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On metric properties of substitutions

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Abstract. The metric theory of substitutions of constant length is developed. Some characterizations of discrete spectrum in this class are given. Local rank one phenomen in a class of some direct produts is examined. The measure-theoretic centralizer of bijective substitutions is determined.

Introduction

Let $r \geqslant 2$ be a natural number. We will denote the set $\{0, 1, \ldots, r-1\}$ by N_r. Let $N_r^* = \bigcup_{n \geqslant 1} N_r^n$. The elements of N_r^* are called blocks. If $B \in N_r^*$, $B = (b_0 \ldots b_{n-1})$ then $B[s, t] = (b_s \ldots b_t)$, $B[s, s] = B[s]$, and n is called the length of B, $|B| = n$. These notations can be extended to the elements of $N_r^{\mathbb{Z}}$ in an obvious way.

Let $\lambda \geqslant 2$ and θ: $N_r \to N_r^{\lambda}$. There is a natural extension of θ to a map from N_r^k into $N_r^{k\lambda}$ and to a map from $N_r^{\mathbb{Z}}$ into itself (denoted also by θ) given as follows

$$\theta(\ldots b_{-1}b_0b_1 \ldots) = \theta(b_{-1})\theta(b_0)\theta(b_1) \ldots .$$

We denote by θ^n the n-fold composition of θ. Such a map is called a substitution of constant length on r symbols if

$$\begin{aligned}&\text{there exists } n \geqslant 1 \text{ such that for each } i, j \in N_r,\\ &\theta^n(i)[k] = j \text{ for some } k \ (k \text{ depends on } i, j).\end{aligned} \tag{1}$$

It is well-known that in N_r there exist elements i and j such that for some p the first symbol of $\theta^p(j)$ is j and the last symbol of $\theta^p(i)$ is i. We define $x_0 \in N_r^{\mathbb{Z}}$ as follows:

$$x_0[-\lambda^{pn}, \lambda^{pn} - 1] = \theta^{pn}(ij) \tag{2}$$

Then x_0 is a fixed point of θ. Let T be the shift on $N_r^{\mathbb{Z}}$. Let $X(\theta) = \overline{T^n(x_0): n \in \mathbb{Z}\}}$ denote the closure of the orbit of x_0. Then $(X(\theta), T)$ is a uniquely ergodic dynamical system. We denote the unique T-invariant measure by μ. We will assume that the substitution regarded θ is noncyclic i.e., $X(\theta)$ is an infinite set.

REMARK. The reader should be able to find some of the terms concerning ergodic theory which are undefined in the paper in the papers listed in the References.

This paper is concerned with two measure-theoretic invariants for the class of substitutions of constant length: the rank and the centralizer. In [11] the first author has proved that for the substitutions on two symbols the rank characterizes the discrete spectrum. In this paper we prove the following:

THEOREM 1. *If θ is a substitution of constant length then the substitution $(X(\theta), T, \mu)$ has discrete spectrum iff T has rank 1.*

Another characterization of discrete spectrum gives

THEOREM 2. *The substitution $(X(\theta), T, \mu)$ has discrete spectrum iff T is rigid.*

In [2], Ferenczi introduced the notion of the local rank 1 property. Since each finite rank automorphism has local rank 1, Theorem 3 below shows that each ergodic product of a substitution of constant length (with partly continuous spectrum) with any aperiodic automorphism is not a finite rank transformation.

THEOREM 3. *Let $(X(\theta), T, \mu)$ be a substitution of constant length with partly continuous spectrum and let τ: $(Z, m) \to (Z, m)$ be any aperiodic automorphism such that $T \times \tau$ is ergodic. Then $T \times \tau$ is not local rank* 1.

COROLLARY 4. *Let T' be a local rank 1 transformation. Suppose T is substitution such that τ and T are factors of T'. If $T \times \tau$ is ergodic then τ is a rotation on a finite group.*

The corollary follows from the fact that T is disjoint from τ. Thus $T \times \tau$ is a factor of T' and hence it must have the local rank 1 property.

The second invariant considered in this paper is the measure-theoretic centralizer, $C(T)$. In [8] J. King introduced the notion of the essential centralizer $EC(T) = C(T)/\{T^n\}$. In [10] the question of whether each finite group is realized as the essential centralizer of some automorphism has been raised. From [8] it follows that it is not possible for automorphisms with rank 1. To solve this problem we introduce some definitions and notations.

Following [15] we say that substitution θ of constant length λ on r symbols is bijective if

$$i \neq j \text{ implies } \bar{d}(\theta(i), \theta(j)) = 1 \quad \text{for} \quad i, j \in N_r \tag{3}$$

where $\bar{d}(b_1 b_2 \ldots b_k, c_1 c_2 \ldots c_k) = \text{card}\,\{i: b_i \neq c_i\}/k$. If this is the case, then we may identify the columns in the matrix $\theta(i)[t]$ with the permutations $\{\theta_0, \ldots, \theta_{\lambda-1}\}$ of the set N_r, where $i \xrightarrow{\theta_t} \theta(i)[t]$. Observe, that the group $G(\theta) \subset S_r$ generated by $\{\theta_0, \ldots, \theta_{\lambda-1}\}$ acts transitively on N_r. Without loss of generality we may assume that $\theta_0 = id$.

Let us denote by $C(\theta)$ the centralizer of $G(\theta)$ in S_r. Now we can formulate the following results:

THEOREM 5. *Let θ be a bijective substitution. Then*

$$EC(T) = C(\theta).$$

Now, let G be a finite group. To realize G as the essential centralizer of some bijective substitution it is sufficient to show that G is isomorphic to the centralizer of some set $\{\sigma_1, \sigma_2, \ldots, \sigma_\lambda\} \subset S_r$ for some $r > 1$, such that $G(\sigma_1, \ldots, \sigma_\lambda)$ acts transitively on N_r. Let us notice that if $G(\sigma_1, \ldots, \sigma_\lambda)$ acts transitively on N_r then $C(G(\sigma_1, \ldots, \sigma_\lambda)) \subset S_r$ has at most r elements. Consider now $\varphi, \psi: G \to S_r$ given by the formulas:

$$\psi(g)(h) = gh \quad \text{and} \quad \varphi(g)(h) = hg^{-1}.$$

It is clear that $\psi(G)$ acts transitively on N_r and $C(\psi(G)) = \varphi(G) \cong G$.

COROLLARY 6. *There exists an automorphism which is not weakly-mixing (in particular which is not prime), with trivial centralizer.*

COROLLARY 7. *There exists a finite extension of some dynamical system with discrete spectrum which has trivial centralizer.*

As a problem, we state the question of what kinds of groups are realized as the $EC(T)$ for some ergodic automorphism T with partly continuous spectrum. For further discussion we refer to the last section.

Proofs. We start with some definitions connected with the notion of substitution of constant length.

Let θ be a substitution of constant length λ on r symbols, and let x_0 be a fixed point of θ defined by (2). The number $h(\theta) = \max\{n \geqslant 1\colon \text{g.c.d.}(n, \lambda) = 1, n \text{ divides g.c.d.} \{t : x_0[t] \neq x_0[0]\}\}$ will be called the height of θ ([1], p. 226).

Substitution θ is called pure if $h(\theta) = 1$ ([1], p. 229). If θ is not pure then there is a pure substitution η of constant length λ such that

$$(X(\theta), T) \cong (X(\eta) \times \mathbb{Z}_h, \sigma)$$

where

$$h = h(\theta) \quad \text{and} \quad \sigma(x, i) = \begin{cases} (x, i+1) & \text{if } i > h-1 \\ (Tx, 0) & \text{if } i = h-1 \end{cases}$$

(see [1], Lemma 17 p. 229).
The substitution η is called a pure base of θ.

If θ is a pure substitution then the column number $c(\theta)$ of θ is defined by ([1], p. 230):

$$c(\theta) = \min_{n \geqslant 1} \min_{0 \leqslant t < \lambda^n} \operatorname{card} \{\theta^n(0)[t], \theta^n(1)[t], \ldots, \theta^n(r-1)[t]\}.$$

If θ is not pure then its column number is defined as the column number of its pure base.

The first fact which we use in our proofs is due to Host and Parreau (Lemma 8 below). Suppose that θ is a substitution of constant length λ on r symbols. Let $i, j \in N_r$. Set

$$d_{ij}^n = \bar{d}(\theta^n(i), \theta^n(j)), \quad n \geqslant 1.$$

Then

$$d_{ij}^{n+1} \leqslant (\lambda^{n+1} - (\lambda^n - \lambda^n \cdot d_{ij}^n)\lambda)/\lambda^{n+1} = d_{ij}^n$$

which implies

$$d_{ij}^n \searrow a_{ij} \geqslant 0.$$

Let

$$a = \inf\{a_{ij} > 0\colon i, j \in N_r\}.$$

We define the equivalence relation $\sim$ on N_r setting

$$i \sim j \quad \text{iff} \quad a_{ij} = 0.$$

If card $N_r/\sim\ = 1$ then the column number of θ is equal to 1 and $(X(\theta), T)$ is an automorphism with discrete spectrum ([1] Th. 7 p. 223).

From now on we will assume that card $N_r/\sim\ \geqslant 2$.

Let η be the substitution on $N_r/\sim$ defined by the formula

$$\eta(\tilde{i}) = \widetilde{\theta(i)}. \tag{4}$$

Let us take $\theta(i) = i_1 \ldots i_\lambda$, $\theta(j) = j_1 \ldots j_\lambda$, $i \sim j$. We will prove that $i_t \sim j_t$ for $t = 1, \ldots, \lambda$. Assume on the contrary, that $i_t \nsim j_t$ for some t. This implies

$$\bar{d}(\theta(i), \theta(j)) \geqslant 1/\lambda.$$

Hence

$$\bar{d}(\theta^{n+1}(i), \theta^{n+1}(j)) = \bar{d}(\theta^n(i_1) \ldots \theta^n(i_t) \ldots \theta^n(i_\lambda),$$

$$\theta^n(j_1) \ldots \theta^n(j_t) \ldots \theta^n(j_\lambda)) \geqslant a\, \lambda^n/\lambda^{n+1} = a/\lambda \not\to 0,$$

a contradiction. It follows that $i_t \sim j_t$ for $t = 1, \ldots, \lambda$. Moreover we have obtained that for all $i \nsim j$,

$$\bar{d}(\eta^n(\tilde{i}), \eta^n(\tilde{j})) \geqslant a/\lambda.$$

By virtue of Lemma 17 p. 229 in [1] we may assume that the height of θ is 1, so the height of η is 1 as well. It is not hard to see that the column number of θ is equal to the column number of η.

LEMMA 8. *The substitutions* $(X(\theta), T)$ *and* $(X(\eta), T)$ *are isomorphic* (*measure-theoretically*).

Proof. Let $\varphi\colon X(\theta) \to X(\eta)$ be defined by the formula $\varphi(x) = \ldots \tilde{i}_{-1}\tilde{i}_0\tilde{i}_1 \ldots$ where $x \in X(\theta)$, $x = \ldots i_{-1}i_0i_1 \ldots$. Obviously φ is measureable and $\varphi \cdot T = T \circ \varphi$. Since $X(\eta)$ is uniquely ergodic, φ is measure-preserving. Denote by c the (common) column number of θ and η. Consider the diagram

$$\begin{tikzcd}
(X(\theta), T) \arrow[r] \arrow[rr, bend right] & (X(\eta), T) \arrow[r] & (Z(\lambda), \tau)
\end{tikzcd}$$

where $(Z(\lambda), \tau)$ is the maximal (common) factor of $(X(\theta), T)$ and $(X(\eta), T)$ with discrete spectrum. $Z(\lambda)$ determines unique T-invariant partitions in $X(\theta)$ and $X(\eta)$, whose atoms have c elements (see [1], Th 4, p. 231). Because $(Z(\lambda), \tau)$ is a canonical factor ([14], φ is an isomorphism. ∎

The substitution η described above, enjoys the following property:

$$i \neq j \quad \text{implies} \quad \bar{d}(\eta^n(i), \eta^n(j)) \geqslant a > 0, \quad n \in \mathbb{N}. \tag{5}$$

From now on we will consider substitutions satisfying (5).

LEMMA 9. *Let θ be a substitution of constant length λ. There exist $L \in \mathbb{N}$ and $\delta > 0$ satisfying the following: If $A = \theta^n(j_1) \ldots \theta^n(j_L)$, $n \in \mathbb{N}$, $B = x_0[p, p + L\lambda^n - 1]$, where x_0 is a fixed point of θ and if $\bar{d}(A, B) < \delta$ then*
(a) $\bar{d}(A, B) = 0$
(b) *λ^n divides p.*

We omit the proof since it is similar to the proof of Lemma 1 in [11].
We now are in a position to prove Theorem 1.

Proof of Theorem 1. The necessity is a well-known general result ([5]), so we will show th sufficiency. Assume that the rank of $(X(\theta), T)$ is 1 and T has partly continuous spectrum. Applying Th. 3.1 from [6] to the time-zero partition of $X(\theta)$ we obtain that for each $\varepsilon > 0$ there exists a block $S \in N_r^*$ of positive measure such that for q large enough

$$x_0[0, q - 1] = P_0 S_1 P_1 S_2 P_2 \ldots P_{p-1} S_p P_p$$

where

$$\sum_{i=1}^{k} |P_i| < \varepsilon \cdot q \quad \text{and} \quad \bar{d}(S, S_i) < \varepsilon, \quad i = 1, \ldots, p.$$

Since $\mu(S) > 0$, S has the form:

$$S = w_0 \theta^n(i_1) \theta^n(i_2) \ldots \theta^n(i_t) w_1,$$

$$L \leqslant t \leqslant \lambda(L + 1), \quad |w_i| < \lambda^n, \quad i = 0, 1,$$

for some n, where L is the number from Lemma 9.

We intend to show that for each natural m the sequence x_0 has a sector of the form

$$W = P_u S_{u+1} P_{u+1} S_{u+2} \ldots P_{u+m-1} S_{u+m} P_{u+m}$$

where

$$|P_{u+i}| < a \cdot \lambda^n/4 \tag{5a}$$

and

$$S_{u+i} = w_0^i \theta^n(i_1) \ldots \theta^n(i_t) w_1^i.$$

Indeed, take $\varepsilon < \min (a/8m \cdot \lambda \cdot (L + 1), \delta/2)$. If (5a) is not valid then

$$q\varepsilon \geqslant \frac{q}{m(t+2)\lambda^n} \cdot \frac{a\lambda^n}{4} = \frac{qa}{4m(t+2)} \geqslant \frac{qa}{4m\lambda(L+1)} > 2 \cdot \varepsilon \cdot q,$$

a contradiction, Moreover, using Lemma 9 we see that $\bar{d}(S, S_k) < \varepsilon$ implies

$$S_k = w_0^k \theta^n(i_1) \ldots \theta^n(i_t) w_1^k,$$

where

$$|w_i^k| = |w_i|, \quad i = 0, 1.$$

Since $|P_{u+k}| < a\lambda^n/4$, there are three possibilities:
(a) for each $k = 1, \ldots, m$, $|w_0| + |w_1| + |P_{u+k}| = 0$;
(b) for each $k = 1, \ldots, m$, $|w_0| + |w_1| + |P_{u+k}| = \lambda^n$;
(c) for each $k = 1, \ldots, m$, $|w_0| + |w_1| + |P_{u+k}| = 2\lambda^n$.

Now, we intend to prove that all these cases imply that x_0 has a sector of the form $\underbrace{QQ \ldots Q}_{m \text{ times}}$.

Ad a Denote $Q = i_1 i_2 \ldots i_t$. Then $W = \underbrace{\theta^n(Q) \ldots \theta^n(Q)}_{m \text{ times}}$ Since x_0 is a fixed point of θ, x_0 has a sector $\underbrace{QQ \ldots Q}_{m \text{ times}}$.

Ad b In this case

$$W = \theta^n(j_0)\theta^n(i_1)\theta^n(i_2)\theta^n(i_3) \ldots \theta^n(i_t)\theta^n(j_i)\theta^n(i_1) \ldots \theta^n(i_t)\theta^n(j_2) \ldots$$
$$\theta^n(j_k)\theta^n(i_1) \ldots \theta^n(i_t)\theta^n(j_{k+1}) \ldots \theta^n(i_t)\theta^n(j_m)$$

where

$$\theta^n(j_k) = w_1^{u+k} P_{u+k} w_0^{u+k+1}, \quad k = 1, \ldots, m.$$

We wish to show that $j_1 = j_2 = \ldots = j_m$. Assume $j_k \neq j_l$.

From the assumptions on S, S_{u+p}, $p = 1, \ldots, m$ we have

$$\bar{d}(S, S_{u+p}) = (\bar{d}(w_0, w_0^{u+p}) \cdot |w_0| + \bar{d}(w_1, w_1^{u+p}) \cdot |w_1|)/S. \tag{6}$$

On the other hand

$$a < \bar{d}(\theta^n(j_k), \theta^n(j_l)) = \bar{d}(w_1^{u+k} P_{u+k} w_0^{u+k+1}, w_1^{u+l} P_{u+1} w_0^{u+l+1})$$

$$= (\bar{d}(w_1^{u+k}, w_1^{u+l}) \cdot |w_1| + \bar{d}(P_{u+k}, P_{u+1}) \cdot |P_{u+k}|$$

$$+ \bar{d}(w_0^{u+k+1}, w_0^{u+l+1}) \cdot |w_0|)/\lambda^n \leqslant (\bar{d}(w_1^{u+k}, w_1^{u+l}) \cdot |w_1|$$

$$+ \bar{d}(w_0^{u+k}, w_0^{u+l}) \cdot |w_0| + \bar{d}(P_{u+k}, P_{u+l}) \cdot |P_{u+k}|$$

$$+ \bar{d}(w_0^{u+k+1}, w_0^{u+l+1}) \cdot |w_0| + \bar{d}(w_1^{u+k+1}, w_1^{u+l+1}) \cdot |w_1|)/\lambda^n.$$

Combining this with (6) applied to $p = k, l, k + 1, l + 1$ we obtain

$$a < \bar{d}(S_{u+k}, S_{u+l}) \cdot |S|/\lambda^n + \bar{d}(S_{u+k+1}, S_{u+l+1}) \cdot |S|/\lambda^n$$

$$+ \bar{d}(P_{u+k}, P_{u+l})|P_{u+k}|/\lambda^n < 2\varepsilon|S|/\lambda^n + 2\varepsilon|S|/\lambda^n + |P_{u+k}|/\lambda^n \leqslant 4\varepsilon|S|/\lambda^n$$

$$+ a/4 \leqslant 4 \cdot \varepsilon \cdot \lambda \cdot (L + 1) + a/4 < 4a\lambda(L + 1)/8m\lambda(L + 1)$$

$$+ a/4 = a/2m + a/4 < a,$$

a contradiction. Put $j = j_k$.

Denote

$$Q = ji_1 \ldots i_t, \quad L \leqslant t \leqslant \lambda(L + 1).$$

Then

$$\underbrace{W = \theta^n(Q)\theta^n(Q) \ldots \theta^n(Q)}_{m \text{ times}}.$$

Since x_0 is a fixed point of θ, x_0 has a sector $\underbrace{QQ\ldots Q}_{m \text{ times}}$.

Ad c Now,

$$W = \theta^n(j_0^1)\theta^n(i_1)\ldots\theta^n(i_t)\theta^n(j_1^0)\theta^n(j_1^1)\theta^n(i_1)\ldots$$

$$\theta^n(i_t)\theta^n(j_2^0)\theta^n(j_2^1)\theta^n(i_1)\ldots\theta^n(i_t)\ldots\theta^n(i_t)\theta^n(j_k^0)\,\theta^n(j_k^1)\theta^n(i_1)\ldots$$

$$\theta^n(i_t)\theta^n(j_{k+1}^0)\theta^n(j_{k+1}^1)\ldots\theta^n(i_t)\theta^n(j_m^0),$$

and

$$\theta^n(j_k^0)\theta^n(j_k^1) = w_1^{u+k}P_{u+k}w_0^{u+k+1}.$$

It is clear that $|P_{u+k}| < a\lambda^n/4$ implies

$$j_1^0 j_1^1 = j_2^0 j_2^1 = \ldots = j_{m-1}^0 j_{m-1}^1 = j^0 j^1.$$

Put $Q = j^1 i_1 \ldots i_t j^0$, $L \leqslant t \leqslant \lambda(L+1)$. Then

$$W = \underbrace{\theta^n(Q)\theta^n(Q)\ldots\theta^n(Q)}_{m \text{ times}}$$

and hence x_0 has a sector $\underbrace{QQ\ldots Q}_{m \text{ times}}$.

We conclude that in each case, x_0 has a sector $\underbrace{QQ\ldots Q}_{m \text{ times}}$, where $|Q| \leqslant \lambda(L+1)+2$. Since m is an arbitrary positive integer and there is only finite number of blocks of length at most $\lambda(L+1)+2$, there is a block Q such that for an infinite number of m's, x_0 has sectors $\underbrace{QQ\ldots Q}_{m \text{ times}}$. Because x_0 is a strictly transitive sequence, there exists at most $|Q|$ blocks of length $|Q|$ with positive measure. By virtue of Lemma 6, p. 225 in [1], x_0 is periodic, a contradiction. ■

Proof of Theorem 2. Each system with discrete spectrum is obviously rigid. Assume that $(X(\theta), T)$ is rigid. Let

$$[i] = \{x \in X(\theta)\colon x[0] = i\}, \quad i \in N_r.$$

There is a sequence $n_k \nearrow \infty$ such that

$$\mu([i]\,\Delta\, T^{n_k}[i]) \xrightarrow[k\to\infty]{} 0.$$

Take n_k such that for $i = 0, 1, \ldots, r-1$,

$$\mu([1] \,\Delta\, T^{n_k}[i]) < \delta/2r,$$

where δ is the number from Lemma 9. Let $A_i = [i] \,\Delta\, T^{n_k}[i]$. Since $(X(\theta), T)$ is uniquely ergodic,

$$\left| \frac{1}{N} \sum_{t=0}^{N-1} \chi_{A_i} \circ T^{-t}(x_0) - \mu(A_i) \right| \xrightarrow[N \to \infty]{} 0.$$

Hence, for N large enough

$$\frac{1}{N} \sum_{t=0}^{N-1} \chi_{T^t A_i}(x_0) < \delta/r, \quad i = 0, 1, \ldots, r-1.$$

Therefore

$$\sum_{i=0}^{r-1} \left(\frac{1}{N} \sum_{t=0}^{N-1} \chi_{T^t A_i}(x_0) \right) < \delta$$

But the left hand of this inequality is equal to

$$\frac{1}{N} \operatorname{card} \{0 \leqslant s \leqslant N-1 \colon x_0[s] \neq x_0[n_k + s]\}$$

$$= \bar{d}(x_0[0, N-1], x_0[n_k, n_k + N - 1]).$$

Thus, for n large enough

$$\bar{d}(x_0[0, L \cdot \lambda^n - 1], x_0[n_k + L\lambda^n - 1]) < \delta.$$

Due to Lemma 9,

$$x_0[0, L\lambda^n - 1] = x_0[n_k + L\lambda^n - 1].$$

Since n is arbitrarily large,

$$x_0[t] = x_0[n_k + t]$$

for all $t \geqslant 0$ i.e., x_0 is a periodic sequence, a contradiction. ■

We can now prove Theorem 3. However, in order to do this we introduce some further notations.

Let S: $(Y, \ell, \nu) \to (Y, \ell, \nu)$ be an ergodic automorphism. Let $P = (P_0, \ldots, P_{r-1})$ be a finite partition of Y.

By P-n-name of $y \in Y$ we mean the block $B \in N_r^n$ defined by

$$B[i] = j \quad \text{iff} \quad S^i(y) \in P_j.$$

For a given $\delta > 0$, $1 > b > 0$, we say B has a δ-structure of a b-δ-ξ_0-block if B is of the form $B = \gamma_0 \xi_1 \gamma_1 \ldots \xi_l \gamma_l$ and $\Sigma_{i=0}^{l} |\gamma_i| < (1 - b)(1 + \delta)|B|$. $\bar{d}(\xi_i, \xi_0) < \delta$.

In [12] there is a condition on S equivalent to the local rank 1 property. Namely

S has local rank 1 iff for every partition P there exist ξ_0 and N such that for $n > N$ $\nu\{B : B$ has a δ-structure of a b-δ-ξ_0-block$\} > 1 - \delta$, where $\nu(B) = \nu\{y \in Y$: $P - |B| -$ name of y is $B\}$.

Given θ and τ as in Theorem 3 we denote by $Q = (Q_0, \ldots, Q_{r-1})$ the time zero partition for $X(\theta)$, and in addition for fixed $\varepsilon > 0$, $s \in \mathbb{N}$ let

$$Q'_{\varepsilon,s} = (A, \tau(A), \ldots, \tau^{s-1}(A), Z - \bigcup_{i=0}^{s-1} \tau^i(A))$$

be the partition associated with an (ε, s)-Rokhlin's tower $(A, \tau(A), \ldots, \tau^{s-1}(A))$.

Let $S = T \times \tau$, $Y = X(\theta) \times Z$, $\nu = \mu \times m$. We will describe the $P = Q \times Q'_{\varepsilon,s}$-$n$-names, $n \in \mathbb{N}$. To this end let us take a new alphabet

$$\mathscr{A} = \{(i, j): i = 0, 1, \ldots, r - 1, \quad j = 0, 1, \ldots, s - 1, s\}.$$

Now, observe that Q-n-name of $x \in X(\theta)$ is equal to $x[0, n - 1]$. Given $z \in Z$ let $\xi = (\xi_0, \ldots, \xi_{n-1})$ be its $Q'_{\varepsilon,s}$-n-name. Then ξ can be described as follows:

$$\xi = (i, i + 1, \ldots, s - 1, s, s, \ldots, s, 0, 1,$$
$$\ldots, s - 1, s, \ldots, s, \ldots j),$$

and the number of s's is less than $\varepsilon \cdot |\xi|$. Now, the P-n-name of $(x, z) = ((x[0], \xi_0), (x[1], \xi_1) \ldots, (x[n - 1], \xi_{n-1}))$.

Proof of Theorem 3. Let us suppose $S = T \times \tau$ has local rank 1 with the corresponding constant b, $0 < b < 1$. We fix $s \in \mathbb{N}$ and $\varepsilon > 0$ (ε will depend on s). Let $Q'_{\varepsilon,2s} = Q'$. From (7) there is a P-block ξ_0, $|\xi_0| = t$ and $N > 0$ such that for $n > N$

$$\nu(D = \{(x, z) \in X(\theta) \times Z\colon P\text{-}n\text{-name of } (x, z) \text{ has an } \varepsilon\text{-structure of a } b\text{-}\varepsilon\text{-}\xi_0\text{-block}\}) > 1 - \varepsilon. \tag{9}$$

From the definition of ν it follows that $Id_{X(0)} \times \tau^i \in C(S)$ for all integer i. Consider the sequence of the following sets

$$D, (Id \times \tau^{-1})(D), (Id \times \tau^{-2})(D), \ldots.$$

For a suitable choice of ε we can assume

$$\nu\left(\bigcap_{i=0}^{s} (Id \times \tau^{-i})(D)\right) > 0.$$

Hence there is $(x, z) \in (Id \times \tau^{-i})(D)$, $i = 0, 1, \ldots, s$ which implies

$$(x, \tau^i(z)) \in D \quad \text{for} \quad i = 0, 1, \ldots, s.$$

For $k = 0, 1, \ldots, s$ we denote by $H_k = \{i_0^{(k)}, \ldots, i_{j_k}^{(k)}\}$ the set of ε-occurency of ξ_0 on P-n-name of $(x, \tau^k(z))$ given by the condition (9), i.e.

$$i \in H_k \quad \text{if} \quad \bar{d}(\xi_0, (x, \tau^k(z))[i, i + |\xi_0| - 1]) < \varepsilon \quad \text{(see Fig. 1).}$$

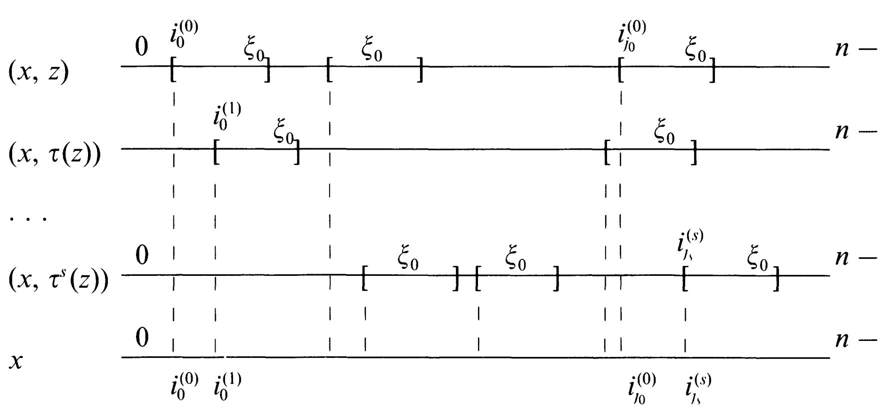

Fig. 1.

The key to our proof consists of the simple observation that $H_k \cap H_{k'} = \varnothing$ for $k \neq k'$. Denote

$$\xi_0 = ((\xi'_0, \xi''_0), \ldots, (\xi'_{t-1}, \xi''_{t-1})),$$

$$\xi' = (\xi'_0, \ldots, \xi'_{t-i})$$

and

$$\xi'' = (\xi''_0, \ldots, \xi''_{t-1}).$$

Let us look at Fig. 2, assume $H_k \cap H_{k+2} \neq \varnothing$, and let $i_j^{(k)} = i_{j'}^{(k+2)}$. We denote the P-n-name of $(x, \tau^k(z))$ here by

$$\xi^{(k)} = (\xi^{(k)'}, \xi^{(k)''}), \quad k = 0, 1, \ldots, s.$$

Let us observe that

$$\bar{d}(\xi^{(k)''}[i_j^{(k)}, i_j^{(k)} + t - 1], \xi^{(k+2)''}[i_{j'}^{(k+2)}, i_{j'}^{(k+2)} + t - 1])$$
$$\leqslant \bar{d}(\xi'', \xi^{(k)''}[\ldots, \ldots]) + \bar{d}(\xi'', \xi^{(k+2)''}[\ldots, \ldots]) < 2\varepsilon.$$

From Fig. 2 it follows that if $\xi^{(k)''}[i] = \xi^{(k+2)''}[i]$ then

$$\xi^{(k)}[i] = 2s. \tag{11}$$

Let $\eta = (2s, 2s, \ldots, 2s)$, $|\eta| = t$.

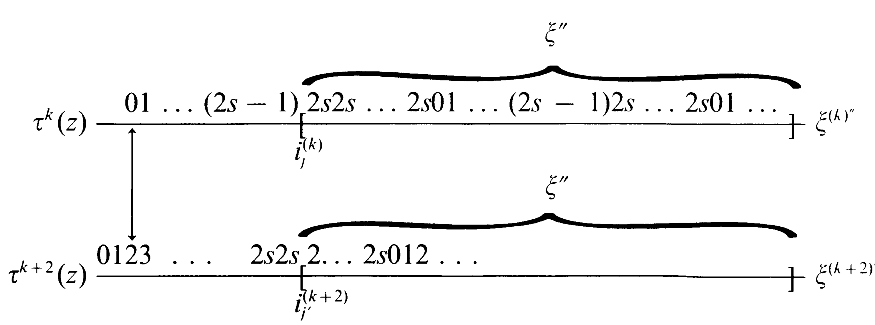

Fig. 2.

From (10) and (11) it easily follows that

$$\bar{d}(\xi'', \eta) < 2\varepsilon.$$

On the other hand, if $\bar{d}(\xi'', \xi^{(k}[i_l^{(k)}, i_j^{(k)} + t - 1]) < \varepsilon, l = 0, 1, \ldots, j_k$, then

$$\bar{d}(\xi^{(k)''}[i_l^{(k)}, i_l^{(k)} + t - 1], \eta) < 4\varepsilon. \tag{12}$$

But if $\xi^{(k)}$ has an ε-structure of a b-ε-ξ_0-block then $\xi^{(k)''}$ has an ε-structure of a b-ε-ξ''-block.

Combining (10) with (8) we obtain a contradiction and the proof that $H_k \cap H_{k'} = \emptyset$, $k \neq k'$, is complete.

By Fig. 2 we have that

$$\varepsilon - fr(\xi', x[0, n - 1]) \geqslant \operatorname{card}\left(\bigcup_{k=0}^{s} H_k\right) \tag{13}$$

where

$$\varepsilon - fr(\xi^1, x[0, n - 1])$$
$$= \operatorname{card}\{0 \leqslant i \leqslant n - 1 : \bar{d}(\xi^1, x[i, i + |\xi'| - 1]) < \varepsilon\}.$$

Let us write $\xi' = \xi_1 \xi \xi_2$, where

$$\xi = \theta^u(i_1) \ldots \theta^u(i_{L'}), \quad L \leqslant L' \leqslant \lambda\,(L + 1),$$

and

$$|\xi|/|\xi'| \geqslant 1/2.$$

From (13) it follows that

$$4\varepsilon - fr(\xi, x[0, n - 1]) \geqslant \operatorname{card}\left(\bigcup_{k=0}^{s} H_k\right).$$

By a suitable choice of $\varepsilon > 0$ we can assume

$$fr(\xi, x[0, n - 1]) \geqslant \sum_{k=0}^{s} \operatorname{card}(H_k).$$

But $\xi^{(k)}$ has an ε-structure of a b-ε-ξ_0-block, $k = 0, \ldots, s$, so card $(H_k) \cdot |\xi'| \geqslant (b - \varepsilon)n$ which implies $fr(\xi, x[0, n - 1]) \geqslant s(b - \varepsilon)n/|\xi'| \geqslant (1/2)\, s(b - \varepsilon) \cdot n/|\xi|$. But from Lemma 9

$$fr(\xi, x[0, n - 1]) \leqslant n/\lambda^u.$$

Hence

$$n/\lambda^u \geqslant fr(\xi, x[0, n-1]) \geqslant (1/2)s(b-\varepsilon)n/|\xi| = (1/2)s(b-\varepsilon)n/L' \cdot \lambda^u.$$

Thus

$$1 \geqslant (1/2) \cdot s(b-\varepsilon)/L' \geqslant (1/2)s(b-\varepsilon)/\lambda(L+1).$$

This is a contradiction to a suitable choice of ε and s. ■

Proof of Theorem 5. Let θ be a bijective substitution. Assume that $\varphi \in C(T)$. For a.e. $x \in X(\theta)$, x is a unique concatenation of blocks of the form $\theta^n(i)$ (we call such a block n-symbol) for each n, i.e.,

$$x = \ldots \theta^n(i_{-1})\theta^n(i_0)\theta^n(i_1) \ldots .$$

It follows that

$$\varphi(x) = \ldots \theta^n(j_{-1})\theta^n(j_0)\theta^n(j_1) \ldots .$$

In this notation the zero-coordinate $x[0]$ of x, satisfies

$$x[0] = \theta^n(i_0)[p] \quad \text{for some } 0 \leqslant p < \lambda^n,$$

where λ is the length of θ. The key observation is that

$$\begin{array}{l} \text{if } x[0] = \theta^n(i_0)[p_n] \text{ and } \varphi(x)[0] = \theta^n(j_0)[p_n + t_n], \\ \text{then } t_n \text{ is constant for } n \text{ large enough } (t_n = t). \end{array} \tag{14}$$

(Actually this means that for all n, $t_n = t \bmod \lambda^n$.) To prove (14) let us fix $\varepsilon > 0$ and let φ_ε be the corresponding finite code of length m, i.e., φ_ε is a measurable map commuting with T, mapping $X(\theta)$ into $N_r^{\mathbb{Z}}$, such that $\varphi_\varepsilon(x)[i]$ depends only on $x[-m+i, m+i]$ and

$$\lim_{n\to\infty} \bar{d}(\varphi_\varepsilon(x)[-n, n] \quad \varphi(x)[-n, n]) < \varepsilon. \tag{15}$$

Let $x \in X(\theta)$. Clearly φ_ε maps each block of length s in x into a block of length $s - 2m$. We fix k such that

$$m/\lambda^k < \varepsilon \tag{16}$$

and

$$\bar{d}(\varphi_e(\theta^k(p)), \varphi_\varepsilon(\theta^k(q))) > 1 - 3\varepsilon \quad \text{for} \quad p \neq q. \tag{17}$$

The inequality (17) follows from (15) and (3).

Let M be a positive integer such that on each block of length M which appears in any $x \in X(\theta)$ there exist blocks ij and ij' and $j \neq j'$.

We divide $\varphi(x)$ into a sequence of groups of blocks consisting of M of k-symbols. Thus there must exist a group A such that

$$\bar{d}(A, \varphi_\varepsilon(x)[\ldots, \ldots]) < 2\varepsilon. \tag{18}$$

To prove (14) it is sufficient to show that

$$t_k/\lambda^k < 1/2\lambda \quad \text{if} \quad 0 \leqslant t_k < \lambda^k/2 \tag{19}$$

and

$$(\lambda^k - t_k)/\lambda^k < 1/2\lambda \quad \text{if} \quad \lambda^k/2 \leqslant t_k < \lambda^k. \tag{20}$$

Indeed, if $t_{k+1} \neq t_k$ then $t_{k+1} = t_k + a\lambda^k$, $a \neq 0$, and $t_{k+1}/\lambda^{k+1} = t_k/\lambda^k + a/\lambda > 1/2\lambda$ if $t_{k+1} < \lambda^{k+1}/2$, or $< 1 - 1/2\lambda$ if $t_{k+1} \geqslant \lambda^{k+1}/2$.

Now, we prove (19). Consider Fig. 3.

In the block A there are blocks $\theta^k(i)\theta^k(j)$ and $\theta^k(i)\theta^k(j')$, $j \neq j'$.

1°. $p = q$. Since

$$\bar{d}(\theta^k(j)[0, t_k - m - 1], \theta^k(j')[0, t_k - m - 1]) = 1,$$

$$\bar{d}(\varphi_\varepsilon(\theta^k(p)), \theta^k(i)\theta^k(j)[\ldots, \ldots]) + \bar{d}(\varphi_\varepsilon(\theta^k(p)), \theta^k(i)\theta^k(j')[\ldots, \ldots])$$

$$\geqslant (t_k - m)/(\lambda^k - 2m).$$

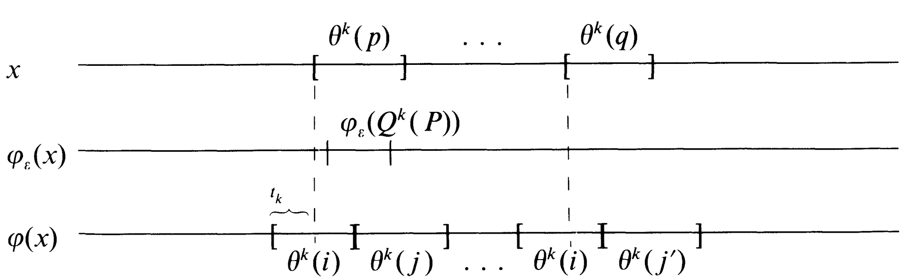

Fig. 3.

This implies that

$$\bar{d}(A, \varphi_\varepsilon(x)[\ldots, \ldots]) > (t_k - m)/M\lambda^k.$$

Combining this inequality with (18) we have

$$2\varepsilon > (t_k - m)/M\lambda^k.$$

Hence

$$t_k/\lambda^k \leqslant 2\varepsilon M + m/\lambda^k < (2M + 1)\varepsilon < 1/2\lambda$$

for a suitable choice of ε.

2°. $p \neq q$. Then

$$\bar{d}(\varphi_\varepsilon(\theta^k(p))[0, \lambda^k - m - t_k - 1], \varphi_\varepsilon(\theta^k(q))[0, \lambda^k - m - t_k - 1])$$

$$\geqslant (\lambda^k - m - t_k - 3\varepsilon(\lambda^k - 2m))/(\lambda^k - m - t_k)$$

and this implies

$$\bar{d}(\varphi_\varepsilon(\theta^k(p))[0, \lambda^k - m - t_k - 1], \theta^k(i)[t_k + m, \lambda^k - 1])$$

$$+ \bar{d}(\varphi_\varepsilon(\theta^k(q))[\ldots, \ldots], \theta^k(i)[\ldots, \ldots])$$

$$\geqslant (\lambda^k - m - t_k - 3\varepsilon(\lambda^k - 2m))/(\lambda^k - m - t_k).$$

We conclude that

$$2\varepsilon > (\lambda^k - m - t_k - 3\varepsilon(\lambda^k - 2m))/M\lambda^k$$

and

$$t_k > \lambda^k - 2\varepsilon M\lambda^k - 3\varepsilon(\lambda^k - 2m) > \lambda^k/2.$$

This is a contradiction. The case (20) is similar to the preceding one and the details are left to the reader. The proof of (14) is complete.

Let us denote $t_k = t$. The constant t depends on x, but since T is ergodic, it follows that t is constant for a.e., x.

If we replace φ by $\varphi \circ T^{-t}$ then we may assume that x and $\varphi(x)$ have thesame structure of n-symbols for each n, i.e., the n-symbols in x and $\varphi(x)$ overlap each other (see Fig. 4).

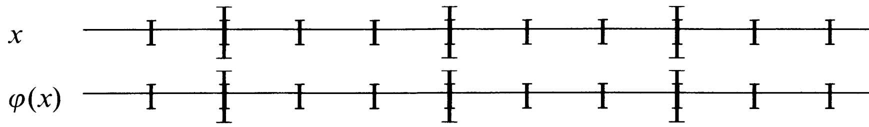

Fig. 4.

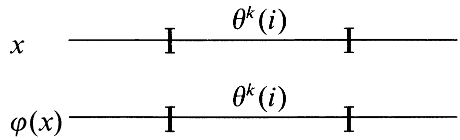

Fig. 5.

Let L be the integer from Lemma 9. For natural k and $i, j \in N_r$ let $A^i_j(\mathbf{k})$ denote the number of times that the situation shown in Fig. 5 occurs in sector $[-L/2\lambda^k, L/2\lambda^k]$ of x and $\varphi(x)$ (i.e., $A^i_j(k)$ denotes the number of the pairs $(\theta^k(i), \theta^k(j))$ such that $\theta^k(i)$ and $\theta^k(j)$ overlap each other).

If k is large enough then it is an easy calculation that for $j \neq j'$, either $A^i_j(k) < 8L\varepsilon$ or $A^i_{j'}(k) < 8L\varepsilon$ for small ε. This implies that for each $i \in N_r$ there is a unique $j \in N_r$ such that

$$A^i_j(k) > 2L(1/r - 4r\varepsilon).$$

Due to this inequality we can define bijections ψ_k, $k \geqslant 1$ on N_r as follows

$$\psi_k(i) = j \quad \text{iff} \quad A^i_j(k) > 2L(1/r - 4r\varepsilon).$$

Using Lemma 9 it is clear that $\psi_k = \psi_{k+1}$ for all k large enough. Let us put $\psi = \psi_k$. Moreover, this map is common for a.e., x, and it is easy to see that

$$\theta\psi = \psi\theta$$

(we recall that the first symbol in $\theta(i)$ is i for each $i \in N_r$).

Collecting results we have obtained that

$$\varphi = \psi \circ T^{-n}$$

for some $\psi \in C(\theta)$ and $n \in \mathbb{Z}$. Clearly each transformation of this form is a member of $C(T)$. Consequently

$$EC(T) = C(\theta)$$

and the proof of Theorem 5 is complete. ■

REMARK. We can apply the above arguments to a little bit more general situation. Namely:

COROLLARY 10. *Asssume that θ and η are two bijective substitution of length λ on r symbols and $(X(\theta), T)$ is isomorphic to $(X(\eta), T)$. Then there exists a permutation ψ of N_r such that*

$$\theta = \psi\eta\psi^{-1}$$

■

Some examples of infinite rank transformations with countable centralizers

In [16] Rudolph raised the question whether there exists an automorphism with infinite rank and trivial centralizer. One can ask more generally whether the centralizer $C(T)$ express how complicated T is. One might expect this to be true on the basic of the formula

$$C(T \times T') = C(T) \times C(T'), \tag{21}$$

but unfortunately this is not true in general – even if $T \perp T'$ (T is disjoint from T' ([4])). Assuming that $T \perp T'$ we are interested in some additional conditions such that (21) holds. In a letter, A. del Junco has observed that (21) is true under the condition that T and T' are simple ([7]).

We intend to exhibit a class of infinite rank automorphisms with countable centralizers. We start with the following:

LEMMA 11. *Let τ: $(Y, \mathscr{L}, \nu) \to (Y, \mathscr{L}, \nu)$ be an ergodic automorphism with pure point spectrum and let T: $(X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be a finite extension of τ, i.e., there is φ: $(X, \mathscr{B}, \mu) \to (Y, \mathscr{L}, \nu)$, $\varphi T = \tau\varphi$,* card *$\varphi^{-1}(y) = r$ for a.e. $y \in Y$.*

Then T is coalescent (i.e., $C(T)$ is a group ([13])).

The lemma easily follows from [14].

THEOREM 12. *Let θ and θ' be two bijective noncyclic substitutions of lengths λ and λ^1 resp. and on 2 symbols, where λ and λ' are odd, g.c.d. $(\lambda, \lambda') = 1$.*

Then $C(T \times T')$ is countable.

A proof may be constructed from the following remarks:

1. $T(T')$ is isomorphic to a $\mathbb{Z}_2$-extension $\tau_\varphi(\tau'_{\varphi'})$, where $\tau(\tau')$ has discrete spectrum with the group of eigenvalues equal to $G\{\exp(2\pi i/\lambda^t)\colon t \geqslant 0\}$ $(G\{\exp(2\pi i/\lambda''^t)\colon t \geqslant 0\})$, and $\tau_\varphi \times \tau'_{\varphi'}$ is ergodic.
2. $\tau_\varphi \times \tau'_{\varphi'}$ is isomorphic to the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-extension of $\tau \times \tau$, $(\tau \times \tau')_{\varphi \times \varphi'}$, $(\tau \times \tau')_{\varphi \times \varphi'}((x, x'), (i, j)) = ((\tau \times \tau')(x, x'), (\varphi(x), \varphi'(x')) + (i, j))$.
3. If $\bar{S} \in C((\tau \times \tau')_{\varphi \times \varphi'})$ and $\bar{S}^2 = Id$, then $\bar{S} = \sigma_{k,l}$, $(k, l) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\sigma_{k,l}((x, x'), (i, j) = ((x, x'), (k, l) + (i, j))$. (Since $-1 \notin S_p(\tau \times \tau')$, $S^2 = Id$ and $S \in C(\tau \times \tau')$ imply $S = Id$ ([14]).)
4. $\bar{S} \in C(\tau_\varphi \times \tau'_{\varphi'})$ and $\bar{S}^2 = Id$ imply $\bar{S} = Id \times Id$, $S = \sigma \times Id$, $S = Id \times \sigma'$, or $S = \sigma \times \sigma'$ where $\sigma(x, i) = (x, i + 1)$ and $\sigma'(x', j) = (x', j + 1)$.
5. For every ergodic automorphism $U\colon (Y, \mu) \to (Y, \mu)$ there is a one-to-one correspondence between nontrivial $S \in C(U)$ with $S^2 = Id$ and measureable U-invariant partitions β of Y with the property that $A \in \beta$ implies card $(A) = 2$ for a.e. $A \in \beta$.
6. For $U = \tau_\varphi \times \tau'_{\varphi'}$, the only measurable U-invariant partitions β with card $(A) = 2$ for $A \in \beta$ are:

   $$\beta_1 = m_{(X \times Z_2)} \times \varepsilon_{(X \times Z_2)} \quad \text{i.e., } ((x, 0), (x', i)) \overset{\beta_1}{\sim} ((x, 1), (x', i))$$

   $$\beta_2 = \varepsilon_{(X \times Z_2)} \times m_{(X \times Z_2)} \quad \text{i.e., } ((x, i), (x', 0)) \overset{\beta_2}{\sim} ((x, i), (x', 1))$$

   $$\beta_0 \ \text{ such that } \ ((x, i), (x', j)) \overset{\beta_0}{\sim} ((x, i + 1), (x', j + 1))$$

   corresponding to $\sigma \times Id$, $Id \times \sigma'$, $\sigma \times \sigma'$ respectively.
7. If $\bar{S} \in C(\tau_\varphi \times \tau'_{\varphi'})$ then either

   $$\bar{S}(\beta_1) = \beta_1 \quad \text{or} \quad \bar{S}(\beta_2) = \beta_2.$$

   Otherwise, since $\bar{S}$ is invertible (Lemma 11), we get either

   $$\bar{S}(\beta_2) = \beta_1 \quad \text{or} \quad \bar{S}(\beta_1) = \beta_2.$$

   In both cases, $\tau_\varphi \times \tau' \cong \tau \times \tau'_{\varphi'}$. Hence $\tau_\varphi \times \tau'$ has τ_φ and $\tau'_{\varphi'}$ as factors. From the proof of Th. I.4 in [4] it follows that $\tau_\varphi \perp \tau'_{\varphi'}$ since $\tau_\varphi \times \tau'_{\varphi'}$ is ergodic. Therefore $\tau_\varphi \times \tau'$ has $\tau_\varphi \times \tau'_{\varphi'}$ as a factor. Since $\tau_\varphi \times \tau'$ is a $\mathbb{Z}_2$-extension of the system with discrete spectrum $\tau \times \tau'$ and $\tau_\varphi \times \tau'_{\varphi'}$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-extension of $\tau \times \tau'$, it is a contradiction.
8. If $\bar{S} \in C(\tau_\varphi \times \tau'_{\varphi'})$ then $\bar{S}^2(\beta_i) = \beta_i$ and $\bar{S}^2(\beta_2) = \beta_2$.

9. If $U = U_1 \times U_2$ has a simple spectrum then $C(U) = C(U_1) \times C(U_2)$ (since each factor of U is then canonical ([14])).
10. $C(\tau_\varphi \times \tau') = C(\tau_\varphi) \times C(\tau')$ and $C(\tau \times \tau'_{\varphi'}) = C(\tau) \times C(\tau'_{\varphi'})$. (In [12] there is a proof that the products $\tau_\varphi \times \tau'$ and $\tau \times \tau'_{\varphi'}$ have simple spectra.)
11. If $\bar{S} \in C(\tau_\varphi \times \tau'_{\varphi'})$ then $\bar{S}^2 = U_1 \times U_2$, $U_1 \in C(\tau_\varphi)$, $U_2 \in C(\tau'_{\varphi'})$.
12. If $S_1, S_2 \in C(\tau \times \tau')$ and $S_1^2 = S_2^2$ then

$$S_1 = S_2 \quad (\text{since } -1 \notin \mathrm{Sp}\,(\tau \times \tau').$$

13. If $\bar{S} \in C(\tau_\varphi \times \tau'_{\varphi'})$ then $\bar{S}$ has at most four square roots ([7], Th. 2.1). ∎

REMARK. If follows from [7] that the automorphisms T, T' used in Theorem 12 are not simple.

REMARKS ON FR_∞ CLASS. Having proved Theorem 3, it is natural to work out some subclass, say FR_∞, of ergodic local rank 1 (l.r. 1) automorphisms.
$T \in FR_\infty$ if $T \times \tau'$ is not l.r. 1 as soon as
$T \times \tau$ is ergodic and τ is aperiodic.
Then if $T \in FR_\infty$ appears as a factor of ergodic l.r. 1 automorphism T', such that in addition T' has another factor τ with $\tau \perp T$, then τ is a rotation on a finite group. Also, for ergodic powers of T, the rank of T^s $(rk(T^s))$ is equal to the rank of $T \times \varrho_s$ (ϱ_s is a rotation on $\{0, 1, \ldots, s-1\}$). Hence $rk(T^{n_k}) \nearrow \infty$ for each sequence $\{n_k\}$, $n_k | n_{k+1}$, as soon as $T \in FR_\infty$. Theorem 3 says that each substitution of constant length with partly continuous spectrum belongs to FR_∞. It turns out that FR_∞ contains l.r. 1 automorphisms with zero rigidity ([9]). In particular, all l.r. 1 mixing automorphisms are in FR_∞.

THEOREM 13. *If T is an ergodic* l.r. 1 *automorphisms and if T has zero rigidity then $T \in FR_\infty$.*

Proof. Suppose $T \times \tau$ has l.r. 1 with constant b, $0 < b < 1$. Then repeating the considerations from the proof of Theorem 3 (with Q to be any finite generator and x any generic point of T) we are able to obtain the following:

> For any $\varepsilon > 0$ there exist a string ξ'_0, $|\xi'_0| = r$ (r sufficiently large) and a sequence of natural numbers $\{i_m\}$, $i_m + r \leqslant i_{m+1}$ such that $\bar{d}(\xi'_0, x[i_m, i_m + r - 1]) < \varepsilon$ and that the density of $\bigcup_{m=0}^{\infty}[i_m, i_m + r - 1]$ is at least $b/2$.

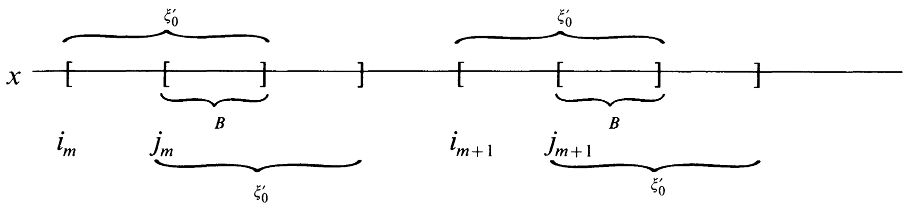

Fig. 6.

Furthermore there is j_m such that

$$\bar{d}(\xi_0', x[j_m, j_m + r - 1]) < \varepsilon, \quad 0 < |j_m - i_m| = s.$$

Denote $B = \xi_0'[j_m - i_m, r - 1]$. Hence we get a sequence $\{j_m\}, j_m = i_m + s$ (see Fig. 6), $j_m + |B| - 1 < j_{m+1}$ such that $\bar{d}(x[j_m, j_m + |B| - 1], B) < 100\varepsilon$ and the density of

$$\bigcup_{s=0}^{\infty} [j_m, j_m + |B| - 1] > b/100.$$

So, the assumptions of Rigidity Criterion ([9]) are fulfilled and T has no zero-rigidity, a contradiction. ■

Using the same arguments we have

THEOREM 14. *If T is not rigid and has rank 1 with bounded number of columns* ([3]) *then T belongs to FR_∞. In particular Chacon transformation is in FR_∞.*

REMARK. Theorem 14 says that for Chacons transformation T, $T \times T$ is not of local rank 1. Hence the problem whether $T \times T$ is L.B. remains open.

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