Compositio Mathematica

S. Klimek
W. Kondracki
W. Oledzki
P. Sadowski

The density problem for infinite dimensional group actions

Compositio Mathematica, tome 68, nº 1 (1988), p. 3-10
<http://www.numdam.org/item?id=CM_1988__68_1_3_0>

Compositio Mathematica 68: 3–10 (1988)

The density problem for infinite dimensional group actions

S. KLIMEK, W. KONDRACKI*, W. OLĘDZKI & P. SADOWSKI
Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland (*author for correspondence)

Received 31 December 1986

Abstract. In this paper we consider a large class of smooth group actions. Under some weak assumptions for the action of the infinite dimensional Hilbert–Lie group we prove that a non-empty set of points of a manifold with trivial isotropy group is an open-dense subset. In the case of a linear action of a compact Lie group on a Hilbert space the orbit space admits a natural structure of a stratification onto smooth Hilbert manifolds. An example shows that this does not in general happen for non linear actions

1. Introduction

One of the ingredients necessary for the construction of field models in Quantum Physics is the knowledge of infinite dimensional group actions on infinite dimensional manifolds. Pursuing this line of though P.G. Ebin, A.E. Fisher and J.P. Bourguignon have studied the action of the group of diffeomorphism of Sobolev class H^{k+1} on the space of Riemannian metrics of class H^k [2, 4, 6]. The geometric structure of the orbit space for the above action may play an essential role in a rigorous quantization of General Relativity. In the case of Yang–Mills theory, the time zero configuration space of fields corresponds to the orbit space of the infinite dimensional group action of automorphism of a principal bundle on the space of connections.

The geometric structure of the orbit space was, in such a setting, studied by W. Kondracki and J. Rogulski [7], cf. also [8], whereas the mixed Yang–Mills–Einstein theory was recently studied by W. Kozak [9] in a Kaluza–Klein context. In all the above cited papers the density of the main stratum, i.e. of the subset of the orbit space corresponding to the set of O-type orbits, was established with the help of difficult technics, in general different for every case. At the same time the slice property for the group action was proved independently.

In this paper we will show that the slice property implies already the density of the main stratum. The theorem of density proved here is presented in a general formulation, namely it holds under weak assumptions and contains, mutatis mutandis, all above cited situations. This theorem is proved in Section 2, where it is also shown that the slice property of infinite dimensional group action implies the countability of the set of orbit types. Thus the slice property for smooth actions leads to simple proofs of theorems previously considered as rather difficult.

In Section 3 we will investigate a linear action of a compact Lie group on a Hilbert space. Using the results of Section 2 we prove that the oribt space for this action admits a stratification structure onto smooth Hilbert manifolds. It can be presented in the form of a countable sum of an ordered and disjoint family of manifolds which fulfil some additional boundary conditions. Such manifolds are called strata. Strata correspond to orbit types. In particular the main stratum which corresponds to O-type orbits is dense in the whole orbit space, as it was mentioned before. In Section 3 we also give an example of a compact Lie group action on a compact manifold for which the orbit space does not admit a stratification structure in a natural way.

2. Density of the set of O-type orbits

Consider an action of a Lie group G on a manifold M. For $x \in M$ let S_x denote the isotropy subgroup of x in G, i.e.,

$$g \in S_x \quad \text{if and only if} \quad gx = x. \tag{2.1}$$

We will also call S_x the symmetry group of x. Let S be a compact subgroup of G such that there exists a point $x \in M$ for which $S = S_x$. Every conjugated subgroup gSg^{-1} is the symmetry of the point gx, we have $gSg^{-1} = S_{gx}$. Let Gx denote the orbit through x of the action of G. Obviously, for any element S' form $(S) = \{$the class of subgroups conjugated with S in $G\}$ there exists a point $x' \in Gx$ such that $S' = S_{x'}$. In this way every orbit Gx of the G-action determines uniquely a conjugacy class of a compact subgroup in G. In general there exist, however, many different orbits corresponding to one conjugacy class (S). We shall call (S) the orbit type. Let $M_{(S)}$ be the set of elements of M with isotropy groups conjugated to a certain S_x, for some fixed $x \in M$. $M_{(S)}$ is called the set of S-type orbits. If for a compact Lie subgroup S of G there exists an $x \in M$ such that $S_x = S$, then $M_{(S)}$ is the set of all points from M with symmetry of S-type – otherwise $M_{(S)}$ is empty. It is seen that $M_{(S)}$ is invariant under the G-action. By $M_{(O)}$ we shall denote the set of all elements of M without any symmetry except the identity in G. We shall call $M_{(O)}$ the set of O-type orbits.

In this section we confine our attention to the O-type orbits in order to show that $M_{(O)}$ is an open and dense subset in M. We shall mostly consider a smooth action of a Hilbert–Lie group G on a Hilbert manifold M endowed with a G-invariant smooth Riemannian metric. Let us note that the compactness of any subgroup of symmetries S_x is assured if we assume the G-action on M to be proper. If an orbit Gx is a submanifold in M then by N_y we shall mean the space of vectors orthogonal to the tangent space T_yGx of Gx in $y \in Gx$. N_y is just a fibre of the normal bundle N of a fixed orbit Gx, a closed smooth subbundle in $TM|_{Gx}$.

DEFINITION 2.1. $(U, Pr)_x$ is called a tubular neighbourhood of a given orbit Gx if and only if U is a G-invariant, open neighbourhood of Gx and Pr is a locally trivial G-equivariant retraction of U onto Gx and a smooth submersion.

Suppose that Gx has a tubular neighbourhood $(U, Pr)_x$ and let

$$\sigma_y = Pr^{-1}(y), \quad y \in Gx, \tag{2.2}$$

denote the slice at y. Thus the tubular neighbourhood has the structure of a fibre bundle (U, Pr, Gx) with σ_y as a fibre at $y \in Gx$.

DEFINITION 2.2. We say that an action of G on M admits slices at each point $x \in M$ if for every orbit Gx, there exists a tubular neighbourhood $(U, Pr)_x$.

There is no need to emphasize that the slice method is one of the best instruments for testing the differential and the topological structure of manifolds under consideration (cf. [2, 3, 5, 10]). As we shall show below, all the advantages of the slice method were not recognized yet.

PROPOSITION 2.1. *Let the action of a Hilbert–Lie group G on a Hilbert manifold M with a G-invariant metric be smooth and proper. Suppose that every orbit of this action is a smooth submanifold in M. Then the action of G on M admits slices at each $x \in M$.*

Proof. The proof which is, in fact, a standard construction of a tubular neighbourhood $(U, Pr)_x$, will consist of a brief outline of the method. For $x \in M$ we have the following, rather straightforward, identifications

$$T_xM = \pi_N^{-1}(x) \oplus T_xGx = T_xN, \tag{2.3}$$

where π_N denotes the canonical projection in the normal bundle N. Since $\exp_*$ is the identity on T_xM, so it is on T_xN, we can therefore find an open neighbourhood $V \subset N$ of zero in the fibre $\pi_N^{-1}(x)$ such that the derivative of $\exp: V \to M$ is an isomorphism at an arbitrary point $v \in V$. Thus $\exp: V \to \exp(V)$ is a local homeomorphism. Since the metric is G-invariant and exp is G-equivariant, one can make V to be G-invariant. We can make use of the following topological lemma [3]: Let X, Y be metric spaces and let $f: X \to Y$ be a local homeomorphism. Let f be a $1:1$ mapping on a closed subset $B \subset X$ and $f^{-1}f(B) = B$. Then there exists an open neighbourhood W of B such that $f: W \to f(W)$ is a homeomorphism. Now let $X = V$, let B be the image of the zero section in N and let $f = \exp$. By the lemma there exists a G-invariant, open neighbourhood W of the image of the zero section in N such that $W \subset V$ and $\exp: W \to \exp(W)$ is a homeomorphism, hence a diffeomorphism. For the neighbourhood $(U, Pr)_x$ of Gx one can set $U = \exp(W)$ and $Pr = \pi_N \circ \exp^{-1}$. ■

PROPOSITION 2.2. *Let any compact group G act on a Riemannian manifold. Let the action be smooth and proper and let M be connected and separable. Then the number of different sets of S-type orbits is at most countable.*

Proof. Observe first that for any action of a compact Lie group on a Riemannian manifold we can find a G-invariant Riemannian metric. By Proposition 2.1., the G-action admits slices for each $x \in M$.

Consider the covering $(U, Pr)_x$, $x \in M$ of M. Since M is metric and separable it is a Lindelöf space. Choose now a subcovering which is countable. Every tubular neighbourhood intersects at most a countable number of $M_{(S)}$, i.e., sets of S-type orbits. The result follows from lemma 2.1 below and from the countability of the number of conjugacy classes of closed subgroups S of a compact Lie group G, cf. [7, 8]. ■

Observe the following lemma as a simple consequence of the G-equivariance of Pr and exp.

LEMMA 2.1. *Let the action G on M fulfil all the hypotheses of Proposition 2.1, and let $(U, Pr)_x$ denote a tubular neighbourhood of Gx. For each $y \in U$ and for each $g \in G$ we have*

$$gy = y \Rightarrow gPry = Pry, \tag{2.4}$$

$$gy = y \iff g_* \exp^{-1}(y) = \exp^{-1}(y). \tag{2.5}$$

THEOREM 2.3. *Under the same assumptions as in the Proposition 2.1 – that is, the action of a Hilbert–Lie group G on a connected Hilbert manifold M with* *a G-invariant Riemannian metric is smooth and proper, moreover every orbit of this action is a smooth submanifold in M, we have:*

If $M_{(O)}$ is non-empty, then it is open and dense in M.

At first, we consider a special case of the above theorem in the following lemma. Next, we can make use of the obvious corollary of lemma below, which presents the linear situation of G-action on the Hilbert space. It is shown how to reduce the problem in Theorem 2.3 to this case.

REMARK. The assumptions of the Theorem 2.3 hold in the physically relevant examples, i.e. Gauge Theory, General Relativity and Kaluza–Klein Theory mentioned at the beginning of the paper. In the case of the Gauge Theory where the action of gauge group on the space of connections is considered cf. [7, 8] it has been proved that this action is smooth, proper and admits G-invariant Riemannian metric. It was also shown that the orbits compose the submanifolds. In this way all assumptions listed here are verified.

We can not use directly Theorem 2.3 in the case of General Relativity and Kaluza–Klein Theory because the groups acting in this context (the group of diffeomorphisms and the group of quasi-gauges) are not the Lie groups. Moreover their actions are not smooth. However, the proof mechanism, under small modifications, is able to work as it was done in [9].

LEMMA 2.2. *Let a compact Lie group G act smoothly on a connected manifold M with a G-invariant Riemannian metric. Let M be geodesically complete and let $M_{(O)}$ be non-empty. Then $M_{(O)}$ is open and dense in M.*

Proof. In virtue of the Proposition 2.1 the G-action admits slices for each $x \in M$. To show that $M_{(O)}$ is open consider the orbit $Gx \subset M_{(O)}$ of an arbitrary point $x \in M_{(O)}$. The tubular neighbourhood $(U, Pr)_x$ is also contained in $M_{(O)}$ because if S_x is the symmetry of x and S_y is the symmetry of $y \in \sigma_x$, $S_y \subset S_x$. This shows that $M_{(O)}$ is the sum of tubular neighbourhoods of its orbits, and therefore it is also open. Let us turn our attention to the density of $M_{(O)}$. Let $x \in M_1 = M \backslash \bar{M}_{(O)}$ and let $y_0 \in M_{(O)}$. We can join y_0 with $x_0 \in Gx$ by a geodesic Γ in such a way that Γ is orthogonal to Gx at x_0. This geodesic realizes the local minimum of distance between y_0 and Gx, its existence is assured by geodesic completeness of M and by the compactness of Gx. Since M_1 is open, σ_{x_0} contains points with non-trivial symmetries only. By the virtue of Lemma 2.1. all vectors in N_{x_0} also have non-trivial isotropy groups (symmetries), which are contained in the isotropy group of x_0. (We implicitly identify the isotropy group S and its lifting S_* to N).

Consider any $g \in S_{x_0}$, such that $g_* v = v$, $v \in N_{x_0}$, and v is the vector tangent to Γ at x_0. Then every point of the geodesic is a fixed point for the G-action. But for $y_0 \in M_{(O)}$ there is one only possible isotropy $g = 0$. ■

COROLLARY 2.4. *If a compact Lie group G acts isometrically on a Hilbert space H and the set of O-type orbits $H_{(O)}$ is non-empty, then $H_{(O)}$ is open and dense in H.*

Proof of the Theorem 2.3. The fact that $M_{(O)}$ is open is a consequence of the slice property of the G-action on M, by the same considerations as in the proof of the Lemma 2.2.

To show the density of $M_{(O)}$ let $M_1 = M \backslash \bar{M}_{(O)}$. Since M is connected there exists $x \in \bar{M}_{(O)} \cap \bar{M}_1$. Consider the slice σ_x in x. $M_1 \cap \sigma_x$ is open in σ_x thus $\exp^{-1}(M_1 \cap \sigma_x)$ is also open and non-empty in N_x. Note that

$$\exp^{-1}(M_1 \cap \sigma_x) \cap \exp^{-1}(M_{(O)} \cap \sigma_x) = \varnothing. \tag{2.6}$$

Let now $(N_x)^S_{(O)}$ be the set of O-type orbits of the linear action S_x on N_x, namely, the differential of the S_x on σ_x. Since the G-action is proper S_x is compact. But on account of Lemma 2.1. by (2.5) we have

$$\exp^{-1}(M_{(O)} \cap \sigma_x) = (N_x)^S_{(O)} \cap \exp^{-1}(\sigma_x), \tag{2.7}$$

where $\exp^{-1}(\sigma_x)$ is open.

By Corollary 2.4. $(N_x)^S_{(O)} \cap \exp^{-1}(\sigma_x)$ is dense in $N_x \cap \exp^{-1}(\sigma_x)$. But the non-empty and open set $\exp^{-1}(M_1 \cap \sigma_x)$ is contained in $(N_x \backslash (N_x)^S_{(O)} \cap \exp^{-1}(\sigma_x)$ which leads to a contradiction. ■

3. Density theorem-linear case

As a result of the previous section we are able to perform a decomposition of the manifold M into the sum of S-type submanifolds:

$$M = \bigcup_{S \subset G} M_{(S)}. \tag{3.1}$$

The set of S-type orbits $M_{(S)}$ is a submanifold in M, which follows from Proposition 2.1 and from the fact that $M_{(S)} \cap U$ is a submanifold in U, where U is a tubular neighbourhood of the orbit through $x \in M_{(S)}$, with $S_x \in (S)$. We have also showed that the sum (3.1) is countable and that $M_{(O)}$

is open and dense in M. It is interesting to ask whether $M_{(S)}$ is dense in $M_{(S)} \cup M_{(S')}$, for any two subgroups of G satisfying $S \subset S'$.

In this section we give the positive answer to the above question in the case of a linear acton of G. We present also a counterexample that even in the case of a smooth action of a finite Lie group on a compact, connected manifold the density theorem for arbitrary $S \subset S'$ is not true.

THEOREM 3.1. *Let a compact Lie group G act smoothly and linearly on a Hilbert space M. Suppose that there exists $x \in M$ such that $S_x = S$. Then $S \subset S'$ implies that $M_{(S)}$ is dense in $M_{(S)} \cup M_{(S')}$.*

Proof. Since G is compact there exists in M a G-invariant inner product. Define

$$M_{\tilde{S}} = \{x \in M\colon Sx = x\}. \tag{3.2}$$

$M_{\tilde{S}}$ is a closed vector subspace in M. Let $N(S)$ be the normalizer of S in G. Observe that $N(S)$ being the compact Lie group is the maximal subgroup conserving $M_{\tilde{S}}$. Thus the action

$$N(S) \times M_{\tilde{S}} \to M_{\tilde{S}} \tag{3.3}$$

is well defined. $N(S)$ does not act effectively on $M_{\tilde{S}}$ but $N(S)/S$ does so. Denote by M_S the set of points of M with the symmetry group exactly equal to S. It is seen that the set of 0-type orbits of the action of $N(S)/S$ on $M_{\tilde{S}}$ is equal to M_S. Thus in virtue of Corollary 2.4 M_S is open and dense in $M_{\tilde{S}}$. Let us denote

$$M_{(\tilde{S})} = \bigcup_{S \subset S'} M_{(S')}. \tag{3.4}$$

We have

$$M_{(S)} = GM_S \subset GM_{\tilde{S}} = M_{(\tilde{S})}, \tag{3.5}$$

which implies that $M_{(S)}$ is also dense in $M_{(\tilde{S})}$. Hence, of course, $M_{(S)}$ is dense in $M_{(S)} \cup M_{(S')}$ which is contained in $M_{(\tilde{S})}$ for $S \subset S'$. ■

Now, we present an example of a compact Abelian Lie group G acting smoothly on a finite dimensional smooth manifold M in such a way that there exist two non trivial isotropy groups $S_1 \subset S_2 \subset G$ wih the property that $M_2 \not\subset \bar{M}_1$, where $\varnothing \neq M_i$ is the set of all $x \in M$ such that an isotropy group of the point x is conjugated with S_i in G, $i = 1, 2$.

Let M be the real projective plane $\mathbb{P}_2$. Take as a group G the $\pi/2$-rotations around certain point $x_0 \in M$. The isotropy group S_2 of x_0 is then simply G. We see that M decomposes into three strata: $M_2 = \{x_0\}$, M_1 which is a circle with isotropy group S_1 being the group of the π-rotations and M_0 with trivial isotropy group, i.e. main stratum. Since M_1 is closed so $M_2 \not\subset \bar{M}_1$, hence the counter-example claimed.

The existence of the counter-examples shows that (besides the linear case, Theorem 3.1) the orbit space of a G-action is not, in general, endowed with a structure of the stratification. However, in the mentioned above physical theories, it was shown that the orbit space admits the natural stratification stucture.

Acknowledgment

Our special thanks are due to the referee for the suggestion of the counter-example (presented here) simpler than the original one.

References

1. Bourbaki, N.: Variétés différentielles et analytiques, *Fascicule de résultats* (§1–§7). Paris: Hermann (1971).
2. Bourguignon, J.P.: Une stratification de l'espace des structures riemanniennes. *Comp. Math.* 30 (1975) 1–41.
3. Bredon, G.E.: *Introduction to Compact Transformation Groups*. New York: Academic Press (1972).
4. Ebin, D.G.: The manifold of Riemannian metrics. *Proc. Symp. Pure Math. Am. Math. Soc.* XV (1970) 11–40.
5. Ebin, D.G. and Marsedn, J.E.: Groups of diffeomorphisms and the motion of an incompressible perfect fluid. *Ann. of Math.* 92 (1970) 102–163.
6. Fisher, A.E.: *The Theory of Superspace. Relativity*. New York: Plenum Press (1970).
7. Kondracki, W. and Rogulski, J.S.: On the stratification of the orbit space for the action of automorphisms on connections. *Diss. Math.* 250 (1986).
8. Kondracki, W. and Sadowski, P.: Geometric structure on the orbit space of gauge connections. *J. Geom. Phys.* 3 (1986) 421–434.
9. Kozak, M.: On the geometry of the configuration space for Kaluza Klein field theory. Warsaw University PhD thesis, to appear.
10. Mitter, P.K. and Viallet, C.M.: On the bundle of connections and the gauge orbit manifold in Yang–Mills theory. *Comm. Maths. Phys.* 79 (1981) 457–472.
11. Narasimhan, M.S. and Ramadas, T.R.: Geometry of SU(2) gauge fields. *Comm. Math. Phys.* 67 (1979) 121–136.
12. Palais, R.S.: Embedding of compact differentiable transformation groups in orthogonal representations. *J. Math. Mech.* 6 (1957) 673–678.
13. Singer, I.M.: Some remarks on the Gribov ambiguity. *Comm. Math. Phys.* 60 (1978) 7–12.
14. Whitney, H.: Elementary structures of real algebraic varieties. *Ann. of Math.* 66 (1957) 546–556.