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Parabolic measure on domains of class Lip $\frac{1}{2}$

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1. *We shall construct a domain* Ω *in* $\mathbb{R}^2_{x,t}$, *whose boundary is given by the graph of a* Lip $\frac{1}{2}$ *function* $x = F(t)$, *so that on* $\partial\Omega$ *the parabolic measure* ω *and the adjoint parabolic measure* $\omega*$ *are concentrated on two disjoint sets, whose projections onto the t-axis have Hausdorff dimensions strictly less than* 1.

We do not known how small the dimensions can be made in the example. But the dimensions must be at least $\frac{1}{2}$, by a theorem of Taylor and Watson [8; p. 337], which states that a set E on a Lip $\frac{1}{2}$ curve $x = F(t)$ has heat capacity zero if and only if its projection onto the t-axis has zero $\frac{1}{2}$-capacity.

In a previous paper [3], we constructed a Lip $\frac{1}{2}$ domain $\{x > F(t)\}$ satisfying the weaker property that the projections of supports of ω and $\omega*$ have zero 1-dimensional Hausdorff measure. There are two technical improvements made here: an explicit construction of $F(t)$ is given and shown to satisfy an explicit inequality in class Lip $\frac{1}{2}$, and a more careful estimation of parabolic measure is necessary. F is a lacunary sum but the gaps are not too large to obtain an estimate of the dimension. The size of the gaps is critical in obtaining an explicit estimate of F in Lip $\frac{1}{2}$.

Because the only diffeomorphisms that preserve the solutions of the heat equation $((\partial^2/\partial x^2) - (\partial/\partial t))u = 0$ are $\{(x, t) \to (ax + b, a^2 t + c)\}$, [2], domains $\Omega = \{x > f(t)\}$, with Lip $\frac{1}{2}$ boundary $x = f(t)$ are very natural for studying solutions of the heat equation. It follows from theorems of Petrowsky [7] that these domains are Dirichlet regular for the solutions of the heat equation.

For a Borel set $E \subset \partial\Omega$, we denote by $\omega^{(x,t)}(E)$ (or $\omega*^{(x,t)}(E)$) the parabolic measure (or the adjoint parabolic measure) of E with respect to Ω, i.e., the solution of the heat equation (or the adjoint heat equation) on Ω with boundary value 1 on E, 0 on $\partial\Omega\backslash E$, in the Brelot–Perron–Wiener sense.

We say that ω is supported on a Borel set $S \subset \partial\Omega$ provided that $\omega^{(x,t)}(\partial\Omega\backslash S) = 0$ for *every* $(x, t) \in \Omega$; and similarly for $\omega*$.

We say that u is a parabolic (or adjoint parabolic) function provided that

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad \left(\text{or } \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0\right).$$

In the case of Laplace's equation, on the boundary of a Jordan domain in $\mathbb{R}^2$, the harmonic measure is concentrated on a set of Hausdorff dimension 1; and any set of Hausdorff dimension less than 1 has zero harmonic measure [6]. For the heat equation, we conjecture that the parabolic measure on the boundary of a Jordan domain is supported on a set of "parabolic dimension" at most 2. We refer to [8] for the definition of parabolic dimension, and recall that the parabolic dimension of the line $\{t = 0\}$ is 1, of the line $\{x = at + b\}$ is 2, and of $\mathbb{R}^2$ is 3.

The parabolic measure for $\{x > 0\}$ is supported only by sets of full linear measure, therefore of parabolic dimension exactly 2. For the example to be constructed, the parabolic measure is supported by a subset of the graph $\{x = F(t)\}$ of parabolic dimension < 2.

2. A. function of class Lip $\frac{1}{2}$

LEMMA 1. *Given $0 < \varepsilon < 10^{-2}$, let $h(t)$ be a C^1 function on $(-\infty, \infty)$, periodic with period 1, so that $h(0) = 0$, $h(t) = h(1 - t)$, $h(t) = t^{1/2}$ for $\varepsilon \leqslant t \leqslant \frac{1}{4}$, $|h| < 1$, $|h'| \leqslant 2\varepsilon^{-1/2}$, and $|h(t) - h(s)| \leqslant 2|t - s|^{1/2}$ for all s, t. Let $N \geqslant \varepsilon^{-5/2}$ be an integer and $f(t) = \Sigma_1^\infty N^{-n} h(N^{2n} t)$. Then*

(a) $|f(t) - f(s)| \leqslant (2 + 6\varepsilon)|t - s|^{1/2}$ *for all s, t.*

Let $\tau = mN^{-2k}$, with an integer $m = 0, 1, \ldots, N^{2k} - 1$. Then

(b) $|f(t) - f(\tau)| \leqslant 3\varepsilon^{1/2} N^{-k}$, *whenever* $0 \leqslant t - \tau \leqslant \varepsilon N^{-2k}$, *and*

(c) $\frac{1}{2}(t - \tau)^{1/2} \leqslant f(t) - f(\tau) \leqslant \frac{3}{2}(t - \tau)^{1/2}$, *whenever* $\varepsilon N^{-2k} \leqslant t - \tau \leqslant \frac{1}{4} N^{-2k}$.

Proof. To prove (a) we observe that

$$|f(t) - f(s)| \leqslant 2 \sum_{n=1}^{\infty} \min(N^{-n}, |t - s|^{1/2}, N^n \varepsilon^{-1/2} |t - s|).$$

We obtain each term inside the minimum by using one of the inequalities on h. The numbers $n = 1, 2, 3, \ldots$ are divided into three groups.

(i) $N^{-n} \leqslant \varepsilon |t - s|^{1/2}$. Using N^{-n} in the minimum, and $N \geqslant 4$, we see that the contribution from this group is at most $2\varepsilon|t - s|^{1/2} N(N - 1)^{-1} < 3\varepsilon|t - s|^{1/2}$.

(ii) $N^n \varepsilon^{-1/2}|t - s| \leqslant \varepsilon |t - s|^{1/2}$. The same estimate applies, if we use the third term in the minimum.

(iii) $\varepsilon |t - s|^{1/2} < N^{-n} < \varepsilon^{-3/2}|t - s|^{1/2}$.

There can be at most one solution n to this pair of inequalities: if $n_1 \leqslant n_2$ and both are solutions, then $(n_2 - n_1)\log N < \log \varepsilon^{-5/2}$, so $n_2 - n_1 < 1$ or $n_2 = n_1$. The contribution from (iii) is thus at most $2|t - s|^{1/2}$.

Adding up for (i), (ii), (iii) gives (a).

In view of (a), inequality (b) is evident.

To prove (c), with $s = \tau$, we observe that $n = k$ belongs to (iii).

However, a more precise estimate can be given: $N^{-k}h(N^{2k}t) - N^{-k}h(N^{2k}\tau) = N^{-k}h(N^{2k}t) = (t - \tau)^{1/2}$. The total contribution from (i) and (ii) is at most $6\varepsilon(t - \tau)^{1/2}$. Since $6\varepsilon < 1/2$, (c) follows.

3. Estimates of parabolic measure

Suppose that $x = F(t)$ is Lip $\frac{1}{2}$ on $(-\infty, \infty)$ with $|F(t) - F(s)| \leqslant M|t - s|^{1/2}$. For $a > 0$, we denote by $\Delta(t, a) = \{(F(s), s): |s - t| < a\}$ and $A(t, a) = (F(t) + 2M\sqrt{a}, t + 2a)$. Lemma 1.4 in [5] can be restated as follows.

LEMMA 2. *There is a positive constant $C(M)$ depending on M only, so that if u is a nonegative parabolic function on $\Omega \equiv \{x > F(t)\}$, vanishing on $\{(F(s), S): |s - t| > a/64\}$, then*

$$u(y, s) \leqslant C(M)u(A(t, a))\,\omega^{(y,s)}(\Delta(t, a))$$

whenever $(y, s) \in \Omega$ satisfies $|s - t| > a/8$ or $|y - F(t)| \geqslant M\sqrt{a}$.

We may view Lemma 2 as a quantitative version of the Harnack inequality.

Given $0 < \varepsilon < 10^{-4}$ and $N \geqslant \varepsilon^{-4}$, let $f(t)$ be the function defined in Lemma 1, $F(t) = 2\sqrt{2}f(t)$ *and* $\Omega = \{x > F(t)\}$. It follows from (a) in Lemma 1 that $|F(t) - F(s)| \leqslant 9|t - s|^{1/2}$ for all s, t. We fix a reference point (100, 100), and denote $\omega^{(100,100)}$ by ω.

LEMMA 3. *There is an absolute constant $c_0 > 0$, so that whenever $\tau = mN^{-2k}$ with $m = 0, 1, \ldots, N^{2k} - 1$,*

$$I_k = \{(F(t), t): |t - \tau| < N^{-2k}\} \text{ and}$$

$$E_k = \{(F(t), t): 0 < t - \tau < \varepsilon N^{-2k}\},$$

we have $\omega(E_k) \leqslant c_0 \varepsilon^{3/2}\omega(I_k)$.

Proof. For a fixed $\tau = mN^{-2k}$, we let $A = (F(t) + 5N^{-k}, \tau + \frac{1}{8}N^{-2k})$, $J_k = \{(F(t), t): 0 < t - \tau < \frac{1}{4}N^{-2k}\}$, and $L_k = \{(F(t), t): |t - \tau| < \frac{1}{16}N^{-2k}\}$. Applying Lemma 2 with $M = 10$, $a = \frac{1}{16}N^{-2k}$ and $u = \omega(E_k)$, we obtain

$$\omega(E_k) \leqslant C_1\,\omega(L_k)\,\omega^A(E_k) \leqslant C_1\,\omega(I_k)\,\omega^A(E_k),$$

where C_1 is an absolute constant.

To estimate $\omega^A(E_k)$, we define Φ to be the transformation: $(x, t) \to (20\sqrt{\varepsilon} + (x - F(\tau)N^k, \varepsilon + (t - \tau)N^{2k})$. Hence $\Phi(A) = (20\sqrt{\varepsilon} + 5, \varepsilon + \frac{1}{8})$; and because of (b) and (c),

$$\Phi(E_k) \subset \{(x, t): \varepsilon \leqslant t \leqslant 2\varepsilon,\ 10\sqrt{\varepsilon} \leqslant x \leqslant 30\sqrt{\varepsilon}\},$$

$$\Phi(J_k) \subset \{(x, t): \varepsilon \leqslant t \leqslant \tfrac{1}{4} + \varepsilon,\ x > \sqrt{2t}\}.$$

Since Φ preserves parabolic functions, $u(Q) \equiv \omega^{\Phi^{-1}(Q)}(E_k)$ is the parabolic measure of $\Phi(E_k)$ with respect to the domain $\Phi(\Omega)$, and $u(\Phi(A)) = \omega^A(E_k)$. Because $u = 0$ on $\partial\Phi(\Omega) \cap \{t < \varepsilon\}$, we have $u = 0$ on $\Phi(\Omega) \cap \{t = \varepsilon\}$. Let

$$\begin{aligned} K(x, t) &= \frac{\partial}{\partial t} W(x, t; 0, 0) \\ &= \frac{1}{\sqrt{4\pi}}\, t^{-3/2}\left(\frac{x^2}{4t} - \frac{1}{2}\right) e^{-(x^2/4t)}, \quad t > 0, \end{aligned}$$

which is parabolic for $t > 0$, and is positive when $x > \sqrt{2t}$. When $(x, t) \in \Phi(E_k)$, $K(x, t) \geqslant c_2\varepsilon^{-3/2}$ for some absolute constant $c_2 > 0$. Applying the maximum principle to $c_2^{-1}\varepsilon^{3/2}K$ and u over the region $\Phi(\Omega) \cap \{\varepsilon \leqslant t \leqslant \varepsilon + \frac{1}{4}\}$, we have

$$\begin{aligned} u(\Phi(A)) &\leqslant C_2^{-1}\varepsilon^{3/2}K(20\sqrt{\varepsilon} + 5, \varepsilon + \tfrac{1}{8}) \\ &\leqslant C_2^{-1}\varepsilon^{3/2}\sup\{K(x, t): 5 \leqslant x \leqslant 6, \tfrac{1}{8} \leqslant t \leqslant \tfrac{1}{4}\} = C_3\varepsilon^{3/2}, \end{aligned}$$

where C_3 is an absolute constant. Since $\omega^A(E_k) = u(\Phi(A))$, we conclude the lemma.

REMARK. If we choose $x = Bf(t)$, with $B > 2\sqrt{2}$, in the construction of Ω, the domain has a bigger Lip $\frac{1}{2}$ constant. We need then to use higher partials $(\partial^n/\partial t^n)W(x, t; 0, 0)$ as the comparison functions in estimation and obtain $\omega(E_k) \leqslant C_B\varepsilon^{\varrho_B}\omega(I_k)$ in the lemma with $C_B > 0$ and $\varrho_B > 3/2$ depending on B.

4. Conclusion

LEMMA 4. *Suppose that ε is the reciprocal of a positive even integer with $\varepsilon < \min\{10^{-4}, (2c_0)^{-2}\}$, c_0 as in Lemma 3, and that $N = \varepsilon^{-4}$. Then there exist sets T, $T^* \subseteq (-\infty, \infty)$ of dimension strictly less than 1, so that ω and ω^* are supported on $\{x = F(t): t \in T\}$ and $\{x = F(t): t \in T^*\}$ respectively. Moreover T and T^* can be chosen to be disjoint.*

Proof. Because $F(t) = F(-t)$, the conclusion for $\omega *$ follows from that for ω by symmetry.

Because a set of the form $\{x = F(t): t \in E\}$, with $|E| = 0$, can be written as the union of two disjoint sets E_1 and E_2 with $\omega(E_1) = 0$ and $\omega*(E_2) = 0$ at any point in Ω [9]; we may modify T and T^* to become disjoint after we prove their existence.

Because F has period one, we need only to study the support of ω on $\partial\Omega \cap \{0 \leqslant t \leqslant 1\}$. We shall fix the reference point (100, 100) and denote by $\omega = \omega^{(100,100)}$.

An increasing sequence $A_0 \subset A_1 \subset A_2 \subset \ldots$ of algebras of subsets of $[0, 1)$ is defined as follows. A_k is the algebra generated by the intervals $[2pN^{-2k}, (2p + 2)N^{-2k})$, where p is an integer and $0 \leqslant 2p \leqslant N^{2k} - 2$. ($A_0$ is the trivial algebra). Since $2pN^{-2k} = 2pN^2 \cdot N^{-2k-2}$, we have $A_k \subset A_{k+1}$. Let $\tau = (2p + 1)N^{-2k}$ as above; then the interval $[\tau, \tau + \varepsilon N^{-2k})$ belongs to the algebra A_{k+1} if $(2p + 1)N^{-2k} + \varepsilon N^{-2k} = 2qN^{-2k-2}$ defines an integer q. Now $q = N^2(2p + 1 + \varepsilon) = \varepsilon^{-8}(2p + 1 + \varepsilon)$, so that q is indeed an integer. The interval $[\tau, \tau + \varepsilon N^{-2k})$, called $B(p, k)$, is contained in a basic interval $\tilde{B}(p, k)$ of A_k and $\lambda(B(p, k)) \leqslant c_0\varepsilon^{3/2}\lambda(\tilde{B}(p, k))$, where λ is the normalized projection of ω on $[0, 1)$ with $\lambda([0, 1)) = 1$. Let B_k be the union of the sets $B(p, k)$, $0 \leqslant p \leqslant N^{2k} - 2$; the conditional probability $\lambda(B_k | A_k) \leqslant c_0\varepsilon^{3/2}$.

Let f_k be the characteristic function of B_k. Hence $g_k \equiv f_k - E(f_k | A_k)$ defines an orthogonal sequence with $|g_k| \leqslant 1$. Using the orthogonality, and Chebyshev's inequality as in [4] we see that $g_2 + g_4 + \cdots + g_{2r} = o(r)$ λ-almost everywhere, or $f_2 + f_4 + \cdots + f_{2r} \leqslant c_0\varepsilon^{3/2}r + o(r)$ λ – a.e. Thus for λ-almost all t, the number $n(r, t)$ of integers $k \leqslant r$, such that $t \in B_{2k}$, is at most $c_0\varepsilon^{3/2}r + o(r)$. (The number c_0 retains the same value.) Fix δ, with $c_0\varepsilon^{3/2} < \delta < \varepsilon/2$ and let $E_m = \{t \in [0, 1): n(r, t) \leqslant \delta r$, for every $r \geqslant m\}$. Then $\lambda(\bigcup_1^\infty E_m) = 1$.

Let $t = \Sigma_1^\infty C_k(t)N^{-2k}$ be the expansion in base N^2, excepting the rational numbers. Then $t \in B_{2k}$ if and only if $C_{2k}(t)$ is odd and $0 \leqslant C_{2k+1}(t) < \varepsilon N^2$.

For large r, E_m is contained in a union of K basic intervals of A_{2r}, where

$$K = O(1) \sum_{n=0}^{[\delta r]} \binom{r}{n} \left(\frac{\varepsilon N^4}{2}\right)^n \left[\left(1 - \frac{\varepsilon}{2}\right) N^4\right]^{r-n}$$

$$= O(1) M^{4r}, \text{ with a constant } M < N.$$

This may be seen as follows. Fixing m, $r \geqslant m$, and $0 \leqslant n \leqslant \delta r$, we consider those $t \in E_m$, so that $n(r, t) = n$, i.e., those $t \in E_m$, such that the event $t \in B_{2k}$ occurs for exactly n values of k, $0 \leqslant k \leqslant r$. These n places can be chosen in $\binom{r}{n}$ ways. When $t \in B_{2k}$, the number of choices for C_{2k} and C_{2k+1} is $(\varepsilon/2)N^4$; when $t \notin B_{2k}$, the number of choices for C_{2k} and C_{2k+1} is $N^4 - (\varepsilon/2)N^4$. This yields the first estimation for K.

To obtain the second estimation, we use Stirling's formula $n! \approx n^n e^{-n}\sqrt{n}$, for large n. The largest term occurs when $n \sim \delta r$ (since $\delta < \varepsilon/2$) and is approximately $N^{4r}\eta^r$, where $\eta = \delta^{-\delta}(1 - \delta)^{-(1-\delta)}(\varepsilon/2)^\delta(1 - (\varepsilon/2))^{1-\delta}$. Now η depends on δ, and increases up to $\eta = 1$ when $\delta = \varepsilon/2$. Especially $\eta < 1$ when $\delta < \varepsilon/2$. From the definition of Hausdorff dimension, we conclude that $\bigcup_1^\infty E_m$ has dimension at most $\log M/\log N < 1$. (Apart from the details introduced to make the σ-algebras match up, the method is due to Besicovitch [1]; certain choices of successive digits in the expansion occur with a frequency different from – in fact, less than – the natural frequency. This pushes the dimension below 1.)

Therefore the domain $\{x < F(t)\}$ so constructed has all the properties promised at the beginning.

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