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Some results on the syzygies of finite sets and algebraic curves

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1. Introduction

Let $X \subseteq \mathbf{P}^r$ be a projective variety, not contained in any hyperplane, and let $I = I_{X/\mathbf{P}^r}$ denote the homogeneous ideal of X. When X is a finite set or an algebraic curve, several authors have given criteria for I to be generated by quadrics (cf. [M1], [St.D], [F], [H], [G]). As the results and conjectures of [G] and [GL1] indicate, however, one expects that theorems on generation by quadrics will extend to – and be clarified by – analogous statements for higher syzygies. Our purpose here is to prove two elementary theorems along these lines.

Let $E_\bullet$ be a minimal graded free resolution of I over the homogeneous coordinate ring S of $\mathbf{P}^r$:

$$0 \to E_{r+1} \to \cdots \to E_2 \to E_1 \to I \to 0,$$

where $E_i = \oplus S(-a_{ij})$. We are interested in knowing when the first few terms of $E_\bullet$ are as simple as possible. Specifically, for a given integer $p \geqslant 1$, we ask whether $X \subseteq \mathbf{P}^r$ satisfies the following property:

$$(N_p) \qquad E_i = \oplus S(-i-1) \text{ (i.e., all } a_{ij} = i+1) \quad \text{for} \quad 1 \leqslant i \leqslant p.$$

Thus:

X satisfies $(N_1) \Leftrightarrow I_{X/\mathbf{P}^r}$ is generated by quadrics;
X satisfies $(N_2) \Leftrightarrow (N_1)$ holds for X, and the module of syzygies among quadratic generators $Q_i \in I_{X/\mathbf{P}^r}$ is spanned by the relations of the form $\Sigma L_i Q_i = 0$, where the L_i are *linear* polynomials;

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and so on. (It would also be natural to define property (N_0) to mean that X imposes independent conditions on hypersurfaces of degrees $\geqslant 2$. All of the results below extend to (N_0), and in fact most are classical in this case.)

Our first result concerns finite sets:

THEOREM 1. *Suppose that $X \subseteq \mathbf{P}^r$ consists of $2r + 1 - p$ points in linear general position, i.e., with no $r + 1$ lying on a hyperplane. Then X satisfies property (N_p).*

This generalizes a result of St.-Donat [St.D] who showed that the homogeneous ideal of $2r$ points in $\mathbf{P}^r$ is generated by quadrics. Much as in [St.D], the result has implications for the syzygies of algebraic curves. In fact, as an immediate consequence of the Theorem, one recovers a result of the first author from [G]:

(*) *Let X be a smooth irreducible projective curve of genus g, and for $p \geqslant 1$ consider the imbedding $X \subseteq \mathbf{P}(H^0(L)) = \mathbf{P}^{g+p+1}$ defined by the complete linear system associated to a line bundle L of degree $2g + 1 + p$. Then X satisfies property (N_p).*

The case $p = 1$ of (*) is due to Fujita [F] and St.-Donat [St.D], strengthening earlier work of Mumford [M1].

To complete the result (*), it is natural to ask for a classification of all pairs (X, L) for which it is optimal. This is the content of

THEOREM 2. *Let L be a line bundle of degree $2g + p$ on a smooth irreducible projective curve X of genus g, defining an embedding $X \subseteq \mathbf{P}(H^0(L)) = \mathbf{P}^{g+p}$. Then property (N_p) fails for X if and only if either*

(i) X is hyperelliptic;

or

(ii) $X \subseteq \mathbf{P}^{g+p}$ has a $(p + 2)$-secant p-plane, i.e., $H^0(X, L \otimes \omega_X^*) \neq 0$.

So for instance if $X \subseteq \mathbf{P}^{g+1}$ is a non-hyperelliptic curve of degree $2g + 1$, then the homogeneous ideal $I_{X/\mathbf{P}^{g+1}}$ of X is generated by quadrics unless X has a tri-secant line; this was essentially conjectured by Homma in [H] (cf. also [S]). The theorem already gives a first indication of the fact that the syzygies of a curve are closely connected with its geometry. The influence of the geometry on the algebra is an intriguing, but largely uncharted, facet of the theory of algebraic curves. We refer the reader to [GL1, §3], where Theorem 2 was announced, for some conjectures in this direction.

The generators and syzygies of the ideal of a sufficiently general collection of points in $\mathbf{P}^r$ have been studied by several authors (cf. [B], [GGR]). However, as far as we can tell the elementary Theorem 1 seems to have escaped explicit notice.

Our exposition proceeds in three parts. Section 1 contains various Koszul-theoretic criteria for a projective algebraic set to satisfy property (N_p). This material is standard folklore, and is included mainly for the benefit of the reader not versed in such matters. In Section 2 we give the proof of Theorem 1. The application to curves occupies Section 3.

Finally, we are grateful to L. Ein and E. Sernesi for valuable discussions.

§0. Notation and conventions

(0.1). We work throughout over an algebraically closed field k of arbitrary characteristic.

(0.2). Given a vector space V of dimension $r + 1$ over k, $S = \mathrm{Sym}(V)$ denotes the symmetric algebra on V, so that S is isomorphic to the polynomial ring $k[x_0, \ldots, x_r]$. We denote by $\boldsymbol{k}$ the residue field $S/(x_0, \ldots, x_r)$ of S at the irrelevant maximal ideal. For a graded S-module T, we write T_j for its component of degree j, and as usual $T(p)$ is the graded module with $T(p)_j = T_{p+j}$.

(0.3). $\mathbf{P}(V)$ is the projective space of one-dimensional quotients of V, so that $H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(1)) = V$. A subscheme $X \subseteq \mathbf{P}(V)$ is a *non-degenerate* if it does not lie in any hyperplane.

§1. Criteria for property (N_p)

This section is devoted to spelling out some criteria – eventually of a Koszul-theoretic nature – for a projective algebraic set to satisfy property (N_p) from the introduction. The experts won't find anything new here, and we limit ourselves for the most part to what we need in the sequel. For a general overview of Koszul-cohomological techniques in the study of syzygies, the reader may consult e.g. [G].

Fix a vector space V of dimension $r + 1$ over the ground field k, and consider a non-degenerate projective scheme

$$X \subseteq \mathbf{P}(V) \;=\; \mathbf{P}^r$$

of pure dimension n, defined by a (saturated) homogeneous ideal $I \subseteq S = \mathrm{Sym}(V)$. Let $R = S/I$ be the homogeneous coordinate ring of X. Then $\mathrm{Tor}_i^S(R, \boldsymbol{k})$ is a graded S-module, which may be computed from a minimal graded free resolution.

$$E_\bullet \colon 0 \to E_{r+1} \to E_r \to \cdots \to E_1 \to S \to R \to 0$$

of R. Thus $\mathrm{Tor}_i^S(R, \boldsymbol{k})_j$ is a finite dimensional k vector space whose dimension is the number of minimal generators of E_i in degree j. The non-degeneracy hypothesis on X implies that if $i > 0$ then $\{E_i\}_j = 0$ for $j \leqslant i$. Hence $X \subseteq \mathbf{P}^r$ satisfies property (N_p) from the introduction if and only if

$$\mathrm{Tor}_i^S(R, \boldsymbol{k})_j = 0 \quad \text{for} \quad j \geqslant i + 2 \quad \text{and} \quad 1 \leqslant i \leqslant p. \tag{1.1}$$

In fact:

LEMMA 1.2. *For $p \leqslant \mathrm{codim}(X, \mathbf{P}^r)$, property (N_p) holds for $X \subseteq \mathbf{P}^r$ if and only if*

$$\mathrm{Tor}_p^S(R, \boldsymbol{k})_j = 0 \quad \textit{for} \quad j \geqslant p + 2. \tag{1.2}$$

Proof: Set $M_i = \max\{j \mid \mathrm{Tor}_i^S(R, \boldsymbol{k})_j \neq 0\}$. In view of (1.1), it suffices to prove that the $\{M_i\}$ are strictly increasing in i for $i \leqslant \mathrm{codim}(X, \mathbf{P}^r)$, i.e., that

$$M_1 < M_2 < \cdots < M_{r-n-1} < M_{r-n}, \tag{1.3}$$

where as above $n = \dim X$. To this end, note first that $\mathrm{Ext}_S^i(R, S) = 0$ for $i < \mathrm{codim}(X, \mathbf{P}^r)$; this is presumably well-known, and in any event follows from the local fact ([BS] p. 25) that $\mathscr{E}xt^i_{\mathcal{O}_{\mathbf{P}^r}}(\mathcal{O}_X, \mathcal{O}_{\mathbf{P}^r}) = 0$ when $i < \mathrm{codim}(X, \mathbf{P}^r)$. Hence if as above $E.$ is a minimal graded free resolution of R, and if $E_i^* = \mathrm{Hom}_S(E_i, S)$, then the sequence

$$0 \to E_0^* \to E_1^* \to \cdots \to E_{n-r-1}^* \to E_{n-r}^* \tag{$*$}$$

is exact. On the other hand, recall that if F is any finitely generated graded S-module, then:

the integers $m_i(F) =_{\mathrm{def}} \min\{j \mid \mathrm{Tor}_i^S(F, \boldsymbol{k})_j \neq 0\}$ $(**)$
are strictly increasing in i.

(This follows from the fact that a minimal graded free resolution $E_\bullet(F)$ of F may be constructed inductively by choosing minimal generators of $\ker\{E_i(F) \to E_{i-1}(F)\}$.) But $(*)$ determines a minimal resolution of $\operatorname{coker}(E^*_{n-r-1} \to E^*_{n-r})$, and (1.3) is then a consequence of $(**)$. ■

In the situations that will concern us, one can get away with checking even a little less:

LEMMA 1.4. *Suppose that the ideal sheaf $\mathscr{J}_{X/\mathbf{P}^r}$ of X is 3-regular in the sense of Castelnuovo-Mumford, i.e., assume that*

$$H^i(\mathbf{P}^r, \mathscr{J}_{X/\mathbf{P}^r}(3 - i)) = 0 \quad \textit{for} \quad i > 0.$$

Then for $p \leqslant \operatorname{codim}(X, \mathbf{P}^r)$, (N_p) holds for $X \subseteq \mathbf{P}^r$ if and only if

$$\operatorname{Tor}^S_p(R, \boldsymbol{k})_{p+2} = 0.$$

Proof: It follows from [M2, Lecture 14] that if $\mathscr{F}$ is an m-regular coherent sheaf on $\mathbf{P}^r$, then the corresponding graded S-module $\Gamma_*(\mathscr{F}) = \oplus_l H^0(\mathbf{P}^r, \mathscr{F}(l))$ is generated by elements of degree $\leqslant m$. Applying this observation inductively to the sheafification

$$0 \to \mathscr{E}_{r+1} \to \mathscr{E}_r \to \cdots \to \mathscr{E}_2 \to \mathscr{E}_1 \to \mathscr{J}_{X/\mathbf{P}^r} \to 0$$

of a minimal graded free resolution of $I = \Gamma_*(\mathscr{J}_{X/\mathbf{P}^r})$, one fnds that $\mathscr{E}_i$ is $(i + 2)$-regular. But this means that $\operatorname{Tor}^S_i(R, \boldsymbol{k})_j = 0$ for $j \geqslant i + 3$, so the lemma follows from (1.2). ■

In order to prove in practice the vanishing occurring (1.2) and (1.4), the crucial point is that one can compute $\operatorname{Tor}^S_i(R, \boldsymbol{k})$ via a resolution of $\boldsymbol{k}$. Specifically, consider the Koszul resolution

$$0 \to S(-r - 1) \otimes_k \Lambda^{r+1} V \to \cdots \to S(-2) \otimes_k \Lambda^2 V$$

$$\to S(-1) \otimes_k V \to S \to \boldsymbol{k} \to 0$$

of $\boldsymbol{k}$. Tensoring by R and taking graded pieces, one finds that $\operatorname{Tor}^S_i(R, \boldsymbol{k})_j$ is given by the homology (at the middle term) of the complex of vector spaces

$$\Lambda^{i+1} V \otimes R_{j-i-1} \to \Lambda^i V \otimes R_{j-i} \to \Lambda^{i-1} V \otimes R_{j-i+1}. \tag{1.5}$$

LEMMA 1.6. *Assume that the ideal sheaf $\mathscr{I}_{X/\mathbf{P}^r}$ of X in $\mathbf{P}^r$ is 3-regular. Then for $p \leqslant \operatorname{codim}(X, \mathbf{P}^r)$, $X \subseteq \mathbf{P}^r$ satisfies property (N_p) if and only if the Koszul-type complex*

$$\Lambda^{p+1} \otimes V \to \Lambda^p V \otimes H^0(\mathbf{P}^r, \mathcal{O}_X(2)) \to \Lambda^{p-1} V \otimes H^0(\mathbf{P}^r, \mathcal{O}_X(3)) \qquad (1.7)$$

is exact at the middle term.

Proof: Since $X \subseteq \mathbf{P}^r$ is non-degenerate, one has $R_1 = V$. Furthermore, $R_m = H^0(\mathbf{P}^r, \mathcal{O}_X(m))$ for $m \geqslant 2$ thanks to the 3-regularity of $\mathscr{I}_{X/\mathbf{P}^r}$. Thus (1.7) is just the special case of (1.5) with $i = p$ and $j = p + 2$, so the assertion follows from (1.4). ■

Suppose finally that X is an irreducible projective variety of dimension $n \geqslant 1$, and that the embedding $X \subseteq \mathbf{P}(V)$ is defined by the complete linear system associated to very ample line bundle L on X (so that $V = H^0(X, L)$). We wish then to interpret the exactness of (1.7) in terms of sheaf cohomology. To this end, consider the natural surjective evaluation map

$$e_L \colon H^0(X, L) \otimes_k \mathcal{O}_X \to L$$

of vector bundles on X, and set $M_L = \ker(e_L)$. Thus M_L is a vector bundle of rank $r = h^0(X, L) - 1$ on X, which sits in an exact sequence

$$0 \to M_L \to H^0(X, L) \otimes_k \mathcal{O}_X \to L \to 0. \qquad (1.8)$$

(Note that M_L is defined whenever L is generated by its global sections.) Taking $(p + 1)^{\text{st}}$ exterior powers and twisting by L^m yields

$$0 \to \Lambda^{p+1} M_L \otimes L^m \to \Lambda^{p+1} H^0(X, L) \otimes_k L^m \to \Lambda^p M_L \otimes L^{m+1} \to 0. \qquad (1.9)$$

LEMMA 1.10. *Assume that L is normally generated, i.e., that the natural maps $S^m H^0(X, L) \to H^0(X, L^m)$ are surjective for all m. Suppose also that $H^i(X, L^{2-i}) = 0$ for $i > 0$. Then for $p \leqslant \operatorname{codim}(X, \mathbf{P}^r)$, $X \subseteq \mathbf{P}(H^0(X, L))$ satisfies property (N_p) if and only if*

$$H^1(X, \Lambda^{p+1} M_L \otimes L) = 0.$$

Proof: The normal generation of L means that $h^1(\mathbf{P}^r, \mathscr{I}_{X/\mathbf{P}^r}(m)) = 0$ for all m, and the cohomological hypothesis on L then implies the 3-regularity of

$\mathscr{I}_{X/\mathbf{P}^r}$. Hence we are in the situation of Lemma 1.6. On the other hand, the maps occurring in (1.7) factor as shown through homomorphisms deduced from (1.9):

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & \\
 & & & H^0(X, \Lambda^{p-1} M_L \otimes L^3) \arrow[dr] & \\
\Lambda^{p+1} V \otimes H^0(X, L) \arrow[rr] \arrow[dr, "u"'] & & \Lambda^p V \otimes H^0(X, L^2) \arrow[rr] \arrow[ur] & & \Lambda^{p-1} V \otimes H^0(X, L^3) \\
 & H^0(X, \Lambda^p M_L \otimes L^2) \arrow[ur] & & & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

Thus the exactness of (1.7) is equivalent to the surjectivity of the indicated homomorphism u (which comes from (1.9) with $m = 1$). But $H^1(X, L) = 0$ thanks to the hypothesis on L, and it follows from (1.9) that u is surjective if and only if $H^1(X, \Lambda^{p+1} M_L \otimes L) = 0$. ■

§2. Syzygies of finite sets

Fix a vector space V of dimension $r + 1$ over k, and let $X \subseteq \mathbf{P}(V) = \mathbf{P}^r$ be a finite set consisting of $2r + 1 - p$ $(1 \Delta p \leqslant r)$ distinct points

$$x_1, \ldots, x_{2r+1-p} \in \mathbf{P}^r.$$

THEOREM 2.1. *Assume that the points of X are in linear general position, i.e. that no $r + 1$ lie on a hyperplane. Then $X \subseteq \mathbf{P}^r$ satisfies property (N_p).*

Proof: It is classical that $2r + 1$ or fewer points in linear general position in $\mathbf{P}^r$ impose independent conditions on quadrics, and it follows that $\mathscr{I}_{X/\mathbf{P}^r}$ is 3-regular. Hence by Lemma 1.6 we are reduced to showing that the complex

$$\Lambda^{p+1} V \otimes V \xrightarrow{a} \Lambda^p V \otimes H^0(X, \mathscr{O}_X(2)) \xrightarrow{b} \Lambda^{p-1} V \otimes H^0(X, \mathscr{O}_X(3)) \quad (2.2)$$

is exact in the middle.

To this end, we start by writing X as the disjoint union $X = X_1 \cup X_2$, where $X_1 = \{x_1, \ldots, x_{r+1}\}$, so that X_2 consists of $r - p$ points. Then

$$H^0(X, \mathcal{O}_X(m)) = H^0(X_1, \mathcal{O}_{X_1}(m)) \oplus H^0(X_2, \mathcal{O}_{X_2}(m))$$

for every m, and the homomorphism b in (2.2) breaks up as the direct sum $b = b_1 \oplus b_2$, where for $i = 1$ or 2

$$b_i\colon \Lambda^p V \otimes H^0(X_i, \mathcal{O}_{X_i}(2)) \to \Lambda^{p-1} V \otimes H^0(X_i, \mathcal{O}_{X_i}(3))$$

is the natural map. Now it is an elementary exercise to show that since $\# X_2 \leqslant r + 1$, property (N_r) holds for $X_2 \subseteq \mathbf{P}^r$ (e.g., 2-regularity of $\mathscr{I}_{X_2/\mathbf{P}^r}$). Hence, as in (1.6), the sequence

$$\Lambda^{p+1} V \otimes V \xrightarrow{a_2} \Lambda^p V \otimes H^0(X_2, \mathcal{O}_{X_2}(2)) \xrightarrow{b_2} \Lambda^{p-1} V \otimes H^0(X_2, \mathcal{O}_{X_2}(3)) \quad (*)$$

is exact. Thus to prove exactness of (2.2), it suffices to show that given element

$$\varphi_1 \in \ker(b_1) \subseteq \Lambda^p V \otimes H^0(X_1, \mathcal{O}_{X_1}(2)) \subseteq \Lambda^p V \otimes H^0(X, \mathcal{O}_X(2)),$$

then one can write $\varphi_1 = a(\xi)$ for some element $\xi \in \Lambda^{p+1} V \otimes V$ with $a_2(\xi) = 0$ (a_2 being the map which appears in $(*)$).

This is in turn verified by an explicit calculation. Specifically, choose a basis $s_1, \ldots, s_{r+1}$ of V so that $s_i(x_j) = \delta_{ij}$, and denote by e_i the evident element of $H^0(X_1, \mathcal{O}_{X_1}(2))$ supported at x_i. Then $\ker(b_1)$ is spanned by elements of the form

$$s_{j_1} \wedge \cdots \wedge s_{j_p} \otimes e_i, \quad \text{where} \quad i \notin \{j_1, \ldots, j_p\}.$$

So fixing indices $j_1 < \cdots < j_p$ and $i \notin \{j_1, \ldots, j_p\}$, it suffices to produce an element $\xi \in \ker(a_2)$ with $a(\xi) = s_{j_1} \wedge \cdots \wedge s_{j_p} \otimes e_i$. To this end, let $v = \Sigma\, \lambda_j s_j \in V$ be a linear form on $\mathbf{P}(V)$ which vanishes on X_2, and set $\xi = s_{j_1} \wedge \cdots \wedge s_{j_p} \wedge s_i \otimes v \in \Lambda^{p+1} V \otimes V$. Then $a_2(\xi) = 0$, and one has $a(\xi) = \pm s_{j_1} \wedge \cdots \wedge s_{j_p} \otimes e_i$ provided that coefficients in v satisfy

$$\lambda_{j_1} = \cdots = \lambda_{j_p} = 0 \quad \text{and} \quad \lambda_i = 1. \qquad (**)$$

Hence to prove the theorem, we are reduced to checking that there is a linear form $v = \Sigma\, \lambda_j s_j$ vanishing on X_2 where the λ_j satisfy $(**)$. Equivalently, we must produce a hyperplane H containing X_2 and any p points of $X_1 - \{x_i\}$, such that $x_i \notin H$. But $X_2 \cup \{p \text{ points of } X_1\}$ consists of r points, which span a unique hyperplane H. By general position H does not contain any further points of X, and we are done. ■

REMARK. Suppose that $X \subseteq \mathbf{P}^r$ consists of $2r + 1 - p$ or fewer points, distinct but not necessarily in linear general position. Then (N_p) may fail for X, but we propose the following.

Conjecture. If $X \subseteq \mathbf{P}^r$ fails to satisfy (N_p), then there is an integer $s \leqslant r$, and a subset $Y \subseteq X$ consisting of at least $2s + 2 - p$ points, such that Y is contained in a linear subspace $\mathbf{P}^s \subseteq \mathbf{P}^r$ in which (N_p) fails for Y.

For instance, the conjecture predicts that the homogeneous ideal of six points in $\mathbf{P}^3$ is generated by quadrics unless five lie in a plane or three on a line. One may formulate an analogous statement concerning the failure of $2r + 1$ points in $\mathbf{P}^r$ to impose independent conditions on quadrics, and this has been verified by the authors.

§3 Syzygies of algebraic curves

Let X be a smooth irreducible projective curve of genus g, and let L be a line bundle of degree $d \geqslant 2g + 1$ on X. Then L is non-special and very ample, and hence defines an embedding.

$$\varphi_L : X \hookrightarrow \mathbf{P}(H^0(X, L)) = \mathbf{P}^{d-g}$$

Furthermore, by a theorem of Castelnuovo, Mattuck and Mumford (cf. [GL1]), L is normally generated, i.e., φ_L embeds X as a projectively Cohen–Macaulay variety. We denote by ω_X the canonical bundle on X. Finally, we shall have occasion to draw on the following standard general position statement:

LEMMA 3.1. *With $X \subseteq \mathbf{P}(H^0(X, L)) = \mathbf{P}^{d-g}$ as above, fix $s \leqslant d - g - 1$ points $x_1, \ldots, x_s \in X$, and set $D = \Sigma x_i$. If the x_i are chosen sufficiently generally, then $L(-D)$ is non-special and generated by its global sections.*

In other words, D spans an $(s-1)$-plane $\Lambda_D \subseteq \mathbf{P}^{d-g}$ such that $X \cap \Lambda_D = D$ (scheme-theoretically). One may prove (3.1) e.g., by a simple dimension count, which we leave to the reader.

To set the stage, we start by showing – in the spirit of [St.D] – that Theorem 2.1 leads to a quick new proof of the result of the first author from [G]:

PROPOSITION 3.2. ([G, (4.a.1)]). *Suppose that* $\deg L = 2g + 1 + p$ $(p \geqslant 1)$. *Then X satisfies property (N_p) for the embedding $X \subseteq \mathbf{P}^{g+1+p}$ defined by L.*

Proof: Let $Y \subseteq \mathbf{P}^{g+p}$ be a general hyperplane section of X. Since $X \subseteq \mathbf{P}^{g+1+p}$ is projectively Cohen–Macaulay, a minimal free resolution of $I_{X/\mathbf{P}^{g+1+p}}$ restricts to one for $I_{Y/\mathbf{P}^{g+p}}$, and in particular $X \subseteq \mathbf{P}^{g+1+p}$ satisfies (N_p) if and only if $Y \subseteq \mathbf{P}^{g+p}$ does. But Y consists of $2(g + p) + 1 - p$ points in linear general position (cf. [L]), and so satisfies (N_p) thanks to (2.1). ∎

The main result of this section is the classification of all pairs (X, L) for which (3.2) is optimal:

THEOREM 3.3. *Suppose that* $\deg L = 2g + p$ $(p \geqslant 1)$, *and consider the resulting embedding*

$$X \subseteq \mathbf{P}(H^0(X, L)) = \mathbf{P}^{g+p}.$$

Then property (N_p) fails for X if and only if either:

(i) *X is hyperelliptic*;

or

(ii) *φ_L embeds X with a $(p + 2)$-secant p-plane, i.e., $H^0(X, L \otimes \omega_X^*) \neq 0$.*

REMARK. Concerning the statements in (ii), note that an effective divisor $D \subseteq X$ of degree $p + 2$ spans a p-plane in $\mathbf{P}^{g+p}$ if and only if

$$h^1(X, L(-D)) = h^0(X, \omega_X \otimes L^*(D)) = 1.$$

Since $\deg(\omega_X \otimes L^*(D)) = 0$, this is in turn equivalent to requiring that D be the divisor of a non-zero section of $L \otimes \omega_X^*$.

Proof of Theorem 3.3. We assume first that X is non-hyperelliptic and that (N_p) fails for X, and we show that $H^0(X, L \otimes \omega_X^*) \neq 0$. To this end, note to begin with that since L is normally generated and non-special, we are in the situation of Lemma 1.10. Hence:

$$H^1(X, \Lambda^{p+1} M_L \otimes L) \neq 0,$$

M_L being the vector bundle defined by (1.8). But M_L has rank $g + p$ and determinant L^*, and therefore $\Lambda^{p+1} M_L \otimes L = \Lambda^{g-1} M_L^*$. Thus

$$H^0(X, \Lambda^{g-1} M_L \otimes \omega_X) \neq 0 \tag{3.4}$$

by Serre duality.

Choose now $g - 2$ points $x_1, \ldots, x_{g-2} \in X$, and put $D = \Sigma x_i$. We assume that the x_i are chosen sufficiently generally so that the conclusion of Lemma 3.1 holds. Then as in (1.8) the vector bundle $M_{L(-D)}$ is defined, and one has an exact sequence.

$$0 \to M_{L(-D)} \to M_L \to \Sigma_D \to 0, \tag{3.5}$$

where $\Sigma_D = \oplus_{i=1}^{g-2} \mathcal{O}_X(-x_i)$ (cf. [GL2, §2]). Observing that $rk(\Sigma_D) = g - 2$ and $\det \Sigma_D = \mathcal{O}_X(-D)$, (3.5) gives rise to a surjective map

$$u_D\colon \Lambda^{g-1} M_L \to M_{L(-D)} \otimes \mathcal{O}_X(-D) \tag{3.6}$$

of vector bundles on X. Twisting by ω_X and taking global sections, one obtains a homomorphism

$$H^0(X, \Lambda^{g-1} M_L \otimes \omega_X) \to H^0(X, M_{L(-D)} \otimes \omega_X(-D)). \tag{3.7}$$

Grant for the time being the following claim:

If $H^0(X, \Lambda^{g-1} M_L \otimes \omega_X) \neq 0$, then for a sufficiently general choice of D the homomorphism (3.7) is non-zero. (3.8)

Then $H^0(X, M_{L(-D)} \otimes \omega_X(-D)) \neq 0$ by (3.4). On the other hand, it follows from the analogue (1.8) for $M_{L(-D)}$ that $H^0(X, M_{L(-D)} \otimes \omega_X(-D))$ is isomorphic to the kernel of the natural map

$$H^0(X, L(-D)) \otimes H^0(X, \omega_X(-D)) \to H^0(X, L \otimes \omega_X(-2D)). \tag{3.9}$$

But since X is non-hyperelliptic, for general D the line bundle $\omega_X(-D) = \omega_X(-x_1 - \cdots - x_{g-2})$ is generated by its global sections, with $h^0(X, \omega_X(-D)) = 2$. Hence by the "base-point free pencil trick" (cf. [ACGH, p. 126]), the kernel of (3.9) is also identified with

$$H^0(X, L(-D) \otimes \omega_X^*(D)) = H^0(X, L \otimes \omega_X^*).$$

Thus $H^0(X, L \otimes \omega_X^*) \neq 0$, as required.

We next turn to the proof of (3.8). Referring to (3.6), it is enough to show that

$$\bigcap_D \ker u_D = 0, \tag{$*$}$$

where the intersection is taken over all effective divisors of degree $g - 2$ for which the conclusion of Lemma 3.1 is valid. For any such divisor D, set $W_D = H^0(X, L)/H^0(X, L(-D))$. Then dim $W_D = g - 2$, and one has a canonical surjective map

$$v_D\colon \Lambda^{g-1}H^0(X, L) \to H^0(X, L(-D)) \otimes \Lambda^{g-2} W_D$$

which fits into an exact commutative diagram

$$\begin{array}{ccccc} 0 \to & \Lambda^{g-1}M_L & \to & \Lambda^{g-1}H^0(X, L) \otimes_k \mathcal{O}_X \\ & \downarrow{\scriptstyle u_D} & & \downarrow{\scriptstyle v_D \otimes 1} \\ 0 \to & M_{L(-D)} \otimes \mathcal{O}(-D) & \to & H^0(X, L(-D)) \otimes \Lambda^{g-2} W_D \otimes_k \mathcal{O}_X \end{array}$$

of bundles on X (cf. [GL2, §2]). In particular ker $u_D \subseteq \ker(v_D \otimes 1)$, so for $(*)$ it suffices to verify

$$\bigcap_D \ker v_D = 0, \qquad (**)$$

where as above the intersection is taken over divisors satisfying (3.1). To this end, fix $g + p + 1$ points $x_1, \ldots, x_{g+p+1} \in X \subseteq \mathbf{P}^{g+p}$, spanning $\mathbf{P}^{g+p}$. By choosing the x_i sufficiently generally, we may assume that for every multi-index $I = \{i_1 < \cdots < i_{g-2}\} \subseteq [1, g + p + 1]$, the divisor

$$D_I =_{\text{def}} x_{i_1} + \cdots + x_{i_{g-2}}$$

satisfies (3.1). But if one then chooses a basis of $H^0(X, L)$ dual to the x_i, one checks immediately that in fact

$$\bigcap_{\# I = g-2} \ker v_{D_I} = 0.$$

This proves $(**)$ and hence also (3.8).

To complete the proof, it remains only to show that property (N_p) actually fails for X if either X is hyperelliptic or if $H^0(X, L \otimes \omega_X^*) \neq 0$. Suppose first that $D \subseteq X$ is a divisor of degree $p + 2$ spanning a p-plane in $\mathbf{P}^{g+p}$. Then as in [GL2, §2], one has an exact sequence $0 \to M_{L(-D)} \to M_L \to \Sigma_D \to 0$ where $rk\, \Sigma_D = p + 1$ and $\det \Sigma_D = \mathcal{O}_X(D) = L^* \otimes \omega_X$. This gives rise to a surjective map $\Lambda^{p+1} M_L \otimes L \to \omega_X$, and hence $H^1(X, \Lambda^{p+1} M_L \otimes L) \neq 0$ by duality. Thus (N_p) fails for X by (1.10).

Finally, suppose X is hyperelliptic. The lines $\overline{P_1P_2}$ spanned by points P_1, P_2 where $P_1 + P_2$ belongs to the g^1_2 sweep out a rational surface scroll Y. One notes (see [G]), letting R_X and R_Y denote $S/I_{X/\mathbf{P}^r}$ and $S/I_{Y/\mathbf{P}^r}$ respectively, that there is a natural injection

$$\mathrm{Tor}^S_{p+1}(R_Y, \boldsymbol{k})_{p+2} \hookrightarrow \mathrm{Tor}^S_{p+1}(R_X, \boldsymbol{k})_{p+2}.$$

By the standard determinantal representation of Y, we have

$$\dim \mathrm{Tor}^S_{p+1}(R_Y, \boldsymbol{k})_{p+2} = \binom{r-1}{p+2}(p+1)$$

On the other hand, the Koszul complex

$$0 \to \Lambda^{p+1} H^0(X, L) \to \Lambda^p H^0(X, L) \otimes H^0(X, L)$$
$$\to \Lambda^{p-1} H^0(X, L) \otimes H^0(X, L^2) \to \cdots$$

has cohomology at the second term isomorphic to $\mathrm{Tor}^S_{p+1}(R_X, \boldsymbol{k})_{p+2}$ and is exact elsewhee if property N_p holds for X. Thus the alternating sum of the dimensions of this complex is $-\dim \mathrm{Tor}^S_{p+1}(R_x, \boldsymbol{k})_{p+2}$. After considerable computation, if $\deg L = 2g + p$, one obtains

$$\dim \mathrm{Tor}^S_{p+1}(R_X, \boldsymbol{k})_{p+2} = \binom{r-1}{p+2}(p-1) - \binom{r-1}{p}.$$

This contradicts the inequality

$$\dim \mathrm{Tor}^S_{p+1}(R_X, \boldsymbol{k})_{p+2} \geqslant \dim \mathrm{Tor}^S_{p+1}(R_Y, \boldsymbol{k})_{p+2}$$

so that property N_p cannot hold. ■

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