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Z. DITZIAN

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The Laplacian and the discrete Laplacian

Z. DITZIAN*
Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

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1. Introduction

The Laplacian Δ in R^d given by

$$\Delta f \equiv \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_d}\right)^2\right) f \tag{1.1}$$

is probably the most investigated partial differential operator. The "discrete" Laplacian is given by

$$\tilde{\Delta}_h f(x) \equiv \sum_{i=1}^{d} (f(x + he_i) + f(x - he_i)) - 2df(x) \tag{1.2}$$

where $\{e_i\}_{i=1}^d$ is an orthonormal system in R^d.

In this article, we will show for $f \in L_\infty(R^d)$ that Δf exists in Sobolev (weak) sense and $\Delta f \in L_\infty(R^d)$ if and only if

$$\sup_h \| h^{-2}\tilde{\Delta}_h f \|_{L_\infty(R)} \leqslant M. \tag{1.3}$$

Extension to $L_p(R^d)$, $1 \leqslant p < \infty$, will also be achieved.

It should be noted that $\tilde{\Delta}_h$ is apparently dependent on the system e_i. However, the result of this paper shows that the condition $\| h^{-2}\tilde{\Delta}_h f \| \leqslant M$ does not depend on the system.

2. An estimate using the Green function and the Green formula

In this section, we will obtain an estimate of $\| h^{-2}\tilde{\Delta}_h f \|$ by $\| \Delta f \|$. For this estimate, we use the Green function $G(\xi, x)$ for the domain $D_a = \{(\xi_1, \ldots, \xi_n): -a < \xi_1 < a, \xi_i \in R \text{ for } i \neq 1\}$.

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This would imply

$$f(x) = -\int_{\partial D_a} f(\xi)\frac{\partial G(\xi, x)}{\partial n_\xi}\,\mathrm{d}\sigma(\xi) - \int_{D_a} \Delta f(\xi)G(\xi, x)\,\mathrm{d}V(\xi) \tag{2.1}$$

where $\partial/(\partial n_\xi)$ is a derivative with respect to the normal to ∂D_a in the outward direction. We recall that the Green function $G(\xi, x)$ on open domain D is a function of $\xi = (\xi_1, \ldots, \xi_d)$ and $x = (x_1, \ldots, x_d)$ satisfying the following:

(A) $G(x, \xi) = G(\xi, x)$.

(B) $\Delta G(\xi, x) = 0$ for $x \neq \xi$ and $x, \xi \in D$.

(C) $G(\xi, x) = (\Gamma(d/2)/(d-2)2\pi^{d/2}R^{d-2}) + H(\xi, x)$ where $R = (\Sigma_{i=1}^d (\xi_i - x_i)^2)^{1/2}$ and H is a harmonic function in D, i.e. $\Delta H(\xi, x) = 0$ for $\xi \in D$ and (fixed) $x \in D$.

(D) $G(\xi, x) = 0$ for $x \in D$ and $\xi \in \partial D$.

In fact, (A) (B), (C) and (D) determine $G(\xi, x)$ uniquely. For the domain D_a and $d \geqslant 3$, we have, following Timofeev [3],

$$G(\xi, x) = \frac{\Gamma(d/2)}{(d-2)2\pi^{d/2}} \sum_{k=-\infty}^{\infty} (r_k^{2-d} - \varrho_k^{2-d}) \tag{2.2}$$

where

$$r_k^2 = (\xi_1 - x_1 - 4ka)^2 + \sum_{i=2}^{d} (\xi_i - x_i)^2 \quad \text{and} \tag{2.3}$$

$$\varrho_k^2 = (\xi_1 - (2a - x_1) - 4ak)^2 + \sum_{i=2}^{d} (\xi_i - x_i)^2.$$

We may also verify that (2.2) satisfies (A), (B), (C) and (D). In particular, we observe that for $\xi_1 = a$, $r_k = \varrho_{-k}$, and for $\xi_1 = -a$, $r_k = \varrho_{-k-1}$.

We now can estimate the discrete Laplacian by the Laplacian. This part of the paper is somewhat computational in contrast to other parts of the proof of the equivalence relation and its extension to other spaces which use "soft" methods.

THEOREM 2.1: *For $f \in C^2(R^d) \cap L_\infty(R^d)$,*

$$\| h^{-2}\tilde{\Delta}_h f \|_{L_\infty(R^d)} \leqslant K \| \Delta f \|_{L_\infty(R^d)} \tag{2.4}$$

where Δf and $\tilde{\Delta}_h f$ are given in (1.1) and (1.2) respectively.

Proof: We first observe that it is sufficient to prove our theorem for $d \geqslant 3$. For $d = 2$, we set $f(x, y) = f_1(x, y, z)$, and using our result for $d = 3$ and f_1, we have

$$\| h^{-2}\tilde{\Delta}_h f_1 \|_{L_\infty(R^3)} \leqslant K \| \Delta f_1 \|_{L_\infty(R^3)}.$$

Since

$$f_1(x, y, z + h) + f_1(x, y, z - h) - 2f_1(x, y, z) = 0$$

and

$$\left(\frac{\partial}{\partial z}\right)^2 f_1(x, y, z) = 0,$$

the above implies (2.4) for f.

We also note that it is sufficient to show

$$|h^{-2}\tilde{\Delta}_h f(0)| \leqslant K \| \Delta f \|$$

as Δf is translation invariant.

To estimate $h^{-2}\tilde{\Delta}_h f$, we use (2.1) to write

$$h^{-2}\tilde{\Delta}_h f(0) = - \int_{\partial D_a} f(\xi) h^{-2}\tilde{\Delta}_h \left[\frac{\partial}{\partial \xi} G(\xi, x)\right] \mathrm{d}\sigma(\xi)$$

$$- \int_{D_a} \Delta f(\xi) h^{-2}\tilde{\Delta}_h G(\xi, x) \, \mathrm{d}V(\xi) \equiv I + J$$

where $\tilde{\Delta}_h$ is taken with respect to the x variables and G is given by (2.2) and (2.3). We now estimate J by

$$|J| \leqslant \frac{\Gamma(d/2)}{(d - 2)\, 2\pi^{d/2}} \| \Delta f \| h^{-2} \left\{ \sum_{k=-\infty}^{\infty} \int_{D_a} |\tilde{\Delta}_h (r_k^{2-d})| \, \mathrm{d}V(\xi) \right.$$

$$\left. + \sum_{k=-\infty}^{\infty} \int_{D_a} |\tilde{\Delta}_h \varrho_k^{2-d}| \, \mathrm{d}V(\xi) \right\}$$

$$= \frac{\Gamma(d/2)}{(d - 2)\, 2\pi^{d/2}} \| \Delta f \| \left(\sum_{k=-\infty}^{\infty} J_r(k) + \sum_{k=-\infty}^{\infty} J_\varrho(k) \right).$$

To find bounds for $J_r(k)$ (or $J_\varrho(k)$), we will evaluate the operator $h^{-2}\tilde{\Delta}_h$ on the function r_k of $(x_1, \dots, x_d)$ at the point $(0, \dots, 0)$. To simplify the expressions in our calculations, we denote for $0 \leqslant j \leqslant d$,

$$r_k(j)^2 = (\xi_1 - \zeta_1(j) - 4ka)^2 + \sum_{i=2}^{d} (\xi_i - \zeta_i(j))^2 \tag{2.5}$$

where

$$\zeta_i(j) = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad \zeta_i(i) = \zeta_i.$$

Using the mean value theorem (in each variable), we have

$$h^{-2}\tilde{\Delta}_h r_k^{2-d}|_{x=0} = -(2-d)d\left\{\sum_{j=2}^{d} r_k(j)^{-d-2}(\xi_j - \zeta_j)^2 + r_k(1)^{-d-2}(\xi_1 - \zeta_1 - 4ka)^2\right\} + (2-d)\sum_{j=1}^{d} r_k(j)^{-d}$$

where since $\tilde{\Delta}_h$ of r_k^{2-d} was evaluated at $(0, \dots, 0)$, $|\zeta_i| < h$. This implies, adding $(2-d)dr_k(0)^{-d}$ to the first sum and subtracting it from the second sum,

$$|h^{-2}\tilde{\Delta}_h r_k^{2-d}| \leqslant |2-d|d\left\{\sum_{j=2}^{d} |r_k(j)^{-d-2}(\xi_j - \zeta_j)^2 - r_k(0)^{-d-2}\xi_j^2| + |r_k(1)^{-d-2}(\xi_1 - \zeta_1 - 4ka)^2 - r_k(0)^{-d-2}(\xi_1 - 4ka)^2|\right\} + |2-d|\sum_{j=1}^{n} |r_k(j)^{-d} - r_k(0)^{-d}|.$$

We denote by $r_k(j^*)$ the expression for $r_k(j)$ in (2.5) but with ζ_j^* replacing ζ_j. We also denote by $r_k(h)$

$$r_k(h)^2 = \min\left\{(\xi_1 - \zeta_1 - 4ka)^2 + \sum_{i=2}^{d} (\xi_i - \zeta_i)^2;\ \sum_{i=1}^{d} \zeta_i^2 \leqslant h^2\right\}. \tag{2.6}$$

We now have for $j > 1$,

$$|r_k(j)^{-d} - r_k(0)^{-d}| \leqslant d|\zeta_j|\,|\xi_j - \zeta_j^*|r_k(j^*)^{-d-2} \leqslant dhr_k(j^*)^{-d-1} \leqslant dhr_k(h)^{-d-1}$$

and

$$|r_k(1)^{-d} - r_k(0)^{-d}| \leqslant d|\zeta_1|\,|\xi_j - \zeta_j^* - 4ka|r_k(j^*)^{-d-2} \leqslant dhr_k(h)^{-d-1}.$$

Similarly, we can show for $j > 1$,

$$|r_k(j)^{-d-2}(\xi_i - \zeta_i)^2 - r_k(0)^{-d-2}\xi_i^2| \leqslant (d + 4)hr_k(h)^{-d-1}$$

and

$$|r_k(1)^{-d-2}(\xi_1 - \zeta_1 - 4ka)^2 - r_k(0)^{-d-2}(\xi_1 - 4ka)| \leqslant (d + 4)hr_k(h)^{-d-1}.$$

Combining the above, we have

$$|h^{-2}\tilde{\Delta}_h r_k^{2-d}| \leqslant h4d^4 r_k(h)^{-d-1}. \tag{2.7}$$

Similarly,

$$|h^{-2}\tilde{\Delta}_h \varrho^{2-d}| \leqslant h4d^4 \varrho_k(h)^{-d-1}$$

where

$$\varrho_k(h)^2 = \min\left\{(\xi_1 - (2a - \zeta_1) - 4ka)^2 + \sum_{i=2}^{d}(\xi_i - \zeta_i)^2\colon \ \sum_{i=1}^{d}\zeta_i^2 \leqslant h^2\right\}. \tag{2.8}$$

We assume $0 < h < a/4$ and estimate first $J_r(0)$.

$$J_r(0) = \int_{D_a\setminus B_{2h}(0)} + \int_{B_{2h}(0)} |h^{-2}\tilde{\Delta}_h r_0^{2-d}|\,\mathrm{d}V(\xi) \equiv J_r'(0) + J_r''(0).$$

To estimate $J_r'(0)$, we utilize (2.7) and write

$$J_r'(0) = \int_{D_a\setminus B_{2h}(0)} |h^{-2}\tilde{\Delta}_h r_0^{2-d}|\,\mathrm{d}V(\xi) \leqslant 2d^3 h \int_{R^d\setminus B_h(0)} r^{-d-1}\,\mathrm{d}V(\xi)$$

$$\leqslant 2d^3\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\,h\int_h^\infty r^{-2}\,\mathrm{d}r = C.$$

For $J_r''(0)$, we write

$$J_r''(0) \leqslant h^{-2}\int_{B_{2h}(0)} |\tilde{\Delta}_h r_0^{2-d}|\,\mathrm{d}V(\xi) \leqslant 4dh^{-2}\int_{B_{3h}(0)} r^{2-d}\,\mathrm{d}V(\xi)$$

$$\leqslant 4d\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\,h^{-2}\int_0^{3h} r\,\mathrm{d}r = C_1.$$

We will be able to show later that

$$K \leqslant (1 + \delta)(C + C_1) \frac{\Gamma(d/2)}{(d - 2)\, 2\pi^{d/2}}$$

where C and C_1 are those of the estimates of $J_r'(0)$ and $J_r''(0)$ and any fixed positive δ.

For $\Sigma_{i=1}^{d} \zeta_i^2 \leqslant h^2$, $(\xi_1, \ldots, \xi_d) \in D_a$, and $k \neq 0$, we have

$$\begin{aligned}
&\left((\xi_1 - \zeta_1 - 4ka)^2 + \sum_{i=2}^{d} (\xi_i - \zeta_i)^2\right)^{1/2} \\
&\geqslant \left((\xi_1 - 4ka)^2 + \sum_{i=2}^{d} \xi_i^2\right)^{1/2} - \left(\sum_{i=1}^{d} \zeta_i^2\right)^{1/2} \\
&\geqslant \left((4|k| - 1)^2 a^2 + \sum_{i=2}^{d} \xi_i^2\right)^{1/2} - h \\
&\geqslant \frac{4|k| - 1}{2} |a| + \frac{1}{2}\left(\sum_{i=2}^{d} \xi_i^2\right)^{1/2} - h \\
&\geqslant (2|k| - 1)a + \frac{1}{2} \sum_{i=2}^{d} \xi_i^2 \geqslant |k|a + \frac{1}{2}\left(\sum_{i=2}^{d} \xi_i^2\right)^{1/2}.
\end{aligned}$$

Using (2.7) and recalling $0 < h < a/4$, we write

$$\begin{aligned}
J_r(k) &\leqslant 2d^2 h \int_{D_a} \left(|k|a + \frac{1}{2}\left(\sum_{i=2}^{d} \xi_i^2\right)^{1/2}\right)^{-d-1} \mathrm{d}V(\xi) \\
&\leqslant 2d^2 ah \int_{R^{d-1}} \left(|k|a + \frac{1}{2}\left(\sum_{i=2}^{d} \xi_i^2\right)^{1/2}\right)^{-d-1} \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d \\
&= Aah \int_0^\infty (|k|a + \tfrac{1}{2}r)^{-d-1} r^{d-2}\, \mathrm{d}r \leqslant A_1 ah \int_0^\infty (|k|a + \tfrac{1}{2}r)^{-3}\, \mathrm{d}r \\
&\leqslant A_2 ah \frac{1}{k^2 a^2} = A_2 \frac{h}{a} \frac{1}{k^2}.
\end{aligned}$$

Therefore,

$$\sum_{k \neq 0} J_r(k) \leqslant \sum_{k \neq 0} a_2 \frac{h}{a} \frac{1}{k^2} \leqslant A_3 \frac{h}{a}.$$

Similarly, we have

$$\sum_{k=-\infty}^{\infty} J_\varrho(k) \leqslant A_4 \frac{h}{a}$$

and, hence combining with the estimate of $J_r'(0)$ and $J_r''(0)$,

$$|J| \leqslant \frac{\Gamma(d/2)}{(d-2)\,2\pi^{d/2}}(C + C_1)\|\Delta f\| + A_5\frac{h}{a}\|\Delta f\|. \tag{2.9}$$

We will show that $I \leqslant B\|f\|/a^2$ which we combine with (2.9) and, as we can choose a to be arbitrarily large, this will complete the proof. We recall that if $\|\Delta f\| = 0$ and $\|f\| < \infty$ then f is a constant and $\tilde{\Delta}_h f = 0$.

To achieve $I \leqslant B\|f\|/a^2$, we observe for $\xi_1 = \pm a$ (that is, $(\xi_1 \ldots \xi_d) \in \partial D_a$), $\Sigma_{i=1}^d \zeta_i^2 \leqslant h^2$, $0 < h \leqslant a/4$, and all k, that

$$r_k = \left((\xi_1 - \zeta_1 - 4ka)^2 + \sum_{i=2}^d (\xi_i - \zeta_i)^2\right)^{1/2} \geqslant \frac{2|k| + 1}{4}|a| + \frac{1}{2}\left(\sum_{i=2}^d \xi_i^2\right)^{1/2}$$

and, similarly,

$$\varrho_k = \left((\xi_1 - (2a - \zeta_1) - 4ka)^2 + \sum_{i=2}^d \xi_i^2\right)^{1/2} \geqslant \frac{|k| + 1}{4}|a| + \frac{1}{2}\left(\sum_{i=2}^d \xi_i^2\right)^{1/2}.$$

We now have for $(x_1, \ldots, x_d) \in B_h(0)$ and $\xi \in \partial D_a$,

$$\left|h^{-2}\tilde{\Delta}_h \frac{\partial}{\partial n_\xi} G(x, \xi)\right| \leqslant \sum_{i=1}^d \left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial n_\xi} G(x, \xi)$$

$$\leqslant \frac{\Gamma(d/2)}{(d-2)\,2\pi^{d/2}} \sum_{i=1}^d \sum_{k=-\infty}^{\infty} \left[\left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial n_\xi} r_k^{2-d} + \left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial n_\xi} \varrho_k^{2-d}\right].$$

Following calculations similar to those done earlier, we have

$$\left|\left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial n_\xi} r_k^{2-d}\right| = \left|\left(\frac{\partial}{\partial x_i}\right)^2\left(\frac{\partial}{\partial \xi_1}\right) r_k^{2-d}\right| \leqslant 2d^3 r_k^{-d-1}.$$

Similarly, we have

$$\left|\left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial n_\xi} \varrho_k^{2-d}\right| \leqslant 2d^3 \varrho_k^{-d-1}.$$

Using the estimate for r_k with $\xi_1 = \pm a$ and $(x_1, \ldots, x_d) \in B_h(0)$, we write

$$\int_{\partial D_a} \left| \left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial n_\xi} r_k^{2-d} \right| \mathrm{d}\sigma(\xi) \leqslant 2 \int_{\xi_1 = \pm a} 2d^3 r_k^{-d-1} \mathrm{d}\sigma(\xi)$$

$$\leqslant 4d^3 S \int_0^\infty \left(\left(\frac{2|k| + 1}{4} \right) a + \frac{1}{2} r \right)^{-d-1} r^{d-2} \mathrm{d}r$$

$$\leqslant B_1 \int_0^\infty ((|k| + \tfrac{1}{2})a + r)^{-3} \mathrm{d}r \leqslant B_2(k + \tfrac{1}{2})^{-2} a^{-2}$$

where B_i are independent of k. Combining the above estimates with similar estimates on ϱ_k and summing on k, we have

$$I \leqslant B \| f \| a^{-2}$$

which completes the proof. ∎

3. The estimate of $\| h^{-2}\tilde{\Delta}_h f \|$ for Δf given in Sobolev sense

We recall first that $\Delta f = \varphi$ in the Sobolev sense if

$$\langle f, \Delta g \rangle = \langle \varphi, g \rangle \equiv \int_{R^d} \varphi(\xi) g(\xi) \mathrm{d}\xi \tag{3.1}$$

for any $g \in \mathscr{D}$ (i.e. $g \in C^\infty(R^d)$ and the support of g is compact). The estimate by Δf is given in the following theorem.

THEOREM 3.1: *If $f \in L_\infty(R^d)$, Δf exists in the Sobolev sense and $\Delta f \in L_\infty(R^d)$, then*

$$\sup_h \| h^{-2}\tilde{\Delta}_h f \|_{L_\infty(R^d)} \leqslant K \| \Delta f \|_{L_\infty(R^d)}. \tag{3.2}$$

Proof: We define $F(\xi)$ by

$$F(\xi) = \langle f(\cdot + \xi), g(\cdot) \rangle \tag{3.3}$$

for $\xi \in R^d$, $f \in L_\infty(R^d)$ and $g \in L_1(R^d)$. Obviously, $g \in \mathscr{D}$ implies $g \in L_1(R^d)$ and $F(\xi) \in C^\infty(R^d)$. Therefore, we may use Theorem 2.1 to obtain

$$\sup_h \| h^{-2}\tilde{\Delta}_h F \|_{L_\infty(R^d)} \leqslant K \| \Delta F \|_{L_\infty(R^d)}. \tag{3.4}$$

We now have

$$|\langle h^{-2}\tilde{\Delta}_h f, g\rangle| = |h^{-2}\tilde{\Delta}_h F(0)| \leqslant K\|\Delta F\|_{L_\infty(R^d)} \leqslant K\|\Delta f\|_{L_\infty(R^d)}\|g\|_{L_1(R^d)}$$

for all $g \in \mathscr{D}$ and all h.

Since $\mathscr{D}$ is dense in $L_1(R^d)$, we can choose for every h and $\varepsilon > 0$ a function g such that $\|g\|_{L_1(R^d)} = 1$ and

$$|\langle h^{-2}\tilde{\Delta}_h f, g\rangle| \geqslant \|h^{-2}\tilde{\Delta}_h f\| - \varepsilon.$$

As both ε and h are arbitrary, we complete the proof. ■

4. The inverse result

In this section, we shall prove the direction opposite to that proved in Theorem 2.1 and 3.1 to complete the "if and only if" relation between Δf and $h^{-2}\tilde{\Delta}_h f$.

THEOREM 4.1: *For $f \in L_\infty(R^d)$ the condition $\|h^{-2}\tilde{\Delta}_h f\|_\infty \leqslant M$ implies that Δf exists in the Sobolev sense and $\|\Delta f\|_\infty \leqslant M$.*

Proof: In fact, this part is quite standard for "Saturation" theorems. Using weak* compactness of the unit ball in L_∞, we choose an accumulation point of $h^{-2}\tilde{\Delta}_h f$ (as $h \to 0+$) in L_∞ which we denote by φ. Obviously, $\|\varphi\|_\infty \leqslant \sup_h \|h^{-2}\tilde{\Delta}_h f\|_\infty$. For $g \in \mathscr{D}$, we have

$$\langle \varphi, g\rangle = \lim_{h\to 0}\langle h^{-2}\tilde{\Delta}_h f, g\rangle = \lim_{h\to 0}\langle f, h^{-2}\tilde{\Delta}_h g\rangle = \langle f, \Delta g\rangle. \quad (4.1)$$

It is sufficient to examine only $g \in \mathscr{D}$ as $\mathscr{D}$ is dense in L_1. This implies that $\varphi = \Delta f$ in the Sobolev sense and thus our proof is complete. ■

5. The situation for other spaces

The analogue of Theorem 2.1 and 3.1 for L_p spaces is given in the following theorem.

THEOREM 5.1: *Suppose that $f \in L_p$, Δf exists in the Sobolev sense and is in L_p, $1 < p < \infty$ or is a finite measure on R^d for $p = 1$. Then*

$$\sup_h \|h^{-2}\tilde{\Delta}_h f\|_{L_p(R^d)} \leqslant K\|\Delta f\|_{L_p(R^d)} \quad \textit{for} \quad 1 < p < \infty \quad (5.1)$$

and

$$\sup_h \| h^{-2}\tilde{\Delta}_h f \|_{L_1(R^d)} \leqslant K \| \Delta f \|_{\mathcal{M}} \tag{5.2}$$

where K is that of Theorem 2.1.

REMARK 5.2: The result (5.1) can be deduced from

$$\left\| \left(\frac{\partial}{\partial x_i} \right)^2 f \right\|_p \leqslant A(p) \| \Delta f \|_p \quad \text{for} \quad 1 < p < \infty \tag{5.3}$$

(see for instance [2]) without referring to Theorem 2.1 but with a constant $K(p)$ that depends on p. However, as $A(p)$ tends to infinity when p tends to 1 or ∞, so will the corresponding $K(p)$ and in this sense, (5.1) is superior.

Proof: We recall that $\mathcal{D}$ is dense in L_q, $1 < q < \infty$ and in C. We imbed L_1 in $\mathcal{M}$ by

$$\int_E f \, \mathrm{d}x = \mu(E)$$

and note that $\mathcal{M}$ is the dual to C and L_p $(1 < p < \infty)$ is the dual to L_q $(q^{-1} + p^{-1} = 1)$. The proof may now be completed following the arguments in the proof of Theorem 3.1 almost verbatim (see also [1]). ∎

We also have the L_p $(1 \leqslant p < \infty)$ analogue of Theorem 4.1.

THEOREM 5.2: *Suppose* $f \in L_p(R^d)$ *and* $\| h^{-2}\tilde{\Delta}_h f \|_p \leqslant M$ *for some* p, $1 \leqslant p \leqslant \infty$. *Then for* $p > 1$, Δf *exists in the Sobolev sense,* $\Delta f \in L_p$ *and* $\| \Delta f \|_p \leqslant M$, *and for* $p = 1$, Δf *exists in Sobolev sense, is a finite measure and* $\| \Delta f \|_{\mathcal{M}} \leqslant M$.

Proof: We repeat the proof of Theorem 4.1 recalling for $p = 1$ that $\mathcal{M}$ is the dual of $C(R^d)$, and L_1 is naturally imbedded in $\mathcal{M}$.

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