightly. Let A satisfy it and suppose that V, W are two irreducible V_1-bisubmodules of A; then V and W are just $i^{-1}(V_\pi \otimes f_\pi)$ and $i^{-1}(V_\mu \otimes f_\mu)$ for some $\pi, \mu \in \overline{A(\mathfrak{O})}^\wedge, f_\pi \in F_\pi, f_\mu \in F_\mu$. (Here we are using the language of Theorem 1.1.) Put $j: F_\pi \otimes F_\mu \cong \Sigma_x F_x$, and let $f_\lambda \otimes f_\mu$ map to Σf_x under any such isomorphism j, with $f_\lambda \in F_\gamma$. We will assume that the product of V and W in A is as large as it could possibly be; i.e., that it is $i^{-1}(\Sigma_x (V_\gamma \otimes f_\gamma)$. We are going to study Dixmier subalgebras of A; if it does not satisfy this condition, it can only have *more* subalgebras than we expect.

Any bisubmodule B of A is $i^{-1}(\Sigma_\pi (V_\pi \otimes S_\pi))$ for some subspaces $S_\pi \subset F_\pi$. Under our hypothesis on A, B is a subalgebra of A if and only if any isomorphism from $F_\pi \otimes F_\mu$ to a direct sum of F_ν maps $S_\pi \otimes S_\mu$ into (but not necessarily onto) the corresponding direct sum of the S_ν. So we ask, in how many ways can subspaces $S_\pi \subset F_\pi$ be chosen to satisfy this tensor-product condition?

The following construction accounts for most of these ways. Let H be any subgroup of $\overline{A(\mathfrak{O})}$, and for each π let S_π be F_π^H, the H-fixed vectors in S_π. It is immediate that this choice of the S_π satisfies the tensor-product condition; on the level of Dixmier algebras, this choice corresponds to the subalgebra A^H of A. It is clear that two subalgebras H, H' of $\overline{A(\mathfrak{O})}$ are conjugate if and only if A^H and $A^{H'}$ are conjugate under $\overline{A(\mathfrak{O})}$. To search for other ways to choose the S_π, it is convenient to use the (possibly new) notion of partial characters, which we now introduce. Let F be any finite group, (π, V) a complex finite-dimensional representation of F, S a subspace of V that is not necessarily F-stable. Make each $f \in F$ act on S by composing the map $\pi(f)$ with the orthogonal projection $V \to S$ with respect to any F-invariant Hermitian inner product on V. Then the partial character χ_S of F with respect to S is the function which assigns to each $f \in F$ the trace of its action map on S. Alternatively, it is the function on F defined by $f \to \Sigma_j \langle \pi(f)v_j, v_j \rangle$, where $\langle\, ,\rangle$ is any F-invariant positive definite Hermitian inner product on V and $\{v_j\}$ is an orthonormal basis of S with respect to this product. One verifies as for ordinary characters that γ_S is independent of the choices of $\langle\, ,\rangle$ and orthonormal basis. The χ_S are *not* class functions, however, as the map $F \to \text{End } S$ defined above is not in general a representation of F.

Partial characters behave just like ordinary characters with respect to direct sums and tensor products. If the subspace S is one-dimensional, and projects onto the subspaces $S_1, S_2, \ldots$ of the irreducible constituents of V, then χ_S is a linear combination of $\chi_{S_1}, \chi_{S_2}, \ldots$. Partial characters satisfy even more orthogonality relations than do ordinary characters: given any basis $b_1, \ldots, b_n$ of an irreducible F-module B, there is another basis $c_1, \ldots, c_n$ of B such that the inner product of the partial characters $\chi_{\mathbb{C}b_i}$ and $\chi_{\mathbb{C}c_j}$ (defined in the obvious way) is nonzero if and only if $i = j$. To see this, first choose a basis $c'_1, \ldots, c'_n$ of B^*, the dual of B, so that the projection of $\mathbb{C}b_i \otimes \mathbb{C}c'_j$ onto the trivial F-module is nonzero if and only if $i = j$, define the c_j by $\langle c_j, b\rangle = c'_j(b)$ for $b \in B$ ($\langle\, ,\rangle$ as above), and use the fact that the sum of the group elements acts by 0 on any irreducible nontrivial module. It follows that whenever the subspaces $S_1, S_2, \ldots$ are linearly independent, then so are their partial characters $X_{S_1}, X_{S_2}, \ldots$ (in fact, it is possible for the X_{S_i} to be independent even if the S_i are dependent).

Our main result on subalgebras is

THEOREM 4.1.1: *Only finitely many choices of subspaces $S_1 = V_1, S_\pi, S_\mu, \ldots$ satisfy the tensor-product condition.*

Proof: Let $S_1, S_\pi, \ldots$ be one such choice and let H be the subgroup of F consisting of all elements which fix the S_ν pairwise. Let C be the complex span of all partial characters with respect to all one-dimensional subspaces of the S_ν. Since the constant function 1 belongs to C, the tensor-product condition and a Vandermonde determinant argument show that there is some partition $P_1 = \{\text{identity coset}\}, P_2, \ldots$ of the left cosets of H in F such that C consists of all functions constant on cosets and agreeing on any two cosets in the same P_i (the definition of H shows that P_1 is as indicated). If v belongs to some S_π, the partial characters $\chi_{\mathbb{C}v}$ belongs to C; conversely, if $\chi_{\mathbb{C}v}$ belongs to C, the independence of the partial characters forces v to be a linear combination of elements of some S_π and thus to belong to S_π. Hence the S_π are completely determined by C. Since there are only finitely many possibilities for C, there are only finitely many for the S_π.

The parametrization of Dixmier subalgebras given by Theorem 4.1.1 is quite inefficient. The following result limits the number of such subalgebras far more sharply.

THEOREM 4.1.2: *Retaining the notations of the preceding proof, we have $S_\pi \neq 0$ whenever $F_\pi^H \neq 0$.*

Proof: We observed above that the partition of F/H induced by C had $P_1 = \{\text{identity coset}\}$. Hence C contains the function which is 1 on H and 0 off H. If F_π has an H-fixed vector $v \neq 0$, then the inner product of this function and $\chi_{\mathbb{C}v}$ is nonzero, whence this function is a linear combination of partial characters at least one of which, say $\chi_{\mathbb{C}w}$, is nonorthogonal to $\chi_{\mathbb{C}v}$. But this forces $w \in F_\pi$, by the orthogonality of partial characters, so $S_\pi \neq 0$.

It is natural to conjecture that the only subspaces satisfying the tensor-product condition are the spaces of invariants under some subgroup, but I do not know how to prove this except in particular cases. Note, however, that if we assume A is completely prime, then we can localize A as in section 3 and apply Jacobson's Galois theory for division rings to deduce this result [Jacobson, 1956]. We will see below how to use Theorem (4.1.2) to prove this for S_3, S_4, and possibly S_5.

We now look for abstract Dixmier algebras, not realizable as subalgebras of the putative A satisfying (4.1.1). In general, the question of their existence boils down to how commutative we can make an analogue of (3.1.3) by an appropriate choice of τ's (not necessarily isomorphisms); that question appears to be very difficult. There is one situation, however, in which the methods of the abelian case prove helpful. Retaining the notation of the last two theorems, let $H \subset F$ and put $S_\pi = F_\pi^{[H,H]}$ for $\pi \in F^\wedge$, where $[H, H]$ is the commutator subgroup of H. Then $H/[H, H]$ acts naturally on the sum of the S_π, and this sum decomposes into one-dimensional constituents under the action. Translating this observation to the level of Dixmier algebras, we get that $A^{[H,H]}$ is graded by the abelian group $H/[H, H]$. In case this group has nontrivial Schur multiplier, we get additional Dixmier algebras by the results of section 3. Since the Schur multiplier of any quotient of H injects into that of H, we can detect these additional algebras by looking at $H^2(H, \mathbb{C}^\times)$. We will say more about this later.

Before discussing particular groups F in the next two subsections, we show how to attach a finite-dimensional algebra to a unipotent Dixmier algebra parametrized by F. We must make the technical hypothesis that tensor products of irreducible F-modules are multiplicity-free; this is satisfied for $F = S_3$ and $F = S_4$, but not $F = S_5$. For each $\mu, \nu \in \hat{F}$, fix an isomorphism $\tau_{\mu\nu}$: $V_\mu \otimes_{V_{\text{triv}}} V_\nu \cong \oplus V_{\chi_i}$, as in (1.1). Given $A = \oplus_{\mu \in F^\wedge} A_\mu$, write each A_μ as a direct sum of copies of V_μ and fix an isomorphism from each copy of V_μ onto V_μ itself. Construct a finite-dimensional $\mathbb{C}$-algebra J as follows. J is spanned as a vector space by linearly independent genetators v_{μ_i}, one for each copy of each V_μ sitting inside A. Any two such copies, say one of V_μ, one of V_ν, must multiply as follows: apply the isomorphisms sending these copies to V_μ, V_ν, then apply $\tau_{\mu\nu}$ to $V_\mu \otimes V_\nu$, getting a sum of V_χ, and finally send this sum into A by a $U(\mathfrak{g})$-bimodule homomorphism. Such a homomorphism may be naturally identified with a $\mathbb{C}$-linear combination of appropriate $v_{\gamma i}$. Define the product of the v_μ; v_ν corresponding to the original two copies to be this linear combination. Extend the definition to arbitrary products by bilinearity. Our algebras A will always have $A_1 = A_{\text{triv}}$ consist of a single copy of $V_1 = V_{\text{triv}}$; we may normalize J so that $v_1 = 1$, the multiplicative unity. J will not be associative in general. If F is abelian, then it is always possible to choose the $\tau_{\mu\nu}$ so that J is associative (and is in fact just the group algebra of F), as we saw in Section 3. Unfortunately, this result fails for non-abelian F. Nor is it true that associativity of J (for some choice of $\tau_{\mu\nu}$) implies associativity of A. Fortunately, we have at least the following fact: A is strongly prime if and only if no linear combination of v_μ's in J annihilates any combination of v_ν's (regardless of the choice of $\tau_{\mu\nu}$).

4.2. The case of S_3

The first thing to be said about this case is that there is an example in which (4.1.1) is satisfied and $\overline{A(\mathfrak{O})} \cong S_3$. Take $\mathfrak{g} = so(8)$, the set of all 8×8

complex skew-symmetric matrices; this is a Lie algebra of type D_4. The unipotent ideal attached to the minimal 10-dimensional nilpotent orbit in $\mathfrak{g}^*$ is the Joseph ideal J, having infinitesimal character (2,1,1,0) in standard coordinates. Put $A = U(\mathfrak{g})/J$. It is well known that A is completely prime and has K-types exactly the Cartan powers of the adjoint representation, each occurring once [Joseph, 1976]. The group D of diagram automorphisms of $\mathfrak{g}$ is isomorphic to S_3 and acts by automorphisms on $U(\mathfrak{g})$. It preserves J since the infinitesimal character of J is the sum of the fundamental dominant weights corresponding to the three outer roots of the Dynkin diagram of $\mathfrak{g}$ and these roots are permuted by D. Hence D acts on A by automorphisms. The Lie subalgebra $\mathfrak{g}^D$ of D is a Lie algebra $\tilde{\mathfrak{g}}$ of type G_2, whence D preserves the $U(\tilde{\mathfrak{g}})$-bimodule action on A. Now $I = U(\tilde{\mathfrak{g}}) \cap J$ turns out to be maximal unipotent in $U(\tilde{\mathfrak{g}})$ [Garfinkle, 1982]; it is attached to the 10-dimensional subregular nilpotent orbit $\mathfrak{O}$ in $\tilde{\mathfrak{g}}^*$. $\mathfrak{O}$ is represented by any sum e of two orthogonal root vectors; it is quite easy to show that $\pi_1(\mathfrak{O}) \cong S_3$ by explicitly calculating $\tilde{G}^e$. Thus A is a Dixmier algebra. It can be seen at once that it contains at least one copy of the $U(\tilde{\mathfrak{g}})$-bimodule V_1 corresponding to the trivial representations of S_3, and at least two copies of the bimodule V_R corresponding to the two-dimensional representation of S_3. The remaining assertions of (4.1.1) will follow from the complete primality of A and the discussion below. Alternatively, one can verify (4.1.1) by calculating the G_2-types of A explicitly, as I will do in a future paper.

Returning to the general case where $\overline{A(\mathfrak{O})} \cong S_3$, denote the trivial, sign, and two-dimensional representations of S_3 by 1, S, R, respectively, and let V_1, V_S, V_R be the corresponding bimodules given by Theorem 1.1. By (4.1.1) there is an algebra A whose decomposition as a bimodule is $V_1 \oplus V_S \oplus 2V_R$. The theory of the last section yields three proper subalgebras of A up to conjugacy, having the decompositions V_1, $V_1 \oplus V_S$, $V_1 \oplus V_R$, and corresponding to the respective conjugacy classes $\{S_3\}$, $\{A_3\}$, $\{2 - \text{subgroups}\}$ of S_3. The only other possibility for a subalgebra of A, by (4.1.2), would be of the form $V_1 \oplus V_S \oplus V_R$. But no subalgebra of A has this form. To see why, suppose the contrary and consider the division ring Fract A with its subdivision rings Fract $(V_1 \oplus V_S \oplus V_R)$, Fract V_1. A simple count shows that the left or right dimension of Fract A over Fract V_1 cannot be a multiple of that of Fract $(V_1 \oplus V_S \oplus V_R)$, a contradiction. (It is an easy consequence of [Barbasch–Vogan, 1985, Theorem III] that, for any group $\overline{A(\mathfrak{O})}$, (Fract $V_1)V_\pi$ has left dimension over Fract V_1 equal to that of π over $\mathbb{C}$, in the notation of Theorem 1.1. This fact is useful later but is not needed now.) Hence the subalgebras of A correspond exactly to subgroups of S_3. In the particular case $A = U(so(8))/J$, the three subalgebras of the form $V_1 \oplus V_R$ correspond to the three conjugate images of $U(so(7))$ in A, while the subalgebra of the form $V_1 \oplus V_S$ is not generated by ad-locally finite elements and so has no such simple description.

Since the group $\mathbb{Z}/2$ has trivial Schur multiplier, the Dixmier algebras of the form $aV_1 \oplus bV_S$ that we have already found are the only ones, and no strongly prime Dixmier algebra contains more than one copy of V_1 or V_S. Consider first Dixmier algebras B that do contain a copy of V_S. They take the form $V_1 \oplus V_S \oplus nV_R$, whence the finite-dimensional algebras J attached to them are spanned by $1, v_S, v_{R_1}, \ldots, v_{R_n}$, where the span J_R of the v_R's is stable under left and right multiplication by V_S. There are two copies of V_R inside A stable under left and right multiplication by the copy of V_S; these copies correspond to the two irreducible A_3-submodules of the two-dimensional S_3-module. We may thus use multiplication in A to define isomorphisms $\tau_{SS}, \tau_{SR}, \tau_{RS}$ making the analogue of (3.1.3) commute whenever it does not involve τ_{RR}. With these choices of $\tau_{SS}, \tau_{SR}, \tau_{RS}$, we see that left and right multiplication by v_S are commuting involutions of J_R. It follows our algebra B is completely reducible over its subalgebra B' of the form $V_1 \oplus V_S$, the irreducible constituents other than B' taking the form V_R.

We therefore ask what the irreducible B'-bimodules are, recalling that the V_1-action is prescribed. We need only consider bimodules of the form $V_1 \oplus V_S$ or V_R (actually, it is easy to show that these are the only possible forms). Then it is not difficult to see that the following possibilities are the only ones. Let C be either copy of V_R inside A used above to define τ_{SR}, τ_{RS}, and define ${}_+B'_\pm$, ${}_\pm C_\pm$ as follows, ${}_+C_-$, for example is isomorphic to $C = {}_+C_+$ as a V_1-bimodule, but the right V_S-action on it is -1 times the right V_S-action on C, while the left V_S-actions on these two bimodules agree. The other definitions are similar. Then the irreducible B'-bimodules are exactly those that have just been defined. We can of course define ${}_-B'_\pm$ as well, but these bimodules are isomorphic to ${}_+B'_\pm$ under the intertwining operator which is 1 on the copy of V_1, -1 on the copy of V_S. I claim now that the decomposition of A over B' is just ${}_+B'_+ \oplus + C_+ \oplus {}_-C_-$. Indeed, the first two pieces occur in A by definition. If the last piece were ${}_+C_-$ or ${}_-C_+$, then every copy of V_R in A would be stable under left or right multiplication by the copy of V_S. Neither of these possibilities holds, as an easy calculation with S_3-modules shows. Hence A decomposes as indicated; now multiplication in A shows that ${}_+C_+ \otimes_{B'} {}_+C_+ \cong {}_-C_-$ and ${}_+C_+ \otimes_{B'} {}_-C_- \cong {}_+B'_+$. The key relation is ${}_-B'_+ \otimes_{B'} {}_+C_+ \otimes_{B'} {}_+B'_- \cong {}_-C_-$, which may be verified as follows. There are V_1-bimodule isomorphisms from ${}_+B'_+$ to ${}_-B'_+$ and ${}_+B'_-$, sending 1 to 1 in both cases. Then multiplication in A combined with these isomorphisms establishes the last isomorphism; one may check directly that it intertwines the actions as indicated. It follows that irreducible B'-bimodules are parametrized by S_3 (not $\hat{S}_3$).

We then get five potential Dixmier algebras including a copy of V_S, having the decompositions $B' \oplus {}_+C_+ \oplus {}_-C_-$, $B' \oplus {}_+B'_-$, $B' \oplus {}_+C_-$, $B \oplus {}_-C_+$, and $B' \oplus {}_+B'_- \oplus {}_+C_+ \oplus {}_+C_- \oplus {}_-C_+ \oplus {}_-C_-$. It turns out that associative algebras exist with each of these decompositions, but that only the first is completely prime (it is of course just A). It is clear what is wrong with $B' \oplus {}_+B'_-$; it contains two copies of V_1, so cannot even be strongly prime. In fact, it is easy to show that its localization is isomorphic to $M_2(\text{Fract } V_1)$. More interestingly, the algebras $B' \oplus {}_+C_-$ and $B' \oplus {}_-C_+$, which turn out to be isomorphic, are strongly but not completely prime. Indeed, their localizations L_1, L_2 are isomorphic to the localization L of $B' \oplus {}_+B_-$ under maps which do *not* respect the G- and $U(\mathfrak{g})$-module structures. (For example, the trivial K-type occurs once in the former localizations, but twice in the latter.) These isomorphisms are defined as follows. The map ${}_+C_+ \otimes {}_-C_- \xrightarrow{\cong} {}_+B'_+$, when localized, yields two elements r, r^{-1} in the localizations of ${}_+C_+$, ${}_-C_-$, which acts as inverses in the localization of A. Then the algebra isomorphism $L \to L_1$ is given by $v \to r \otimes v \otimes r^{-1}$ composed with the localized isomorphism from ${}_+C_+ \otimes {}_+B'_- \otimes {}_-C_-$ to ${}_+C_-$. The other isomorphism is defined similarly. The last and largest algebra, like the second, fails to be strongly prime because of its two copies of V_1. The upshot of the above discussion is that the only completely prime Dixmier algebras containing a copy of V_S are the two subalgebras of A with that property.

We consider now Dixmier algebras having no copy of V_S. Our main result is the following one.

THEOREM 4.2.1: *The only completely prime Dixmier algebras of the form $V_1 \oplus_n V_R$ have $n = 1$.*

To prove this we will use the following fact from algebraic geometry.

PROPOSITION 4.2.2: *Every morphism from complex projective space $\mathbb{P}^n$ to itself has a fixed point.*

Proof: Recall that the graded cohomology algebra $H^*(\mathbb{P}^n, \mathbb{Z})$ of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}[t]/(t^{n+1})$, where t has degree 2. We have $t = c_1\mathfrak{O}(1)$, the first Chern class of the line bundle dual to the canonical line bundle. If $\varphi: \mathbb{P}^n \to \mathbb{P}^n$ is a morphism, then $\varphi^*(c_1\mathfrak{O}(1)) = c_1\mathfrak{O}(k) = c_1\mathfrak{O}(1)^{\otimes k}$ for some $k \geqslant 0$ [Hartshorne, 1977, Theorem 7.1.1], whence the trace of φ^* on $H^*(\mathbb{P}^n, \mathbb{Z})$ is $1 + k + \cdots + k^n > 0$. By the Lefschetz fixed-point-formula [Spanier, 1966], φ must have a fixed point.

Proof of Theorem 4.2.1: We are given a completely prime A with attached J spanned by $1, v_{R_1}, \ldots, v_{R_n}$. The copies of V_R inside A, when localized, have dimension 2 over the localization of V_1, so complete primality forbids the product of two such copies to sit inside V_1. Hence the map $(a_1, \ldots, a_n) \to (b_1, \ldots, b_n)$ defined in homogeneous coordinates by $(a_1 v_{R_1} + \cdots + a_n v_{R_n})^2 = c_1 + b_1 v_{R_1} + \cdots + b_n v_{R_n}$ in F is a well-defined morphism from $\mathbb{P}^{n-1}$ to itself, whence by (4.2.2) it has a fixed point. This point corresponds to a subalgebra of A of form $V_1 \oplus V_R$, which is also completely prime. The localization L of this subalgebra acts on the localization of A, on the left, decomposing the latter into a direct sum $L \oplus C_1 \oplus \cdots \oplus C_r$ where each C_i is necessarily thc localization of a sum of copies of V_R. But each C_i must have left dimension 1 over L, which in turn has left dimension 3 over Fract V_1. This is a contradiction.

We therefore concentrate on associative algebra structures on $V_1 \oplus V_R$; we know that at least one exists. The crucial question is whether a surjection $\tau_{RR}\colon V_R \otimes_{V_1} V_R \to V_1 \oplus V_R$ (now not an isomorphism) can be chosen so as to make the following diagram commute.

$$\begin{array}{ccc} V_R \otimes V_R \otimes V_R & \xrightarrow{\tau_{RR} \otimes id} & (V_1 \oplus V_R) \otimes V_R \\ {\scriptstyle id \otimes \tau_{RR}} \downarrow & & \downarrow {\scriptstyle (\text{left action}) \oplus \tau_{RR}} \\ V_R \otimes (V_1 \oplus V_R) & \xrightarrow[(\text{right action}) \oplus \tau_{RR}]{} & V_R \oplus V_1 \oplus V_R \end{array} \tag{4.2.3}$$

A simple calculation shows that if (4.2.3) commutes for one choice of τ_{RR}, then it commutes for every such choice. In this case we get a family of Dixmier algebras parametrized by $(\mathbb{C}^\times)^2$ modulo the equivalence relation $(a, b) \sim (ac, bc^2)$. The reason for this is that any such algebra is determined by its associated J, whose defining relation is $v_R^2 = av_R + b$. The J corresponding to (a, b) is equivalent to the one corresponding to (ac, bc^2) under the intertwining operator $1 \to 1, v_R \to cv_R$. At least one of these algebras is completely prime, but I do not know about the others. If (4.2.3) fails to commute for any τ_{RR}, then the following diagram does commute for every τ_{RR}:

$$\begin{array}{ccccc} V_R \otimes V_R \otimes V_R & & \xrightarrow{\tau_{RR} \otimes id} & & (V_1 \oplus V_R) \otimes V_R \\ & & & & \downarrow {\scriptstyle (\text{left action}) \oplus \tau_{RR}} \\ {\scriptstyle id \otimes \tau_{RR}} \downarrow & & & & V_R \oplus V_1 \oplus V_R \\ & & & & \downarrow \\ V_R \otimes (V_1 \oplus V_R) & \xrightarrow[(\text{right action}) \oplus \tau_{RR}]{} & V_R \oplus V_1 \oplus V_R & \longrightarrow & V_1 \oplus V_R \end{array} \tag{4.2.4}$$

Here both maps $V_R \oplus V_1 \oplus V_R \to V_1 \oplus V_R$ send (v_r, v, w_r) to $(v_1 v_r + w_r)$. In this case, there is only one algebra structure on $V_1 \oplus V_R$ up to isomorphism, namely the one we already have. We will discuss the commutativity of (4.2.3) for the example $A = U(so(8))/J$ below.

In case there are non-completely-prime algebra structures on $V_1 \oplus V_R$, Theorem 4.2.1 leaves open the question of whether there is any upper bound on the number of copies of V_R in a strongly prime algebra. We mention the following

PROPOSITION 4.2.5: *No strongly prime algebra of the form $V_1 \oplus_n V_R$ has $n \geqslant 5$.*

Proof: Consider the J attached to such a subalgebra. We may identify its elements with matrices by identifying v with left multiplication by v on J. The variety of all matrices with corank at least 2 in a matrix algebra $\mathbb{C}_m$ has codimension 4, so any vector subspace of $\mathbb{C}_m$ of dimension at least 5 meets this variety and thus has an element annihilating vectors in any hyperplane of $\mathbb{C}^m$. Hence if $n \geqslant 5$, some pair of linear combinations of v_R in J have product 0. We saw above that this implies that the algebra is not strongly prime.

We summarize the above discussion in the following statement.

THEOREM 4.2.6: *In the case $\overline{A(\mathfrak{O})} \cong S_3$, the four Dixmier subalgebras of the algebra A satisfying (4.1.1) include the only two strongly prime ones that have a copy of V_S. All are completely prime. There is a strongly but not completely prime Dixmier algebra of the form $V_1 \oplus V_S \oplus V_R$, which is graded by $\mathbb{Z}/2$. Depending on whether a certain diagram commutes, there is either exactly one prime algebra at the form $V_1 \oplus V_R$ or a whole $\mathbb{P}^1$ family of them. A strongly prime Dixmier algebra of the form $V_1 \oplus_n V_R$ has $n \leqslant 4$ while a completely prime one has $n \leqslant 1$.*

We now demonstrate that associativity of the finite-dimensional algebra J attached to a Dixmier algebra A is not necessary for the associativity of A. Indeed, take any A satisfying (4.1.1) for S_3 and consider the possibilities for J. It has a basis $\{1, v_S, v_{R_1}, v_{R_2}\}$ with $v_S^2 = 1$. We observed above that there are exactly two copies of V_R in A stable under left and right multiplication by V_S, so we may as well assume that v_{R_1} and v_{R_2} are eigenvectors for left multiplication by v_S with different eigenvalues. In case J is semisimple it must be isomorphic to $\mathbb{C}^4$ or $\mathbb{C}_2$. In the first case v_{R_1} and v_{R_2} commute, so we see that $v_S v_{R_1} v_{R_2} = v_S v_{R_2} v_{R_1} = v_{R_1} v_{R_2} = -v_{R_2} v_{R_1} = 0$, contradicting strong primality. In the second case we may assume that $v_S = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Then from Lie theory the span of v_{R_1} and v_{R_2} is $\left[\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right]$ and contains two elements with product 0, a contradiction. If J is not semisimple then by various results of [Jacobson, 1956] its Jacobson radical is a direct summand. Tedious but routine arguments show that in all cases $v_{R_1}v_{R_2} = 0$, a contradiction. Hence J cannot be associative for any choice of isomorphisms.

We conclude this subsection by investigating the commutativity of (4.2.3) for $A = U(so(8))/J$. We can embed $\tilde{\mathfrak{g}}$ in $so(7)$ and $so(7)$ in $\mathfrak{g} = so(8)$ in such a way that each $\tilde{\mathfrak{g}}$-weight vector of nonzero weight $so(7)/\tilde{\mathfrak{g}}$ is a sum of orthogonal $\mathfrak{g}$-root vectors and the 0-weight space of $so(7)/\tilde{\mathfrak{g}}$ sits inside a Cartan subalgebra of $\mathfrak{g}$. The $U(\tilde{\mathfrak{g}})$-bimodule generated by the unique $\tilde{\mathfrak{g}}$-stable complement of $\tilde{\mathfrak{g}}$ inside $so(7)$ is isomorphic to V_R and appears in the subalgebra of A of type $V_1 \oplus V_R$, as noted above, so we can use multiplication in this module to define the τ_{RR} in (4.2.3). A little reflection shows that (4.2.3) commutes if and only if $(uv)_1 w = u(vw)_1$ for all $u, v, w \in C$, where $(xy)_1$, $(xy)_1$, $(xy)_R$ denote the projections of xy to V_1 and V_R. This condition is satisfied if $u = v = w = r$, r a weight vector in C, for then r, r^2, $(r^2)_1 = 1/3((12) \cdot r + (13) \cdot r^2 + (23) \cdot r^2)$, and $(r^2)_R = r^2 - (r^2)_1$, are built up out of products of commuting root vectors in $\mathfrak{g}$, so commute with r (here (12), (13), (23) denote transpositions in S_3). Hence (4.2.3) commutes when restricted to the V_1-bisubmodule of $V_R \otimes V_R \otimes V_R$ generated by the $r \otimes r \otimes r$, but I do not know whether this bisubmodule is all of $V_R \otimes V_R \otimes V_R$. If u, v, w are weight vectors of C of different weights, then $(uv)_1 w$ and $u(vw)_1$ are represented by different vectors in $U(\mathfrak{g})$. It seems to be quite tedious to calculate whether or not these vectors are congruent modulo J.

4.3. The cases of S_4 and S_5

Here no algebras satisfying (4.1.1) are known, but we do know two particularly interesting quotients of enveloping algebras attached to nilpotent orbits with these fundamental groups. The kernels of these quotients are maximal ideals with infinitesimal characters specified by [Barbasch–Vogan, 1985, 1.15b], so these quotients are determined by the orbits attached to them. For S_4, the orbit is the one with weighted Dynkin diagram 0200 in the dual of a Lie algebra of type F_4; for S_5, the orbit is the one with diagram $000\overset{0}{2}000$ in the dual of a Lie algebra of type E_8. These orbits play a distinguished role both in Lusztig's work and in [Barbasch–Vogan, 1985, Section 9]; the latter one is the only one in any semisimple Lie algebra dual with fundamental group S_5.

Let A, B satisfy (4.1.1) for $\overline{A(\mathcal{O})} \cong S_4, S_5$, respectively. Then it is easy to show that the only Dixmier subalgebras of A are the algebras of invariants under subgroups of S_4. We omit the details, but the only tools needed are dimension counting, Theorem 4.1.2, and (in one case) the decomposition of A as a bimodule over one of its subalgebras. I believe that the same property is true of B, but have not checked all the details in this case. There are, however, two interesting Dixmier algebras which are not realizable as subalgebras of A or B. They arise by considering bimodules over the subalgebras A^{V_4} and B^{V_4}, where V_4 is any copy of the Klein 4-group sitting inside S_4 and S_5, respectively. One can show tht there are eight bimodules of these algebras which form a group under the tensor product having five elements of order two. Then one can construct these subalgebras by piecing together one-dimensional representations of this group, as indicated above. (A similar situation exists for any $\overline{A(\mathfrak{O})}$ admitting the dihedral group of order 8 as a subgroup.) These algebras are interesting for the following reason. We have seen how finite groups with nontrivial Schur multiplier give rise to unipotent Dixmier algebras if the groups have noncyclic abelian quotients. However, it is possible for $H^2(G, \mathbb{C}^\times)$ to be larger than $H^2(G/[G, G], \mathbb{C}^\times)$, the smallest example being $G = A_4$ and another example being $G = S_4$ [Jacobson, 1974]. It is not clear what the consequences of this phenomenon are for unipotent Dixmier algebras in general, but perhaps the nontrivial element of $H^2(A_4, \mathbb{C}^\times)$ should correspond to one of the algebras constructed above in some reformulation of Vogan's conjecture.

5. A revised class of geometric objects

We saw in Section 3 that orbit covers (even ramified ones) are not sufficiently numerous to parametrize unipotent Dixmier algebras. Equivalently, it is not enough to consider only sheaves of commutative algebras on orbits in order to understand Dixmier algebras. So we consider the simplest kind of sheaves of noncommutative algebras here, namely completely prime algebras of finite type over $S(\mathfrak{g})$ equipped with a locally finite admissible G-action. Since the image of $S(\mathfrak{g})$ is central in such algebras, they become division algebras finite-dimensional over their centers when localized.

Let A be one such localized algebra, and consider first the case where A is commutative. Let I be the kernel of the natural map $S(\mathfrak{g}) \to A$; it is a prime ideal with irreducible associated variety $V(I)$. Assume that $S(\mathfrak{g})/I \hookrightarrow A$ is G-stable and that its G-action coincides with the adjoint action, so that G acts on $V(I)$; also assume that the G-action on $V(I)$ is transitive. (These properties hold for the applications we have in mind.) Then A corresponds to a variety $\mathfrak{O}$, the G-action on $V(I)$ lifts to a transitive action on $\mathfrak{O}$, and we get a finite covering map $\mathfrak{O} \to V(I)$. Such maps are parametrized by conjugacy classes of $\pi_1^G(V(I))$, which coincides with $\pi_1(V(I))$ if G is simply connected.

Now consider the general case. It is well known that by passing to a suitable matrix ring A_m over A one obtains a crossed product algebra (L, Γ, f), where L, a maximal subfield of A_m, is Galois over $K = Z(A)$, $\Gamma = \mathrm{Gal}(L/K)$, and $f \in H^2(\Gamma, L^\times)$ [Herstein, 1968]; of course the action of G on A extends to A_m. Assume that L is G-stable, and so dealt with by the preceding paragraph; it is difficult to see how to make any progress without this assumption. For each $\sigma \in \Gamma$, let $l_\sigma \in A_m$ be an invertible element such that conjugation by l_σ acts on L by σ. Then conjugation by $g \cdot l_\sigma$ acts on L by some $\tau(g) \in \Gamma$ for each $g \in G$, so we get a homomorphism $G \to \Gamma$. Under our standard assumption that G is connected, this homomorphism must be trivial. Thus we get a map $\phi_\sigma\colon G \to L^\times$ defined by $g \cdot k_\sigma = \phi_\sigma(g)k_\sigma$. A simple computation shows that ϕ satisfies Noether's equations, so is a 1-cocycle. Moreover, if one replaces k_σ by lk_σ for some $l \in L^\times$, one obtains a new ϕ_σ which is easily seen to be cohomologous to the old. Thus ϕ_σ may be identified with an element of $H^1(G, L^\times)$. Applying a typical $g \in G$ to the equation $k_\sigma k_\tau = f(\sigma, \tau)k_{\sigma\tau}$, one deduces that

$$(g, f(\sigma, \tau))/f(\sigma, \tau) \quad = \quad \phi_\sigma(g) \cdot \sigma_r(g)^\sigma/\phi_{\sigma\tau}(g), \tag{5.1}$$

where the Galois group action has been indicated by a superscript. It follows that the map $\sigma \to \phi_\sigma$ belongs to $Z^1(\Gamma, H^1(G, L^\times))$.

Thus to each algebra A there corresponds an element of $Z^1(\Gamma, H^1(G, L^\times))$. Conversely, given L with its G-action and an element of $Z^1(\Gamma, H^1(G, L^\times))$ one can construct an algebra A satisfying the above hypotheses if and only if two things happen; first, we can choose the $f(\sigma, \tau)$ satisfying (5.1) *and* the "factor set condition" of belonging to $H^2(\Gamma, L^\times)$; and second, the resulting G-action on A_m descends to A. If these conditions are met, then the $f(\sigma, \tau)$ are determined exactly up to some element of $H^2(\Gamma, (L^\times)^G) = H^2(\Gamma, \mathbb{C}^\times)$ (the equality following since L is completely prime); note, however, that modifying the $f(\sigma, \tau)$ by an element of $H^2(\Gamma, \mathbb{C}^\times)$ may affect the question of whether the G-action on A_m descends to A. We conclude, then, that algebras A satisfying the above hypotheses correspond 1 − 1 (once L has been fixed) to elements of $Z^1(\Gamma, H^1(G, L^\times)) \times H^2(\Gamma, C^\times)$ satisfying the two technical conditions above.

Of course, the second factor is just what we were looking for, but the first depends on G, whereas the number of Dixmier algebras attached to an orbit $\mathfrak{O}$ depends only on $\overline{A(\mathfrak{O})}$. One may hope to avert this difficulty by specializing to the 0 element of the first factor, as it is easy to see that algebras corresponding $0 \times H^2(\Gamma, \mathbb{C}^\times)$ satisfy the first technical condition. Unfortunately, they usually do not satisfy the second. To see why, let A_m be one of them and define the k_σ as above. Then there is $l \in L^\times$ such that $g \cdot k_\sigma = [l/(g \cdot l)]k_\sigma$ for all $g \in G$. Replacing the k_σ by lk_σ, we may assume that the k_σ are all G-fixed. Then so are the $f(\sigma, \tau)$, so they all belong to $\mathbb{C}^\times$. But now it is easy to see that A_m^G is just the complex span of the k_σ, whereas we must have $A_m^G = \mathbb{C}_m$ if the second condition is to be met. Hence the complex span of the k_σ must be isomorphic to $\mathbb{C}_m$ for some m. This is usually not the case; it can happen only if $|\Gamma|$ is a perfect square, for example. We conclude that we can use the above class to parametrize unipotent Dixmier algebras only if we do one of three things: choose some other distinguished element of $Z^1(\Gamma, H^1(G, L^\times))$, consider algebras A_m such G does not act on A, or drop the hypothesis that G acts on L. None of these alternatives seems palatable to me at the present time.

6. Filtrations of Dixmier algebras

The main tool for getting a Dixmier algebra A to resemble the geometric object which should correspond to it is to pass to the graded algebras gr A attached to various filtrations of A. Vogan's conjecture in [Vogan, 1986] made assertions relating such graded algebras to rings of regular functions on coadjoint G-orbits; we will study such rings of regular functions in a future paper. For now, we study the properties we can get gr A to have. We will broaden our horizons by studying arbitrary Dixmier algebras (not necessarily prime or unipotent). As an introduction to this new point of view, we mention the following result.

PROPOSITION 6.0: *Every prime Dixmier algebra A is primitive.*

Proof: Let $A_1 = U(\mathfrak{g})/I \hookrightarrow A$. Since A has finite length as an A_1-bimodule, its ideals satisfy the descending chain condition, so A has a minimal ideal J. If K is another minimal ideal, then primality of A forces $0 \neq JK = J = K$, so J is unique. Since A is Noetherian as a left A_1-module, J has a maximal proper left A_1-submodule J_1. Then J/J_1 is clearly an irreducible left A-module. If its annihilator is nonzero, it must contain the minimal ideal J. But then $J^2 \subseteq J_1$, contradicting $J^2 = J$. Hence A is left (and right) primitive.

This answers in the affirmative a question raised in [Vogan, 1986] whether certain Dixmier algebras constructed as differential operator rings are primitive.

In what follows we call a filtration of A such that gr A has some property P a P filtration. All of our filtrations will have finite-dimensional filtered levels.

6.1. Completely prime filtrations

From our present point of view the three most important properties of any ring of regular functions on an orbit are finite generation, commutativity, and complete primality (the last coming from irreducibility of orbits). The example of A_2' given in section 3.3 shows that it is not always possible to find a gr A with the last two properties, let alone all three. In this section and the next, however, we will show that we can find gr A with the first and second or first and third properties. In the last section we indicate some of the very pleasant consequences of all three properties (plus others). We begin with a lemma on finite-dimensional representations.

LEMMA 6.1.1: *Let V be a finite-dimensional holomorphic G-module, $\oplus_{i=1}^{n} V_i$ its decomposition into irreducibles, $v = \Sigma_{i=1}^{n} v_i$, $v_i \in V_i - \{0\}$. Then v is G-conjugate to a sum of $\mathfrak{h}$-weight vectors of V including highest weight vectors for each V_i (relative to some Borel $\mathfrak{b} \subset \mathfrak{g}$).*

Proof: Write each v_i as a sum of weight vectors in V_i and let γ_i be the $\prec$-maximal weight of any vector occurring in the sum. By downward induction on $\gamma = \Sigma_{i=1}^{n} \gamma_i$, it suffices to assume that γ is not already the sum of the highest weights of the V_i and prove that v is conjugate to a sum of weight vectors including some whose weights add to $\delta > \gamma$. Then some γ_i, say γ_1, is not a highest weight, so the corresponding vector $w_1 \in V_1$ is not annihilated by some simple root vector $x_\alpha \in \mathfrak{g}$. Let $\{v_{1_j}\}, \ldots, \{v_{n_j}\}$ be bases for the $\gamma_1 + \alpha, \gamma_2, \ldots, \gamma_n$ weight spaces of $V_1, V_2, \ldots, V_n$. There are sets of polynomials $\{p_{1_j}\}, \ldots, \{p_{n_j}\}$ in one complex variable z such that the components of $(\exp zx_\alpha) \cdot v_1 \in G \cdot v_1, \ldots, (\exp zx_\alpha \cdot v_n) \in G \cdot v_n$ in the weight spaces $\gamma_1 + \alpha, \ldots, \gamma_n$ are given by $\Sigma_j\, p_{1_j}(z)v_{1_j}, \ldots, \Sigma_j\, p_{n_j}(z)v_{n_j}$ and such that the p_{k_j} are not all 0 for any fixed n. Choosing any $z_0 \in \mathbb{C}$ not a common zero of the p_{k_j} for any k, we see that $(\exp z_0 x_\alpha) \cdot v$ has the desired property.

Our main result is

THEOREM 6.1.2: *Any completely prime Dixmier algebra A has a finitely-generated completely prime filtration.*

Proof: Let $<$ extend $\prec$, as above. Define the height of a finite-dimensional irreducible G-module to be k_0 times the sum of the coefficients of the simple roots in its highest weight, k_0 a fixed integer chosen so that all heights are integral. This is always nonnegative and is 0 only for the trivial module. Define the height of any weight similarly. Put A_n = sum of the K-types of

A of height at most n. Then A_n is finite-dimensional, since A is admissible, and $A_m A_n \subseteq A_{m+n}$ since every constituent of a tensor product has height at most the sum of the heights of the factors. Hence $\{A_n\}$ defines a filtration of A. To show that gr A is completely prime, it suffices to assume that $\bar{x} \in A_n/A_{n-1}$, $\bar{y} \in A_m/A_{m-1}$ are nonzero and prove that $\overline{xy} \neq 0$. We may assume that the representatives x and y of $\bar{x}$ and $\bar{y}$ in A_n and A_m are sums of vectors belonging to K-types of heights n and m, respectively. Let λ, μ be the $<$-highest K-types occurring in the sums for x and y. By (6.1.1), there is a $g \in G$ such that, when $g \cdot x$ and $g \cdot y$ are written as sums of weight vectors, λ and μ are the two $<$-highest weights occurring. Then $g \cdot xy = (g \cdot x)(g \cdot y)$ is a sum of weight vectors including one of weight $\lambda + \mu$, since the product of the λ- and μ-weight vectors in A is nonzero and is not canceled out by any of the $<$-lower weight vectors appearing in $g \cdot xy$. Every vector in A_{n+m-1} is a sum of weight vectors whose weights have heights at most those of the highest weight vectors in A_{n+m-1}, or at most $n + m - 1$. Hence $g \cdot xy \notin A_{n+m-1}$, so $(g \cdot \bar{x})(g \cdot \bar{y})$ and $\overline{xy}$ are both nonzero in gr A. To show that gr A is finitely generated, we first note that, by a result in [Hochschild–Mostow, 1973] the ring of $\mathfrak{n}$-invariants $A^{\mathfrak{n}}$ is finitely generated, $\mathfrak{n}$ the nilradical of the Borel $\mathfrak{b}$ used to define highest weights. (Actually, Hochschild and Mostow state their result on finite generation only for commutative algebras, but their proof easily carries over to the noncommutative Noetherian case.) Let $x_1, \ldots, x_r$ be weight vectors generating $A^{\mathfrak{n}}$; then any weight vector x in $A^{\mathfrak{n}}$ is a linear combination of products of x_i whose weights coincide with that of x, and $A^{\mathfrak{n}}$ is the sum of its weight vectors. It follows that the canonical images $\bar{x}_1, \ldots, \bar{x}_r$ of $x_1, \ldots, x_r$ in gr A generate $(\text{gr } A)^{\mathfrak{n}}$. Enlarge $\{\bar{x}_1, \ldots, \bar{x}_r\}$ to a set $\{\bar{x}_1, \ldots, \bar{x}_s\}$ whose complex span is G-stable; then the subalgebra of gr A generated by $\bar{x}_1, \ldots, \bar{x}_s$ is G-stable and contains $(\text{gr } A)^{\mathfrak{n}}$, hence fills out gr A.

It would be interesting to know whether the filtration can be chosen Noetherian as well. An example is given in [McConnell, 1984] which shows that this property does *not* follow from the others in the theorem.

The filtration has the further property that the image of each simple factor $\mathfrak{g}_i$ of $\mathfrak{g}$ inside A lies in $A_m - A_{m-1}$, m an integer depending only on $\mathfrak{g}$. We may refine the proof of Theorem 6.1.2 to reduce the value of m, but we cannot in general assume that $m = 1$. For example the image of $\mathfrak{g} = sl(2)$ inside the Weyl algebra A_1 contains squares in A_1, so cannot lie in the first level of any completely prime filtration of A_1. This observation is important because Vogan insists in [Vogan, 1986] that filtrations of A should be "good", thus in particular have the image of $\mathfrak{g}$ sitting inside the first level. Now Theorem 6.1.2 suggests that this may be an unreasonable requirement which does not do full justice to the algebraic structure of A. Note, however, the following advantage of at least having all images of $x \in \mathfrak{g}_i - \{0\}$ in $A_m - A_{m-1}$; the left actions of the images $\bar{x} \in \operatorname{gr} A$ commute, so we get a left action of $S(\mathfrak{g}) = \operatorname{gr}_{\text{standard}} U(\mathfrak{g})$ on gr A (even though the restriction of $\{A_n\}$ to the image of $U(\mathfrak{g})$ in A may not agree with the standard filtration). Hence we may speak of the support of gr A in Spec $S(\mathfrak{g})$, though this may not be well behaved if gr A is not finitely generated over $S(\mathfrak{g})$.

In the example of section 3.3, the filtration may be described explicitly. Identify representations of $SL(2) \times SL(2)$ with ordered pairs of non-negative integers (i, j). Put $A = A_2'$, as in section 3.3. Then the filtration is given by

$$A_n = \text{sum of the } K\text{-types } (i, j) \text{ with } i + j \leqslant n.$$

Hence it is just the usual filtration by the degrees of the operators. As pointed out in section 3.3, gr A is a polynomial ring on two pairs of generators. Each pair of generators commutes and generators in different pairs anticommute. Note that in this case the induced filtration on the image of $U(sl(2) \times sl(2))$ agrees with the usual one except for a scale factor of 2, and that the associated graded algebra of this image is isomorphic as a commutative algebra with G-action to the ring of regular functions on the principal orbit $\mathfrak{O} \subset \mathfrak{g}^*$.

6.2. Commutative filtrations

The filtration of the last section has many nice properties, but also the disadvantage that the recipe for it is rather inflexible. We will see below that it is easier to construct many commutative filtrations for even more general algebras A. One could hope to study several of these at once to gain insight into the structure of A.

THEOREM 6.2.1: *Every Dixmier algebra A has a finitely-generated commutative (hence also Noetherian) filtration.*

Proof: Let $a_1, \ldots, a_n$ be a basis for a finite-dimensional G-stable generating subspace V of A as a left $U(\mathfrak{g})$-module. Choose K so large that $[a_i, a_j] = a_i a_j - a_j a_i \in U_K(\mathfrak{g}) \cdot V$ for all i, j. Define a sequence of subspaces A_n of A inductively, as follows. Identify all $U_n(\mathfrak{g}) + \operatorname{Ker}(U(\mathfrak{g}) \to A)$ with their images in A; denote these for simplicity by $V_n(\mathfrak{g})$. Put

$$A_n = \begin{cases} V_n(\mathfrak{g}) & n < k+1 \\ V_n(\mathfrak{g}) + V & n = k+1 \\ \Sigma_{i=1}^{n-1} A_i A_{n-i} + \Sigma_{j=1}^{n-1} [A_j, A_{n+1-j}] & n > k+1 \end{cases} \qquad (5.2.2)$$

Then it is clear that each A_n is G-stable, and the formulas for A_1 and A_n, $n > k + 1$, show that $\{A_n\}$ is a filtration of A making gr A commutative. We now show that gr A is generated as an algebra by the images of the a_i and of any basis of $\mathfrak{g}$ in it; we do this by showing that these images generate all A_n/A_{n-1} by induction on n. Let $x \in A_n$. Then $x = \Sigma\, x_i$, where each x_i can be written as an expression built up from the a_i's and the basis of $\mathfrak{g}$ by taking products and commutators (call such an expression a word in the a_i's and $\mathfrak{g}$), Define the level of any word inductively as follows: the level of a basis element of $\mathfrak{g}$ is 1, the level of any a_i is $k + 1$, the level of a product uv is the sum of the levels of u and v, and the level of a commutator $[uv]$ is the sum of the levels of u and v minus 1. Then any word of level m belongs to A_m. Modifying x by an element of A_{m-1}, we may assume that each $x_i \in A_n - A_{n-1}$ and has level n as a word in the a_i's and $\mathfrak{g}$. If the expression for x_i includes any commutator $[uv]$ with u, v products of a_i's and basis elements, then we may systematically use the conditions $[\mathfrak{g}, a_i] \in V$, $[a_i, a_j] \in U_k(\mathfrak{g})V$ to write down an equivalent word for x_i of level at most n without this commutator. Continuing, we may rewrite the word for x_i so it involves only products, whence the inductive assumption shows that the images of the a_i and the basis generate $\bar{x} \in A_n/A_{n-1}$.

Note that this filtration automatically satisfies the requirement $\mathfrak{g} \subset A_1$ for goodness mentioned above. We may easily modify the construction to make $\{A_n\}$ completely good; i.e., to make gr A finitely generated over $S(\mathfrak{g})$. This would be the right way to study A as a $U(\mathfrak{g})$-module, but not necessarily the right way to study it as an algebra. We study the filtration given by Theorem (6.2.1) for A'_2 by first studying the corresponding filtration $\{B_n\}$ of A_1, the first Weyl algebra. This is *not* the usual filtration by degree, even though that filtration (given by Theorem (6.1.2)) is in fact commutative. Instead, the first few B_n's are

$$B_0 = \mathbb{C},\ B_1 = B_0 + \mathbb{C}\left\langle x^2, x\frac{d}{dx}, \frac{d^2}{dx^2}\right\rangle,$$

$$B_2 = B_1 + \mathbb{C}\left\langle x^4, x^3\frac{d}{dx}, \ldots, \frac{d^4}{dx^4}, x, \frac{d}{dx}\right\rangle,$$

and

$$B_3 = B_2 + \mathbb{C}\left\langle x^6, \ldots, \frac{d^6}{dx^6}, x^3, \ldots, \frac{d^3}{dx^3}\right\rangle,$$

the brackets $\langle , \rangle$ denoting the complex span. It follows that gr A_1 is given by $\mathbb{C}[x^2, xy, y^2, a, b]$ modulo the relations $a^2 = b^2 = ab = ba = 0$, $xya = x^2b, y^2a = xyb$. The ring of regular functions $\mathbb{C}[x^2, xy, y^2]$ attached to A_1 is more visible than it was for the standard filtration but certain subtleties in the structure of A have been blotted out by the presence of nilpotent elements. Similarly, the first few levels of the filtration $\{C_n\}$ of A_2' given by Theorem 6.2.1 are

$$C_0 = \mathbb{C}, C_1 = C_0 + \mathbb{C}\left\langle x^2, x\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, y^2, y\frac{\partial}{\partial y}, \frac{\partial^2}{\partial y^2}\right\rangle,$$

$$C_2 = C_1 + \left\langle x^4, \ldots, \frac{\partial^4}{\partial x^4}, y^4, \ldots, \frac{\partial^4}{\partial y^4}, x^2y^2, \ldots, \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}, x, \frac{\partial}{\partial x}, y, \frac{\partial}{\partial y}\right\rangle,$$

and

$$C_3 = C_2 + \left\langle x^6, \ldots, \frac{\partial^6}{\partial x^6}, y^6, \ldots, \frac{\partial^6}{\partial y^6}, x^4y^2, \ldots, xy, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right\rangle.$$

The associated graded algebra gr A_2' is given by $\mathbb{C}[x_1^2, x_1y_1, y_1^2, x_2^2, x_2y_2, y_2^2, a_1, \ldots, a_S]$ modulo the relations $a_ia_j = 0$, $x_1y_1a_1 = x_1^2a_2$, $y_1^2a_1 = x_1y_1a_2, x_2y_2a_3 = x_2^2a_4$, etc. (the reader may work out the rest of the relations for himself). Again, the attached ring of regular functions is more obvious than it is in the other filtration. A soupçon of the anticommutativity in A_2' is still present, but somewhat blurred by the welter of nilpotent elements.

We conclude this section by giving another criterion for a unipotent Dixmier algebra parametrized by an abelian group to be completely prime. For simplicity of notation we state it only for algebras parametrized by $\mathbb{Z}/2$, but it will be clear how to generalize it to the arbitrary abelian group case.

THEOREM 6.2.3: *Let $A = A_1 \oplus A_g$ be a $\mathbb{Z}/2$-graded unipotent strongly prime Dixmier algebra. Let $\in$ be the automorphism of A which is* 1 *on* A_1, -1 *on* A_g. *Suppose that for each nonzero a_1, $a_2 \in A_1$, $a_g \in A_g$ there is a commutative finitely-generated $\in$-stable filtration of A such that*

(1) *the canonical images* $\bar{a}_1, \bar{a}_2, \bar{a}_g$ *of* a_1, a_2, a_g *in gr* A *are not nilpotent, and*
(2) *if* N *is the nilradical of gr* A *and* B *the localization of* $(gr\ A)/N$, *then* $\dim B^G = 1$ (*it is automatic that* $\dim((gr\ A)/N)^G = 1$).

Then A *is completely prime if and only if* A_1 *is.*

Proof: As shown in section 3, it suffices to assume that A_1 is completely prime and prove that $(A_1^*)^{-1}A_g$ has no square roots of unity. We also showed that if this fails, there are many nonzero choices of $a_1, a_2 \in A_1$, a_g, $a_h \in A_g$, such that $a_1a_2 = a_ga_h$, $a_1a_h = a_ga_2$. We invoke the technical hypothesis above for just one choice of a_1, a_2, a_g satisfying these equations. By Goldie's Theorem, B is a finite direct sum of fields. I claim that this sum has only one term. Indeed, it is clear that G maps the finite-dimensional $\mathbb{C}$-vector space I spanned by the idempotents in B to itself. If G does not act trivially on this space, then some vector v in it is an $\mathfrak{h}$-weight vector of nonzero weight. Then v must be a root of some polynomial equation with coefficients in $\mathbb{C}$, but as every term of this equation has a different $\mathfrak{h}$-weight, some power of v must be 0. This forces $v = 0$, a contradiction. Hence G acts trivially on I and $\dim I = \dim B^G = 1$, as claimed. It follows that every nonnilpotent element in gr A is a non-zero-divisor, whence the images $\bar{a}_1, \bar{a}_2$, $\bar{a}_g, \bar{a}_n$ of a_1, a_2, a_g, and a_n in gr A still satisfy $\bar{a}_1, \bar{a}_2 = \bar{a}_g\bar{a}_h$, $\bar{a}_1\bar{a}_h = \bar{a}_g\bar{a}_2$. We then get $[(\overline{a_g}^1\bar{a}_1)^2 - 1]\bar{a}_2 = (\overline{a_g}^1\bar{a}_1)^2 - 1 = 0$ and $\overline{a_g}^1\bar{a}_1$ is a square root of 1 in the localization L of gr A. Hence $\overline{a_g}^1\bar{a}_1 = \pm 1 + m$ for some $m \in M = LN$, the ideal generated by N in L (since $L/M \cong B$, a field). Then $(\pm 1 + m)^2 = 1$, $m^2 = \pm 2m$, but the two sides of this last equation have different indices of nilpotence unless $m = 0$. Hence $\bar{a}_g = \pm\bar{a}_1$. This means that, for some level n of the filtration, a_1 and a_g both belong to $A_n - A_{n-1}$ while $a_1 + a_g$ or $a_1 - a_g$ belongs to A_{n-1}. The ϵ-stability of the filtration then forces $a_1, a_g \in A_{n-1}$, a contradiction.

The reader can easily verify that the filtrations produced by (6.2.1) can be chosen to be ϵ-stable. If A can be filtered so that the induced filtration on $A_1 = U(\mathfrak{g})/J$ agrees with the standard one, and if gr J is radical in $S(\mathfrak{g})$, then nonnilpotence of $\bar{a}_1$ and $\bar{a}_2$ in gr A will be automatic. The alert reader will have noticed that it is impossible to filter A_2' so as to make any $\bar{a}_g$ nonnilpotent, but Theorem (3.2.2) saves the day in this case.

6.3. Results of Moeglin

As pointed out in section 3.3, A_2' has no commutative completely prime filtration. We also noted that, although A_2' has a natural module, it does not realize the ring of $\mathfrak{g}$-finite endomorphisms on this module. These two phenomena are related. Colette Moeglin has also studied Dixmier algebras A in two recent preprints [Moeglin, 1986a, b]. She makes very strong assumptions about the existence of commutative completely prime filtrations of A, and then concludes that such A have very nice concrete realizations as rings of $\mathfrak{g}$-finite endomorphisms of degenerate Whittaker modules. More precisely, let I be a primitive ideal in $U(\mathfrak{g})$ with associated variety $\bar{\mathfrak{O}} \subset \mathfrak{g}^*$; then $\bar{\mathfrak{O}}$ is well known tobe the closure of a single nilpotent orbit $\mathfrak{O}$. Choose $e \in \mathfrak{O}$; by the Jacobson–Morozov theorem, e embeds in a standard triple $\{h, e, f\}$ satisfying the bracket relations of $sl(2)$. It is known that there is a unipotent subgroup N of G such that h normalizes n, $K(f, \cdot\,)$ restricted to $\mathfrak{n}$ is a character of the latter (K the Killing form), and dim $N \cdot f = 1/2$ dim $\mathfrak{O}$ (Moeglin gives a construction of one such N). Define a *decreasing* filtration $\{U_n'(\mathfrak{g})\}$ of $U(\mathfrak{g})$ by $U_n'(\mathfrak{g}) = \oplus_{m \in \mathbb{Z}} U_{m-1}(\mathfrak{g})^{[m]}$, where $\{U_K(\mathfrak{g})\}$ is the standard filtration and the superscript $[m]$ denotes the m-eigenspace of ad h acting on $U(\mathfrak{g})$. We say that a left $\mathfrak{g}$-module M is a (possibly degenerate) Whittaker module (relative to f, N, I) if (1) M is an irreducible quotient of $U(\mathfrak{g})/I + U(\mathfrak{g})\{x - K(f, x) | x \in \mathfrak{n}\}$ with annihilator I, and (2) $\{U_n'(\mathfrak{g})\}$ induces a filtration $\{M_n\}$ of M making gr M isomorphic as an algebra and an N-module to the ring $R(\widetilde{N \cdot f})$ of regular functions on the universal cover of $N \cdot f$. Here we make gr M into an algebra by declaring that (gr m)(gr m') = gr um', where u lifts $m \in M_n$ to $U_n'(\mathfrak{g})$ and gr x denotes the canonical image of x in gr M.

We say that I admits a Whittaker model if there is a Whittaker module M relative to f, N, I for some (equivalently, any) choice of f, N. Then Moeglin's two main results state the following. I admits a Whittaker model if and only if $U(\mathfrak{g})/I$ has a G-stable commutative completely prime filtration having bounded difference from the usual one, such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{gr}\, U(\mathfrak{g})/I & \overset{\alpha}{\hookrightarrow} & R(\widetilde{G \cdot f}) \\ \beta \uparrow & \nearrow{\scriptstyle i} & \\ S(\mathfrak{g}) & & \end{array} \tag{6.3.1}$$

Here α is an injective homomorphism, i the comorphism of the canonical map $\widetilde{G \cdot f}$ $G \cdot f \hookrightarrow \mathfrak{g} \overset{K}{\longleftrightarrow} \mathfrak{g}^*$, and β is induced from $\mathfrak{g} + I \overset{\mathrm{gr}}{\longrightarrow}$ gr $U(\mathfrak{g})/I$. Furthermore, let E_I denote the set of Dixmier algebras A with $\mathrm{Ker}(U(\mathfrak{g}) \to A) = I$ having a filtration making the following analogue to

(6.3.1) commute:

$$\begin{array}{ccc} \operatorname{gr} A & \hookrightarrow & R(\widetilde{G\cdot f}) \\ \uparrow & \nearrow_{i} & \\ S(\mathfrak{g})/\operatorname{gr} I & & \end{array} \tag{6.3.2}$$

Then every $A \in E_I$ is a subalgebra of the ring of G-finite maps from the Whittaker module M to itself. For any such A, the group of automorphisms of A fixing $U(\mathfrak{g})/I$ pointwise is finite, and there is a bijection between conjugacy classes of this group and rational subalgebras of A (subalgebras recoverable from their quotient fields as the set of G-finite vectors). Moeglin further shows that in case $G = Sp(n, \mathbb{C})$ many completely prime primitive ideals admit a Whittaker model.

Other criteria for Dixmier algebras to have a concrete realization have been given by Barbasch and Vogan, this time as rings of G-finite maps on an irreducible highest weight module. We saw in section 4 that if (4.1.1) is satisfied for some finite group Γ, say by A, then A has enough subalgebras to correspond to subgroups of Γ up to conjugacy. Moreover, there is reason to believe that these Dixmier algebras are the nicest ones attached to Γ, in some sense. If so, it seems likely from Moeglin's work that many such A should admit a concrete realization via Whittaker modules.

Acknowledgements

This paper is based on my Ph.D. thesis and some further results, done under the supervision of David Vogan. I would like to express my deepest gratitude to him for all he has done for me. Thanks are also due to Bertram Kostant for helpful discussions. The author was supported by a post-doctoral fellowship from the Mathematical Sciences Research Institute, Berkeley, California, during the time this paper was written.

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