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Singularities and Kodaira dimension of the moduli space of flat Hermitian–Yang–Mills connections

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0. Introduction

The study of the moduli space of Hermitian–Yang–Mills connections over a compact Kähler manifold M (or equivalently of stable holomorphic vector bundles over M) has drawn increasing attention in recent times. Such a moduli space is known to carry the natural structure of a Hausdorff complex analytic space [L-O, Nor], or of a quasiprojective algebraic scheme when M is algebraic [Gie, Ma1]. However, very little is known about the finer structure of this moduli space except in a few special cases (e.g., when M is an algebraic curve or projective space).

In this paper we study the moduli space of *flat* – or more generally *central* – Hermitian–Yang–Mills connections over a compact Kähler manifold. Equivalently, we study the moduli space of stable holomorphic vector bundles E with $c_2(\mathscr{E}nd\,E) = 0$. We show that the singularities of the moduli space are all of a particularly simple type: quadratic algebraic. And when M is algebraic we show that the Kodaira dimension of the moduli space cannot be maximal. We show also that the Kodaira–Spencer deformation theory for such E is extremely simple: all higher order obstructions to deformation vanish.

Our key observation, from which all other results essentially follow, is that if two (0,1)-forms with values in $\mathscr{E}nd\,E$ are harmonic (relative to the Hermitian–Yang–Mills metric) then so is their product.

This paper is organized as follows. In Section 1 we study the deformation theory of stable holomorphic vector bundles E with $c_2(\mathscr{E}nd\,E) = 0$ and in Section 2 we apply the results from Section 1 to the study of the moduli space.

1. Construction of entire holomorphic deformations

Let (M, ω) be a compact Kähler manifold of complex dimension m and let E be an ω-stable holomorphic vector bundle of rank r over M. Uhlenbeck and Yau have shown that E admits a Hermitian–Yang–Mills metric and that the following Chern number inequality holds:

$$2r \int_M c_2(E) \wedge \omega^{m-2} \geqslant (r-1) \int_M c_1(E)^2 \wedge \omega^{m-2}$$

or, equivalently,

$$\int_M c_2(\mathscr{E}nd E) \wedge \omega^{m-2} \geqslant 0.$$

We shall consider bundles for which this inequality is actually equality. Namely, we shall assume that $c_2(\mathscr{E}nd E) = 0$. This condition implies that the Hermitian–Yang–Mills connection on E has central curvature; it then follows easily that E is stable with respect to any other Kähler metric ω' on M. Examples of such stable bundles E are as follows: (1) any stable bundle over a compact Riemann surface, and (2) any bundle associated to an irreducible unitary representation of $\pi_1(M)$. Nontrivial examples occur only when $\pi_1(M)$ is nontrivial [U-Y].

In this section we shall show that the deformation theory of E is extremely simple, and in particular that all higher order obstructions to deformation vanish.

We define an algebraic subscheme T (possible non-reduced) of the complex vector space $H^1(M, \mathscr{E}nd E)$ as follows:

$$T \stackrel{\text{def}}{=} \{t \in H^1(M, \mathscr{E}nd E) | t \wedge t = 0\}.$$

We call T the *scheme of unobstructed infinitesimal deformations*. We note that T is defined by quadratic homogenous equations and that consequently the inclusion $T \hookrightarrow H^1(M, \mathscr{E}nd E)$ induces an isomorphism of Zariski tangent spaces at the origin. We sometimes denote also by T the complex space which underlies the scheme T.

THEOREM 1.1 (Main Theorem). *There exists a holomorphic family $\{E_t\}$ of the holomorphic vector bundles over M parameterized by T such that*

- $E_0 = E$
- *this family is universal at $t = 0$ and is the Kuranishi family for E. (c.f. [Sun, Kur, F-K]).*

We call the family $\{E_t\}$ in the theorem an *entire holomorphic* deformation of E because the family is parameterised by an affine algebraic scheme and not merely the germ of a complex space.

Conjugate description. We now present a useful conjugate description of the vector spaces $H^i(M, \mathscr{E}nd\,E)$ and of T.

Let h denote the Hermitian–Yang–Mills metric on the bundle $\mathscr{E}nd\,E$ (which is induced from the Hermitian–Yang–Mills metric on E). Then h is a flat metric because the Chern equations $c_2(\mathscr{E}nd\,E) = 0 = c_1(\mathscr{E}nd\,E)$ force the curvature to vanish (c.f. [U-Y]). Consequently the following equality of Laplacians holds: $\Box = \overline{\Box}$ (c.f. [Siu]). Here $\Box$ (resp. $\overline{\Box}$) denotes the $\bar{\partial}$-Laplacian (resp. ∂-Laplacian) on $\mathscr{E}nd\,E$ relative to the flat Hermitian–Yang–Mills metric on $\mathscr{E}nd\,E$ and the Kähler metric on M. We immediately infer the following:

(*) *The product of two harmonic* (0, 1)*-forms with values in* $\mathscr{E}nd\,E$ *is a harmonic* (0, 2)*-form with values in* $\mathscr{E}nd\,E$.

We infer also that the complex conjugate of a harmonic (p, q)-form with values in $\mathscr{E}nd\,E$ is again harmonic. (Note: Complex conjugation of $\mathscr{E}nd\,E$ is defined by taking the Hermitian-adjoint of an endomorphism.) Hence

$$H^0(M, \Omega^i_M \otimes \mathscr{E}nd\,E) \cong \overline{H^i(M, \mathscr{E}nd\,E)}.$$

The conjugate description of T is given by

$$\bar{T} = \{s \in H^0(M, \Omega^1_M \otimes \mathscr{E}nd\,E) | s \wedge s = 0\}$$

Proof of theorem. For each $s \in H^0(M, \Omega^1_M \otimes \mathscr{E}nd\,E)$ we define the Cauchy–Riemann operator $\bar{\partial}_s = \bar{\partial} + \bar{s}$ on the differentiable vector bundle E_{C^∞} which underlies E. Then $\partial s = 0$ since $\Box = \overline{\Box}$ and it follows that $\bar{\partial}_s^2 = \bar{s} \wedge \bar{s}$. Therefore, $\bar{\partial}_s$ is integrable iff $s \wedge s = 0$. Thus we have a family of integrable Cauchy–Riemann operators on E_{C^∞} depending holomorphically on $T(\mathbf{C})$, the set of complex points of the scheme T.

We now recall the Kuranishi's construction:

Let $s_1, s_2, \ldots, s_k$ be a basis for $H^0(M, \Omega^1_M \otimes \mathscr{E}nd\,E)$. Then, in our case, $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k$ is a basis for the space of harmonic (0, 1)-forms with values in $\mathscr{E}nd\,E$.

The Kuranishi map ϕ is by definition the holomorphic map

$$\phi: \mathbf{C}^k \to \{\text{smooth (0, 1)-forms with values in } \mathscr{E}nd\,E\}$$

which satisfies the equation

$$\phi(t) = \sum_{i=1}^{k} t_i \bar{s}_i - \bar{\partial}^* G(\phi(t) \wedge \phi(t)) \tag{1}$$

for $t = (t_1, \ldots, t_k) \in \mathbf{C}^k$ where $\mathbf{G}$ is the Green's operator (relative to the Hermitian–Yang–Mills metric on E and the Kähler metric on M) acting on (0, 2)-forms with values in $\mathscr{E}nd\,E$. (In the general case, ϕ is defined only on a neighborhood of the origin in $\mathbf{C}^k$).

In the case at hand, we see that the Kuranishi map is given simply by

$$\phi(t) = \sum_{i=1}^{k} t_i \bar{s}_i \tag{2}$$

since for this ϕ, $G(\phi(t) \wedge \phi(t)) = 0$ by (*) and hence (1) is satisfied.

Now the Kuranishi family of E is parameterized by the analytic subspace of $\mathbf{C}^k$ given by

$$\mathrm{pr}(\phi(t) \wedge \phi(t)) = 0 \tag{3}$$

where pr denotes orthogonal projection onto the harmonic forms.

Since in our case $\phi(t) \wedge \phi(t)$ is harmonic for all t by (*), (3) reduces to $\phi(t) \wedge \phi(t) = 0$, or equivalently

$$\sum_{1 \leqslant i,j \leqslant k} t_i t_j \bar{s}_i \wedge \bar{s}_j = 0$$

which is just a coordinatized description of our space T, as desired.

Finally, our versal Kuranishi deformation is universal since E is simple.

Q.E.D.

Order of growth. We now show that our family $\{E_t\}$ constructed above has a very low order of transcendental growth. For simplicity we assume that E is unobstructed: $H^2(M, \mathscr{S}\ell E) = 0$. Then the total bundle $\mathbf{E}$ over $M \times \overline{H^0(M, \Omega^1_M \otimes \mathscr{E}nd\,E)}$ is given as follows: differentiably, $\mathbf{E}$ is the pull-back of E via the projection onto M. And the Cauchy–Riemann operator is given by $\bar{\partial} + \bar{s}$ where $\bar{\partial}$ is the "background" Cauchy–Riemann operator obtained by pulling back the holomorphic structure on E. Now let $\mathbf{P}^r$ be a complex projective space that contains $\overline{H^0(M, \Omega^1_M \otimes \mathscr{E}nd\,E)}$ as the complement of a hyperplane. Then the holomorphic vector bundle $\mathbf{E}$ need not extend to a holomorphic vector bundle over $M \times \mathbf{P}^r$ (although it extends differentiably) because the Cauchy–Riemann operator has a simple pole along the divisor at infinity (relative to the trivial differentiable extension). In other words, $\mathbf{E}$ has finite order in the sense of Cornalba–Griffiths [C–G, Definition V, p. 21].

EXAMPLE: In the case where E is the trivial line bundle, our family $\{E_t\}$ is just the pullback to the universal cover of the Picard torus (which is an abelian variety in case M is projective algebraic) of the universal line bundle of degree zero. Thus our family will not, in general, be algebraic.

REMARK: In view of the correspondence between flat unitary bundles and unitary representations of $\pi_1(M)$, one might expect that the Kodaira-Spencer theory for deformations of unitary representations should be simple also. This is indeed the case, for it follows essentially from the ideas presented in this section that *all higher order obstructions to deformation vanish for a unitary representation of the fundamental group of a compact Kähler manifold.*

2. Applications to the structure of the moduli space

As in the previous section, we let (M, ω) be a compact Kähler manifold and E a stable holomorphic vector bundle with $c_2(\mathscr{E}nd E) = 0$. Let $\mathscr{M}(M, E)$ denote the moduli space of all stable holomorphic vector bundles over M which are diffeomorphic (as bundles) to E (c.f. [L–O] or [Nor] for the construction). $\mathscr{M}(M, E)$ has the natural structure of a Hausdorff (possibly non-reduced) complex analytic space, uniquely determined by the requirement that there exist locally with respect to $\mathscr{M}(M, E)$ a universal bundle.

DEFINITION 2.1. A complex affine scheme which is defined by finitely many quadratic homogenous polynomials is said to have quadratic algebraic singularities. A complex space is said to have quadratic algebraic singularities if it is locally isomorphic to an affine scheme of the type described above.

THEOREM 2.1 (Singularities of the moduli space). *The moduli space $\mathscr{M}(M, E)$ has quadratic algebraic singularities.*

Proof. Let E' be any stable bundle which is diffeomorphic to E. Then the germ of the moduli space $\mathscr{M}(M, E)$ at the point $[E']$ is known to be isomorphic to the (germ at the origin of the) Kuranishi family of E' (c.f. [L–O, Nor]), which by the previous section has quadratic algebraic singularities. Q.E.D.

We now specialize to the case in which M is projective algebraic. In this situation Gieseker [Gie] has defined an alternative notion of stability (with respect to an ample line bundle) which we will call *Gieseker-stability*. Gieseker-stability is defined for arbitrary torsion-free coherent analytic

(= algebraic) sheaves, not just locally free ones. There is also a notion of *Gieseker-semistability*. For our purposes it will suffice to note the following implications: (1) stable ⇒ Gieseker-stable and (2) Gieseker-semistable ⇒ semistable. Maruyama [Ma1] has shown that the set of all isomorphism classes of Gieseker-semistable torsion-free coherent sheaves over M with fixed Chern class form a coarse moduli space which has the natural structure of a projective algebraic scheme.

Let E be a stable holomorphic vector bundle over M which represents a smooth point in the (possibly reducible) projective variety obtained by reducing Maruyama's moduli scheme of all Gieseker-semistable torsion-free coherent sheaves having the same Chern class as E.

Let $\mathcal{N}(M, E)_{\mathrm{red}}$ denote the irreducible component of this variety which contains E.

THEOREM 2.2 (Kodaira dimension of the moduli space). *The projective variety $\mathcal{N}(M, E)_{\mathrm{red}}$ is not of general type. Moreover, if $\mathcal{N}(M, E)_{\mathrm{red}} \dashrightarrow Y$ is any dominating rational map to a projective variety Y then Y cannot be of general type.*

Proof. Let $\{E_t\}$ be the entire holomorphic deformation of E that was constructed in the previous section. The reduction T_{red} of the parameter space T of our family is complex affine space $\mathbf{C}^r$ since $[E]$ represents a smooth point in $\mathcal{N}(M, E)_{\mathrm{red}}$. Thus by restricting this family we obtain a new family, also denoted by $\{E_t\}$, parameterised by $\mathbf{C}^r$. Not all of the bundles E_t are Gieseker-semistable, so we do not necessarily obtain a *holomorphic* map from complex affine space into the moduli space. We do, however, obtain a *meromorphic* map from complex affine space into the moduli space and hence into $\mathcal{N}(M, E)_{\mathrm{red}}$. This is because our family is locally algebraic in the sense of Lemma 2.1 below, and because Gieseker-semistability is a Zariski-open condition for algebraic families [Ma2]. Also, our meromorphic map $\mathbf{C}^r \dashrightarrow \mathcal{N}(M, E)_{\mathrm{red}}$ is equidimensional and nondegenerate. The theorem now follows from Lemma 2.2 below. Q.E.D.

LEMMA 2.1. *Every holomorphic family of vector bundles is locally equal to the pullback via a holomorphic map of an algebraic family.*

Proof. (compare [N–S, proof of Theorem 3, p. 565]) By tensoring each bundle E_t in our family by a sufficiently ample line bundle, we can assume that our family is locally (with respect to t) the quotient of a fixed trivial vector bundle $\mathcal{O}_M^{\oplus N}$. By shrinking our parameter space T if necessary, we may assume that our family is for all t such a quotient. Then we obtain a holomorphic map into Grothendieck's Quot scheme such that our family of quotients is the pullback of the universal one which is parameterised by Quot. Since the Quot scheme and the universal quotient are algebraically defined, the lemma follows. Q.E.D.

LEMMA 2.2. *Let* $\mathbf{C}^r \dashrightarrow X$ *be a nondegenerate equidimensional meromorphic map from complex affine space to a complex projective variety X. If* $X \dashrightarrow Y$ *is a dominating rational map to a complex projective variety Y then Y cannot be of general type.*

Proof. First consider the case where $Y = X$. By results of [Gri, Kie] it follows that X cannot be of general type. Next, given Y as in the statement of the lemma, we can replace our complex affine space by a linear subspace and reduce to the first case. Q.E.D.

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