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The divisor of curves with a vanishing theta-null

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§0. Introduction

Denote by $\mathscr{M}_g$ the moduli space of curves of genus g, by $\bar{\mathscr{M}}_g$ its natural compactification by stable curves. We recall that $\bar{\mathscr{M}}_g$–$\mathscr{M}_g$ is the union of $[g/2] + 1$ divisors $\Delta_0, \Delta_1 \ldots \Delta_{[g/2]}$. A general point of Δ_0 is a curve obtained by identifying two points in a curve of genus $g - 1$. A generic point Δ_i is obtained by identifying a point in a curve of genus i with a point in a curve of genus $g - i$.

A theta-characteristic on a curve is a line bundle such that its square is the dualizing sheaf. A non-singular curve has 2^{2g} theta-characteristics corresponding to the points of order two in its jacobian variety. A theta-characteristic is said to be even or odd depending on whether the vector space of its sections is even or odd. The parity of a theta-characteristic is constant in any family ([M], [H]). Any non-singular curve has $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta characteristics (see for instance [R, F] p. 4 and 176–177). The latter have necessarily a section. But a generic curve has no theta-characteristics with 2 or more independent sections (as these line bundles do not satisfy Gieseker–Petri Theorem). It follows that, for $g \geqslant 3$, the locus of curves which have an even theta characteristic with space of sections of (projective) dimension at least one is a divisor in $\mathscr{M}_g$ (see [Fa], [H], [T]). Denote it by $\mathscr{M}_g^1$ and let $\bar{\mathscr{M}}_g^1$ be its closure in $\bar{\mathscr{M}}_g$. The purpose of this work is to prove that $\mathscr{M}_g^1$ is irreducible.

We note that any divisor in $\bar{\mathscr{M}}_g$ intersects Δ_1. The first step is to determine the intersection of $\bar{\mathscr{M}}_g^1$ and Δ_1. This is the union of 2 pieces (see (1.2)). One of them is the locus of curves obtained by identifying a generic point in a curve of $\mathscr{M}_{g-1}^1$ with a point in a generic elliptic curve. By induction hypothesis, this locus is irreducible. Moreover, the intersection of this piece with any other component of $\bar{\mathscr{M}}_g^1 \cap \Delta_1$ contains certain degenerate hyperelliptic

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curves. To finish the proof, we check then that the monodromy interchanges the even dimensional effective theta-characteristics on these curves.

In the last paragraph of the this work, we compute the class of the divisor $\bar{\mathcal{M}}_g^1$ in $\mathcal{M}_g$.

Most notations will be standard. A curve will always be semistable, g will denote its arithmetic genus. A (non necessarily invertible) torsion-free rank-one sheaf on a (singular) curve which is the limit of theta-characteristics on nearby curves will be called a limit theta characteristic. The dimension of a (limit) theta-characteristic is the projective dimension of the space of sections of the line bundle or torsion-free rank-one sheaf.

We shall assume that the reader is well acquainted with the theory of Eisenbud and Harris of limit linear series on curves of compact type ([E, H]1) and that of admissible coverings of Harris and Mumford [H, M]).

§1. Determination of the intersection of $\mathcal{M}_g^1$ with the boundary components of $\bar{\mathcal{M}}_g$

(1.1). Remark. Every component of the intersection of $\bar{\mathcal{M}}_g^1$ with a Δ_i has codimension one in Δ_i. This follows from the fact that $\mathcal{M}_g^1$ has pure codimension one in $\mathcal{M}_g$ and that $\bar{\mathcal{M}}_g$ is a normal variety. We shall also prove that a generic curve in Δ_i does not have a one dimensional (limit) theta-characteristic. By a result of Harris ([H], Th. 1.10), every one dimensional (limit) theta - characteristic extends to a one-dimensional (limit) theta-characteristic on a locus of codimension at most one in every family of curves containing the given curve. It follows that any curve with a (limit) even theta-characteristic of dimension at least one lies in $\bar{\mathcal{M}}_g^1$.

Hence, in order to determine the intersection of $\bar{\mathcal{M}}_g^1$ and Δ_i, one may forget about those configurations of C which move in a locus of codimension 2 or more in Δ_i. Moreover, for loci of codimension one, it is only necessary to determine if the generic curve in the locus lies in $\bar{\mathcal{M}}_g^1$.

(1.2). Proposition. *For $i \geqslant 1$ the intersection of $\bar{\mathcal{M}}_g^1$ with Δ_i consists of 4 pieces. The generic curve C in each piece is obtained from a curve C_i of genus i and a curve C_{g-i} of genus $g - i$ by identifying a point of each to a single point, say P. These data satisfy one of the following conditions (for $j = i$ and $g - i$):*

α_j) C_j has a one dimensional even theta characteristic L_j. In this case the one dimensional even theta characteristics on C are determined by their aspects (see [E, H]1) *$|L_j| + (g-j)P$ on C_j and $|L_{g-j} + 2P| + (j-2)P$ (where L_{g-j} is any even theta-characteristic) on C_{g-j}.*

β_j) The point P is in the support of an effective theta-characteristic L_j on C_j. Then, the aspects of the theta-characteristics on C are $|L_j + P| + (g - j - 1)P$ and $|L_{g-j} + 2P| + (j - 2)P$ where L_{g-j} is any odd theta characteristic on C_{g-j}.
Note: α_1, α_2, β_1 are empty.

(1.3). REMARK. The rational map from any component of β_j to $\mathcal{M}_j$ is generically surjective. This follows from the fact that the locus in $\mathcal{M}_g$ of curves with an odd theta-characteristic of dimension greater than 0 has codimension 3 in $\mathcal{M}_g$ (cf. [T]).

Proof of (*1.2*): By the theory of limit linear series ([E, H]1), the curve C will be in $\bar{\mathcal{M}}_g^1$ if and only if there is a limit linear series of degree $g - 1$ and dimension one on C such that the corresponding line bundles on the components C_i, C_{g-i} of C differ from an even theta-characteristic on C only up to tensoring with a divisor supported on the intersection point P. Write (a_j, b_j) for the vanishing orders at P of the aspect of the limit linear series on C_j. Then $a_j + b_{g-j} \geqslant g - 1$. One has $2b_jP + 2D_j \in |K_{C_j}(2(g - j)P|$ for an effective D_j and $h^0(D_j + (b_j - a_j)P) \geqslant 2$. Hence, $2D_j \in |K_{C_j}(2c_jP)|$ for $c_j = g - j - b_j$. Therefore,

$$c_i - c_{g-i} = g - i - b_{g-i} + i - b_i \leqslant g - b_i - a_{g-i} - 1 \leqslant 0.$$

Moreover, if equality between the right hand side and the left hand side holds, $b_j = a_j + 1$ for $j = 1$ for $j = i$ and $g - i$.

Assume $c_j < 0$, then $2(D + P) \in |K_{C_j}|$ for an effective D. So, P is in the support of an effective theta-characteristic on C_j. If C_j does not have a one-dimensional theta-characteristic, then P is in the support of one of the effective odd theta characteristics on C_j. Hence, β_j is satisfied.

The locus of curves with a theta-characteristic of dimension at least 2 has codimension 3 in $\mathcal{M}_p$ for every p (cf. [T]) and we are interested in loci of codimension one in Δ_i. Hence, if P is generic in C_j, C_j is in $\mathcal{M}_j^1$ as asserted in α_j.

Moreover, as the parity of a theta characteristic on a reducible curve is the product of the parities of the theta-characteristics on its components ([F], p. 14), it follows that the theta-characteristics on C are of the form stated in (1.2).

Assume next $c_i = c_{g-i} = 0$. Then, D_j are effective theta-characteristics and $h^0(D_j + P) \geqq 2$. Hence, by Riemann–Roch, $h^0(D_j - P) \geqq 1$ and this implies again that C_j satisfies one of the conditions of (1.2).

Using the same approach, one finds that a generic curve with three components is not in $\bar{\mathcal{M}}_g^1$. The same will follow for a reducible curve with two components one of which is singular, once we know that, for $p \leqslant g - 1$, the divisor Δ_0 of the moduli space of curves of genus p is not contained in $\mathcal{M}_p^1$. Assume this were not the case. Then, in particular, a curve obtained by identifying a generic point of a generic curve of genus $p - 1$ with a point of a rational curve with a node would be in $\bar{\mathcal{M}}_p^1$. But the analysis we have just done shows that this is impossible.

§2. Irreducibility of $\bar{\mathcal{M}}_g^1$

We are going to prove the irreducibility of $\bar{\mathcal{M}}_g^1$ by induction on g, the result being easy for $g \leqslant 5$.

(2.1). REMARK. In a neighborhood of a given point, the irreducibility of $\bar{\mathcal{M}}_g^1$ is equivalent to the irreducibility of the scheme which parametrizes curves and even theta-characteristics of dimension at least one (see [T]). This follows from the fact that this scheme has dimension $3g - 4$ at all of its points ([H], Th.1.10), all fibres of the map to $\bar{\mathcal{M}}_g^1$ are finite and a generic point in any component of $\bar{\mathcal{M}}_g^1$ has only one theta characteristic of dimension one. ([T], Th.2.16).

(2.2). REMARK. On an hyperelliptic non-singular curve of genus g with Weierstrass points $R_1, R_2, \ldots R_{2g+2}$ the theta-characteristics have the form

$$rg_2^1 + \sum_{j=1}^{g-1-2r} R_{k_j}, \quad -1 \leqslant r \leqslant [g/2] \quad \text{and} \quad k_j < k_{j+1}.$$

The dimension of this theta-characteristic is r. This expression is unique if $r \geqslant 0$, while in case $r = -1$ the two expressions obtained by taking complementary sets of indices for the Weierstrass points give rise to the same line bundle (use Riemann–Hurwitz Formula for the double covering of $\mathbb{P}^1$).

(2.3). REMARK. Every divisor in $\bar{\mathcal{M}}_g$ intersects Δ_1.

Proof: It is enough to show that the generators of the Picard group of $\bar{\mathcal{M}}_g$ map to independent elements in Δ_1.

Denote by $\bar{\mathcal{M}}_k^1$ the moduli space of 1-pointed stable curves of genus k. From [A, C] Prop. 1, the Picard group of the moduli functor for $\bar{\mathcal{M}}_k^1$ ($k \geqslant 3$) has as basis the classes λ', ψ', δ_0', δ_1', $\ldots$, δ_g'. Here $\lambda' = \Lambda^g \pi_*(\omega_\pi)$, $\psi' = \sigma^*(\omega_\pi)$ where $\pi : X \to S$, $\sigma : S \to X$ is a family of 1-pointed stable curves. Also, δ_0' represents the boundary class of curves with a non disconnecting node and

δ'_i the class of curves with a node which disconnects them in a piece of genus i containing the section and a piece of genus $g - i$.

There is a clutching morphism from $\bar{\mathcal{M}}^1_{g-1} \times \bar{\mathcal{M}}^1_1$ onto Δ_1. Consider the composite map h

$$\bar{\mathcal{M}}^1_{g-1} \times \bar{\mathcal{M}}^1_1 \longrightarrow \Delta_1 \hookrightarrow \bar{\mathcal{M}}_g.$$

We consider the pull-back by h of the classes λ, δ_0, δ_1, δ_2, . . . , $\delta_{g/2}$ (or $\delta_{(g-1)/2}$). These may be expressed as tensor products of the classes in the Picard functors of $\bar{\mathcal{M}}^1_{g-1}$ and $\bar{\mathcal{M}}^1_1$ with non-zero coefficients in λ', δ'_0, ψ' and $\delta' g - 2$, δ'_1 and δ'_{g-3}, . . . , $\delta'_{g/2-1}$ (or $\delta'_{(g-3)/2}$ and $\delta'_{(g-1)/2}$) where all classes refer to classes in $\bar{\mathcal{M}}^1_{g-1}$ (cf. [A, C] Lemma 1 and [K] Th.4.2.).

(2.4). THEOREM: *The divisor $\mathcal{M}^1_g$ is irreducible.*

Proof: From (1.2), the intersection of $\bar{\mathcal{M}}^1_g$ and Δ_1 consists of 2 pieces $\alpha_{g-1} = \alpha$ and $\beta_{g-1} = \beta$. By induction hypothesis, α may be assumed to be irreducible.

Moreover, an infinitessimal calculation (see [T]), shows that only one sheet of the scheme parametrising pairs of a curve and a one-dimensional theta-characteristic contains a point (C, L) where C is a generic point of any of these components and C is a one-dimensional theta-characteristic on it.

Consider the curve C obtained by identifying a Weierstrass point P in a generic hyperelliptic curve C' of genus $g - 1$ with a point in a generic elliptic curve E. This curve belongs to α. From (1.3), (2.2) and the fact that the monodromy on the set of hyperelliptic curves interchanges. Weierstrass points, it also belongs to any of the components of β.

Denote by R_1, R_2, R_3 the point of E which differ from P by 2-torsion. Denote by R_4, . . . , R_{2g+2} the Weierstrass points of C' different from P. These are the ramification points of the limit g^1_2 on C and hence are the limits on C of the Weierstrass points on non-singular hyperelliptic curves.

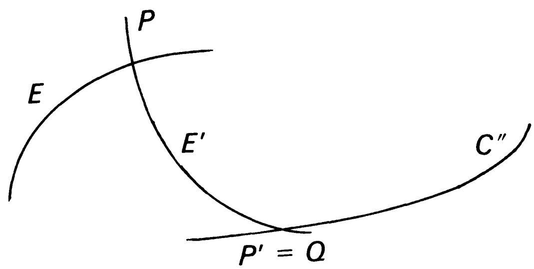
The limit theta characteristics of type α on C are those whose aspect on E differ from an even theta-characteristic on E in $(g - 1)P$, namely those whose aspect on E is $\mathcal{O}_E(P_i + (g - 2)P) = \mathcal{O}_E(P_j + P_k + (g - 3)P)$, $\{i, j, k\} = \{1, 2, 3\}$. The limit theta-characteristics of type β are those whose aspect on E corresponds to the line bundle $\mathcal{O}_E((g - 1)P = \mathcal{O}_E(P_1 + P_2 + P_3 + (g - 4)P)$.

The limit of a family of even effective theta-characteristics on non-singular hyperelliptic curves is a theta-characteristic of type α or β on C depending only on how many of the Weierstrass points which are fixed points in the moving theta-characteristic (see (2.2)) have as limits points on E.

From (2.2) and the fact that the monodromy on the set of hyperelliptic curves acts transitively on the set of Weierstrass points, it follows that it acts transitively on the set of theta-characteristics of a fixed given dimension on a non-singular hyperelliptic curve.

Hence, the monodromy on the set of hyperelliptic curves interchanges any theta-characteristic of type β with one of type α, except may be for the limit of $((g-1)/2)g_2^1$ (when $g \equiv 3$ mod. 4). By induction hypothesis, the monodromy on α acts transitively on the even effective limit theta-characteristics on C. Hence the proof will be complete when we show that the monodromy on $\bar{\mathcal{M}}_g^1$ brings the theta-characteristic $((g-1)/2)g_2^1$ to one of lower dimension.

To this end degenerate C to the curve



Here E' is elliptic, P and P' differ by 2-torsion on E', C'' is hyperelliptic and Q is a Weierstrass point on it which has been identified with the point P' on E.

Consider the family of curves C_X obtained from the join of the curves E and E' at the point P and the curve C'' by identifying a variable point X in E' with the fixed point Q in E''. Denote by P_4, P_5 the two points of E' which differ by 2-torsion from P and P'. Consider the theta-characteristic on C_X with aspects $(g-1)P$ on E, $Q+(g-2)X$ on E' and $(g-1)Q$ on C''. For any X, there is a limit linear series of dimension at least one on C_X whose aspects correspond to these line bundles. For $X = Q$, this is the limit of the theta characteristic $((g-1)/2)g_2^1$ on nearby hyperelliptic curves. For $X = R_5$, C_X is also hyperelliptic and the above aspects are the limit of a theta-characteristic of the form $R_1 + R_2 + R_3 + R_4 + ((g-5)/2)g_2^1$ on nearby curves. This completes the proof of Theorem (2.4).

§3. Computation of the cohomology class of $\bar{\mathcal{M}}_g^1$ in $\bar{\mathcal{M}}_g$

Denote by λ and δ_i for $i = 0 \ldots [g/2]$ the basic divisor clases in Pic $(\bar{\mathcal{M}}_g)$.

(3.1). PROPOSITION. *The class of $\bar{\mathcal{M}}_g^1$ is*

$$2^{(g-3)}[(2^g+1)\lambda - 2^{(g-3)}\delta_0 - \Sigma(2^{g-i}-1)(2^i-1)\delta_i].$$

Proof: Write $a\lambda - \Sigma_{i=0}^{[g/2]} b_i\delta_i$ for the class of $\bar{\mathcal{M}}_g^1$.

Take two generic curves C_1, C_2 of genus i and $g - i$ respectively, $i \leqslant g - i$. Identify a fixed generic point P in C_1 with a moving point Q in C_2. One obtains in this way a curve F in $\Delta_i \subset \bar{\mathcal{M}}_g$ isomorphic to C_2. This satisfies $F\lambda = 0$, $F\delta_j = 0$ for $j \neq i$, $F\delta_i = 2 - 2(g - i)$ ([H, M] p. 86). On the other hand, by (1.2), the points of F in $\bar{\mathcal{M}}_g^1$ are obtained when Q is in the support of an effective theta-characteristic on C_2. For every such curve and every effective theta characteristic on C_1, one obtains a one dimensional theta-characteristic on the reducible curve. As a generic curve of genus p has $2^{(p-1)}(2^p - 1)$ effective theta characteristics, one finds $F\bar{\mathcal{M}}_g^1 = (g - i - 1)$ $2^{(g-i-1)}(2^{(g-i)} - 1)2^{(i-1)}(2^i - 1)$. This gives the expression for b_i, $i > 0$.

The value of a and b_0 can now be obtained from the knowledge of b_1 and b_2 by using Theorem (2.1) in [E, H]2 and (1.2) above.

REMARK (of the referee): the coefficient of λ, $2^{g-3}(2^g + 1)$, is what one should expect from classical arguments. In fact, the product of the even theta-characteristics is a modular form of weight $2^{g-2}(2^g + 1)$ (see [Fr]). Moreover, $\mathcal{M}g$ is tangent to the pull-back of the divisor in $\mathcal{A}_g$ of abelian varieties having a theta-null. The latter statement may be checked using the interpretation of the tangent spaces to $\mathcal{A}_g$ and $\mathcal{M}g$ at a point corresponding to the (jacobian of the) curve C as the duals of $S^2(H^0(C, K_C))$ and $H^0(C, 2K_C)$ respectively (see [O, S]).

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