FABIO BARDELLI
LUISELLA VERDI

On osculating cones and the Riemann-Kempf singularity theorem for hyperelliptic curves, trigonal curves, and smooth plane quintics

Compositio Mathematica, tome 65, n° 2 (1988), p. 177-199

<http://www.numdam.org/item?id=CM_1988__65_2_177_0>
On osculating cones and the Riemann–Kempf singularity theorem for hyperelliptic curves, trigonal curves, and smooth plane quintics

FABIO BARDELLI & LUISELLA VERDI

1 Dipartimento di Matematica, Università, Strada Nuova 65, 27100 Pavia, Italy;
2 Istituto Matematico “U. Dini”, Viale Morgagni 67-A, 50134 Firenze, Italy

Received 31 December 1986; accepted in revised form 2 October 1987

Introduction

Let $C$ be a smooth projective curve of genus $g \geq 3$, $\varphi_k : C \to \mathbb{P}^{g-1}$ its canonical map, $\tilde{C} = \varphi_1(C)$, $I(\tilde{C}) = \bigoplus_d I_d(\tilde{C})$ the homogeneous graded ideal of $\tilde{C}$. It is well known that: $\varphi_k$ is an embedding and $I(\tilde{C})$, by the Enriques–Petri theorem, is spanned by $I_2(\tilde{C})$ except in the following cases (see [S.D.] or [A.C.G.H.]):

i) $C$ is hyperelliptic: it is the only case in which $\varphi_k$ is not an embedding, but it is composed with the unique $g^1_2$ on $C$.

ii) $C$ is trigonal: in this case (and for $g \geq 5$) $I(\tilde{C})$ is spanned by $I_2(\tilde{C})$ and $I_3(\tilde{C})$. The variety defined by $I_2(\tilde{C})$ is the smooth rational 2-dimensional scroll $R$ spanned by the trisecants of $\tilde{C}$. Each trisecant intersects $\tilde{C}$ in a divisor of the unique $g^1_3$ on $\tilde{C}$;

iii) $C$ is a plane quintic: in this case $I(\tilde{C})$ is spanned by $I_3(\tilde{C})$ and $I_5(\tilde{C})$. The variety defined by $I_3(\tilde{C})$ is the Veronese surface $S$ in $\mathbb{P}^5$, which is spanned by the conics passing through any five coplanar points of $\tilde{C}$. The 5-tuples of coplanar points of $\tilde{C}$ constitute the unique $g^2_5$ on $\tilde{C}$.

The exceptional cases i), ii), iii) are due to the presence on $C$ of a unique $g^1_2, g^1_3, g^2_5$ respectively.

Let $C^k$ be the cartesian product of $C$ $k$-times, $C^{(k)}$ be the symmetric product of $C$ $k$-times, $J(C)$ be the jacobian of $C$, $\mu_k : C^k \to J(C)$ be the Abel–Jacobi map, $W_k = \mu_k(C^k) \subseteq J(C)$. By the Riemann–Kempf singularity theorem (denoted by R.K.t. in the sequel, see [K] or [A.C.G.H.]), the $g^1_2$ is represented by a (unique) singular point $T_2$ of $W_2$, the $g^1_3$ is represented by a (unique) singular point $T_3$ of $W_3$, the $g^2_5$ is represented by a (unique) triple point $T_5$ of $W_5$. If we denote by $TC_{T_i} W_i$ the tangent cone to $W_i$ at $T_i$, we have that (by R.K.t.): $\mathbb{P} TC_{T_2} W_2 = \tilde{C}$; $\mathbb{P} TC_{T_3} W_3 = R$; $\mathbb{P} TC_{T_5} W_5 = \text{Ch}(S)$ where
Ch(S) is the chordal variety of the Veronese surface S; in the cases i), ii), iii), respectively. We note that $\text{SingCh}(S) = S$ (see 4.0 iii)).

In this paper we study the osculating cones of order $r$ $\tilde{\mathcal{O}}C(r)_{T_i}(W_i)$ at $T_i$; they are schemes whose underlying points sets, denoted simply by $OC(r)_{T_i}(W_i)$, are constituted by the points of the lines of $T_i(W_i)$ whose intersection multiplicity with $W_i$ at $T_i$ is greater than or equal to $r$. We introduce them in Section 1, where some useful properties are reviewed. Sections 2, 3 and 4 are devoted to the proof of the following

**Proposition 1.** For $C$ hyperelliptic, $g \geq 3$ one has
i) $\mathbb{P}\tilde{\mathcal{O}}C(3)_{T_3}(W_3) = \mathbb{P}\tilde{\mathcal{O}}C(4)_{T_3}(W_3) = \tilde{C}$;
ii) $\mathbb{P}\tilde{\mathcal{O}}C(5)_{T_5}(W_5) = \{\varphi_k(B_i), \ldots, \varphi_k(B_{2g+2})\}_{\text{red}}$ where $B_i = 1, \ldots, 2g + 2$ are the ramification points of the double cover $\pi: C \rightarrow \mathbb{P}^1$ associated to the $g_2^1$.

**Proposition 2.** For $C$ trigonal, $g \geq 4$, and with two distinct $g_1^1$'s if $g = 4$, one has
i) $\mathbb{P}\tilde{\mathcal{O}}C(3)_{T_3}(W_3) = R$;
ii) $\mathbb{P}\tilde{\mathcal{O}}C(4)_{T_3}(W_3) = \tilde{C}$.

**Proposition 3.** For $C$ a plane quintic one has
i) $\mathbb{P}\tilde{\mathcal{O}}C(4)_{T_5}(W_5) = \mathbb{P}\tilde{\mathcal{O}}C(5)_{T_5}(W_5) = \text{Ch}(S)$;
ii) $\mathbb{P}\tilde{\mathcal{O}}C(6)_{T_5}(W_5) \cap S = \tilde{C}$ counted twice.

In each of the previous cases $C$ can be reconstructed from some of the osculating cones $\tilde{\mathcal{O}}C(r)_{T_i}(W_i)$ and their singular loci, in particular for $g = 3$ in the hyperelliptic case, $g = 4$ in the trigonal case, and in the plane quintic case the results above imply the Torelli theorem for these families of curves. The results of Section 2 and 3 of this paper have been announced in [B.V.].

1. Osculating cones and some useful properties

Let $U = \{z \in \mathbb{C}^n: |z| < \varepsilon\}$, $W \subseteq U$ an analytic variety defined by an ideal $I(W)$ of holomorphic functions on $U$ and with $0 \in W$. Let $\gamma: \Delta \rightarrow U$ be an analytic arc of curve with $\gamma(0) = 0$, $\forall f \in I(W)$ $f \circ \gamma(0) = 0$. The intersection multiplicity of $W$ and $\gamma$ at 0 is defined as:

$$ (W \cdot \gamma)_0 = \min_{f \in I(W)} \{\text{ord}_0 (f \circ \gamma)\} $$

(1.1)

(see Sh., p. 73).
For any $f \in I(W)$ we write the power series expansion of $f$ at 0 as

$$f = \sum_{i=1}^{\infty} f_i \text{ with } f_i \text{ homogeneous polynomial } \deg f_i = i.$$  \hfill (1.2)

**DEFINITION 1.3.** The osculating cone of order $r$ to $W$ at 0, denoted by $\tilde{OC}(r)_0(W)$ henceforth, is the scheme defined by the ideal spanned by the following set of forms:

$$\{f_k : \forall f \in I(W), \forall k < r\}.$$  \hfill (1.4)

Then we have:

The set of points underlying the scheme $\tilde{OC}(r)_0(W)$ will be denoted simply by $OC(r)_0(W)$ and is equal to:

$$\{v \in C^n : \text{the line } l = \{\lambda \cdot v \}_{\lambda \in \mathbb{C}} \text{ is such that } (W \cdot l)_0 \geq r\}.$$  \hfill (1.5)

Let $TC_0W$ be the tangent cone (as scheme) to $W$ at 0 and $ITC_0W$ its defining ideal (which is spanned by the initial forms $f^m \forall f \in I(W)$). We have:

If $ITC_0W$ is spanned by forms of degree $k$, then $\forall Q \in ITC_0(W)$ with $\deg Q = k$, there exists $f \in I(W)$ such that $Q = f^m$. \hfill (1.6)

Let $k = \min_{f \in I(W)} \{\deg f^m\}$, then $\tilde{OC}(k + 1)_0(W) \supseteq TC_0W$; and if $ITC_0W$ is spanned by forms of degree $k$ one has

$$\tilde{OC}(k + 1)_0(W) = TC_0W.$$  \hfill (1.7)

### 2. The hyperelliptic case

Let $C$ be a hyperelliptic curve of genus $g \geq 3$, $\pi: C \to \mathbb{P}^1$ the double cover associated to the unique $g_2^1$ on $C$. It is known that $|K_C| = \Sigma_{g-1} g_2^1$. Let $P$ be a ramification point of $\pi$, so $2P \in g_2^1$ and $(2g - 2)P \in |K_C|$. Let $\sigma \in H^0(C, \mathcal{O}_C(2P))$ with $\text{div}(\sigma) = 2P$ and $\omega = \sigma^{g-1} \in H^0(C, \mathcal{O}_C(K_C))$, so that $\text{div}(\omega) = (2g - 2)P$. Let $\sigma, \tau$ be a basis for $H^0(C, \mathcal{O}_C(2P))$ and $f = \tau/\sigma \in \mathcal{M}(C)$ be the rational function giving the map $\pi$. It is easy to see that $q = \sigma^{g-2} \in H^0(C, \mathcal{O}_C(K_C \setminus -2P))$ and that $\{q, qf, \ldots, qf^{g-2}\}$ is a basis for
$H^0(C, \mathcal{O}_C(K_C - 2P))$. In the same way one checks that $\{\omega, \omega f, \ldots, \omega f^{g-1}\}$ is a basis for $H^0(C, \mathcal{O}_C(K_C))$. We will set $\omega_i = \omega f^i$, $i = 0, \ldots, g - 1$. Let $i: C \to C$ be the hyperelliptic involution of $C$. The set of holomorphic differentials

$$V = \{x \in H^0(C, \mathcal{O}(K_C)) | \text{div}(x) = (2g - 2)P\}$$

is a 1-dimensional vector space containing $\omega$ and invariant for $i^*$, so from $i^2 = id_{|C}$ it follows $i^*(\omega) = -\omega$ (see [G.H.]). It is also clear that $i^*(f^k) = f^k \forall k \in \mathbb{N}$ and in particular $i^*(\omega_j) = -\omega_j = 0, \ldots, g - 1$.

The Abel–Jacobi map $\mu_2: C^2 \to J(C)$ is given by

$$C^2 \ni (P_1, P_2) \mapsto \left(\sum_{i=1}^{g} \int_{P_i}^{P_j} \omega_0, \ldots, \sum_{i=1}^{g} \int_{P_i}^{P_j} \omega_{g-1}\right) \in J(C)$$

where $P$, the base point, is the point chosen above. Let $\Gamma = \{(P_1, P_2) \in C^2 : P_2 = i(P_1)\}$. Then it is clear that $\mu_2(\Gamma) = 0 \in J(C)$ and so $T_2$, the singular point of $W_2 = C^2$, is 0.

**Proof of Proposition 1.** Over an open set $U_k \subset C$ with local coordinate $t_k$, we will call $\Omega(t_k) dt_k$ the local expression of $\omega_i$. Let us assume that $U_2 = i(U_1)$ and that $i: U_1 \to U_2$ is given by $t_2 = -t_1$, so that

$$\Omega^2(t_2) = \Omega^1(t_1) \quad \text{for} \quad (t_1, t_2) \in (U_1 \times U_2) \cap \Gamma.$$  \hspace{1cm} (2.1)

Let $(P_1, P_2) \in U_1 \times U_2$, then $\forall \theta \in I(W_2 \cap A)$, where $A \subset J(C)$ is a sufficiently small open neighborhood of $0 \in J(C)$, and after eventually shrinking $U_1 \times U_2$ in such a way that $\mu_2(U_1 \times U_2) \subset A$, we have $g = \theta \circ \mu_2 \equiv 0$ over $U_1 \times U_2$ and so $\partial^{h+k} g / \partial t_1^h \partial t_2^k = 0$ over $U_1 \times U_2$. We let $\theta_i, \theta_{ij}, \theta_{ijk}$ and so on denote the partials of $\theta$ with respect to the variables carrying the lower indices.

**Remark 2.2.** $\forall \theta \in I(W_2 \cap A)$ we have $\theta_i(0) = 0$, $i = 0, \ldots, g - 1$: in fact by R.K.t. (see [K] or [A.C.G.H.]) $\mathbb{P}TC_0 W_2$ is the rational normal curve $\hat{C}$ in $\mathbb{P}^{g-1}$: in particular it is not degenerate (not contained in any hyperplane).

We want to evaluate $\partial^{h+k} g / \partial t_1^h \partial t_2^k$ at one point $(t_1, t_2) \in (U_1 \times U_2) \cap \Gamma$, therefore the $\theta$’s and their partials will be evaluated at 0; and from the relation (2.1), after setting $t_1 = t$ and leaving out the upper indices of the
\( \Omega^i(t) \)'s, one gets:

\[
\frac{\partial^2 g}{\partial t_i \partial t_j} \bigg|_{t_2 = -t_1} = \sum_{i,j=0}^{g-1} \partial_{ij}(0) \Omega_i(t) \Omega_j(t) = 0. \quad (2.3)
\]

\[
\frac{\partial^3 g}{\partial t_i \partial t_j \partial t_k} \bigg|_{t_2 = -t_1} = \sum_{i,j,k=0}^{g-1} \partial_{ijk}(0) \Omega_i(t) \Omega_j(t) \Omega_k(t) + \sum_{i,j=0}^{g-1} \partial_{ij}(0) \Omega'_i(t) \Omega_j(t) = 0. \quad (2.4)
\]

\[
\frac{\partial^4 g}{\partial t_i \partial t_j \partial t_k \partial t_l} \bigg|_{t_2 = -t_1} = \sum_{i,j,k,l=0}^{g-1} \partial_{ijkl}(0) \Omega_i(t) \Omega_j(t) \Omega_k(t) \Omega_l(t) + 2 \sum_{i,j,k=0}^{g-1} \partial_{ijk}(0) \Omega'_i(t) \Omega'_j(t) \Omega_k(t) + \sum_{i,j=0}^{g-1} \partial_{ij}(0) \Omega'_i(t) \Omega'_j(t) = 0. \quad (2.5)
\]

By differentiating (2.3) and after interchanging indices one can see that the second summand of (2.4) is 0 ∀ \( t \) and so that

\[
\sum_{i,j,k=0}^{g-1} \partial_{ijk}(0) \Omega_i(t) \Omega_j(t) \Omega_k(t) = 0 \quad (2.6)
\]

By differentiating (2.6) one gets easily that the second summand of (2.5) is 0 ∀ \( t \) and so

\[
\sum_{i,j,k,l=0}^{g-1} \partial_{ijkl}(0) \Omega_i(t) \Omega_j(t) \Omega_k(t) \Omega_l(t) + \sum_{i,j=0}^{g-1} \partial_{ij}(0) \Omega'_i(t) \Omega'_j(t) = 0. \quad (2.7)
\]

(2.3) and (2.6) say that

\[
\tilde{C} \subset \mathbb{P}\tilde{O}\mathcal{C}(4)_0(W_2) \subset \mathbb{P}\tilde{O}\mathcal{C}(3)_0(W_2). \quad (2.8)
\]
But $\mathbb{P}\tilde{O}C(3)_{0}(W_{2}) = \mathbb{P}TC_{0}W_{2}$ because by R.K.t. $I(TC_{0}W_{2})$ is spanned by quadrics and so (1.7) applies. On the other hand $\mathbb{P}TC_{0}W_{2}$ is the rational normal curve $\tilde{C}$ and so, in view of (2.8), one gets i) of Prop. 1. Now we want to show that the set of ramification points of $\pi$ is exactly the set of common zeroes of all the second summands $\Sigma_{g}$ of (2.7), for $g$ varying in $I(W_{2} \cap A)$. Let $U$ be the open set of $C$ endowed with the local coordinate $t$ introduced above and $\Omega(t)dt = \omega_{i}$. Since $\omega_{i} = \omega f^{i}$ by substituting in (2.3), we get $\Omega^{2}(t) \Sigma_{g}^{-1} \delta_{ij}(0) f^{i+j}(t) = 0$ and, from $\Omega^{2}(t) \neq 0$, we find

$$\sum_{i+j=0}^{g-1} \delta_{ij}(0) f^{i+j}(t) = 0. \quad (2.9)$$

By differentiating we get

$$\sum_{i+j=0}^{g-1} \delta_{ij}(0) (i + j) f^{i+j-1}(t) = 0. \quad (2.10)$$

$\Sigma_{g}$ on $U$ is equal to

$$\sum_{i+j=0}^{g-1} \delta_{ij}(0) [(\Omega'(t))^{2} f^{i+j}(t) + (i + j)(\Omega'(t)\Omega(t)f'(t)f^{i+j-1}(t)$$

$$+ ij\Omega(t)^{2}f'(t)^{2}f^{i+j-2}(t)]. \quad (2.11)$$

In view of (2.9) and (2.10), (2.11) is simply

$$\Omega(t)^{2}f'(t)^{2} \sum_{i+j=0}^{g-1} \delta_{ij}(0) ij f^{i+j-2}(t). \quad (2.12)$$

By R.K.t. the ideal of $TC_{0}W_{2}$ is spanned by the minors of the matrix

$$\begin{vmatrix}
\sigma & \omega & \omega f & \omega f^{2} & \ldots & \omega f^{g-2} \\
0 & \sigma f & \sigma f^{2} & \ldots & \sigma f^{g-2}
\end{vmatrix}$$

$$\tau = \sigma \cdot f
\begin{vmatrix}
\omega f & \omega f^{2} & \omega f^{3} & \ldots & \omega f^{g-1}
\end{vmatrix}$$

(in fact $\omega = \sigma \varphi$) that is, after setting $z_{i} = \omega_{i}$, by the minors of the matrix

$$\begin{pmatrix}
z_{0}, z_{1}, \ldots, z_{g-2} \\
z_{1}, z_{2}, \ldots, z_{g-1}
\end{pmatrix}.$$
It follows that $z_0z_2 - z_1^2 \in I(TC_0W_2)$ and so, by (1.6), there exists a $\mathfrak{g} \in I(W_2 \cap A)$ such that $z_0z_2 - z_1^2 = \mathfrak{g}^n$. Therefore $\mathfrak{g}_{0,2}(0) = 1$, $\mathfrak{g}_{1,1}(0) = -2$, $\mathfrak{g}_{ij}(0) = 0$ for all the other indices $i, j$. (2.12), for $\mathfrak{g} = \mathfrak{g}$, is equal to

$$-2\Omega(t)^2(f'(t))^2. \quad (2.13)$$

One can see that (2.13) is zero exactly at the ramification points $B_1, \ldots, B_{2g+2}$ of $\pi$: on $C \setminus P$ this is obvious because $\Omega(t)$ never vanishes on $C \setminus P$; at $P$ we have $\text{ord}_P \Omega(t) = 2g - 2$, $\text{ord}_P f(t) = -2$, $\text{ord}_P f'(t) = -3$ and therefore

$$\text{ord}_P(\Omega(t)f'(t))^2 = 4g - 10 > 0 \text{ for } g \geq 3 \quad (2.14)$$

so (2.13) vanishes at $P$. Thus it suffices to show that $\forall \mathfrak{g} \in I(W_2 \cap A)$, (2.12) is 0 at $B_1, \ldots, B_{2g+2}$. Let $B_i$ be one of the ramification points $B_i \neq P$: $f$ is holomorphic in a neighborhood $Y$ of $B_i$ so $\sum_{j=0}^{g-1} \mathfrak{g}_{ij}(0) ijf^{i+j-2}(t)$ is holomorphic on $Y$, and therefore (2.12) is zero at $B_i$ because it contains the factor $f'(t)^2$ which vanishes to second order at any ramification point that is regular for $f$. We now compute

$$\text{ord}_P \left\{ \left(\Omega(t)f'(t)\right)^2 \sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0) ijf^{i+j-2}(t) \right\}. \quad (2.15)$$

If $U$ is a neighborhood of $P$ and the local coordinate $t$ is such that $f|_U = 1/t^2$, the relation (2.9) becomes

$$\sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0)t^{-2(i+j)} = 0 \quad (2.15)$$

and from this we deduce

$$\mathfrak{g}_{g-1,g-2}(0) = \mathfrak{g}_{g-1,g-1}(0) = 0. \quad (2.16)$$

From (2.16) the lowest degree for $t$ in $\sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0) ijf^{i+j-2}$ is $-4g + 12$, and so by (2.14)

$$\text{ord}_P \left\{ \left(\Omega(t)f'(t)\right)^2 \sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0) ijf^{i+j-2}(t) \right\} \geq 2. \quad (2.17)$$
Therefore (2.12) is zero at $P \forall \vartheta \in I(W_2 \cap A)$. It follows that the first summands of (2.7), for $\vartheta$ varying in $I(W_2 \cap A)$, vanish simultaneously exactly at the ramification points of $\pi$, and so we get that $\mathcal{P}OC(5)_T(W_2) = \{\varphi_k(B_1), \ldots, \varphi_k(B_{2g+2})\}$ as points sets. Moreover by the previous arguments, (2.13) and (2.17)

$$\min_{\vartheta \in I(W_2 \cap A)} \{|\vartheta\}| = 2 \quad \text{for } i = 1, \ldots, 2g+2$$

(at $P$ take for instance $\vartheta_2 = \begin{pmatrix} z_{g-3} & z_{g-2} \\ z_{g-2} & z_{g-1} \end{pmatrix}$),

so, since $\varphi_k$ has degree 2, we get

$$\min_{\vartheta \in I(W_2 \cap A)} \{|(\tilde{C} \cdot \{\vartheta_4 = 0\})_{\varphi_k(B_i)}\} = 1$$

for $i = 1, \ldots, 2g+2$, whence ii) of Prop. 1 is easily deduced.

**Remark 2.18.** Proposition 1 gives the Torelli theorem for the family of hyperelliptic curves of genus 3.

### 3. The trigonal case

Let $C$ be a trigonal curve of genus $g \geq 4$ and with two distinct $g_1^1$’s if $g = 4$. Let $\omega_0, \ldots, \omega_{g-1}$ be a basis for $H^0(C, \mathcal{O}_C(K_C))$, $P_0$ be a base point on $C$. The Abel–Jacobi map $\mu_3: C^3 \to J(C)$ is defined by

$$C^3 \ni (P_1, P_2, P_3) \mapsto \left( \sum_{i=1}^{3} \int_{P_0}^{P_i} \omega_0, \ldots, \sum_{i=1}^{3} \int_{P_0}^{P_i} \omega_{g-1} \right) \in J(C).$$

Let $\Gamma = \{(P_1, P_2, P_3) \in C^3: P_1 + P_2 + P_3 \in g_1^1 \text{ (a fixed } g_1^1\})\}$. $\mu_3(\Gamma)$ is a singular point $T_3$ of $W_3 = \mu_3(C^3)$ and, after modifying $\mu_3$ by a suitable translation, we will assume that $T_3 = 0 \in J(C)$.

**Proof of Proposition 2.** Over an open set $U_k \subseteq C$ with local coordinate $t_k$ we will denote by $\Omega^k(t_k) dt_k$ the local expression of $\omega_i$. Let $(P_1, P_2, P_3) \in U_1 \times U_2 \times U_3$ and let $\gamma: \Delta \to U_1 \times U_2 \times U_3$ be an analytic arc of curve given by $t_i = h_i \cdot s + \tilde{t}_i, i = 1, 2, 3$ where $h_i \in \mathbb{C}$, $\tilde{t}_i \in U_i$, $s \in \Delta$, and $(t_1, t_2, t_3) \in \Gamma$. 
It is clear that $\forall \mathfrak{g} \in I(W_3 \cap A)$, where $A \subseteq J(C)$ is a sufficiently small open neighborhood of $0 \in J(C)$, we have $g = \mathfrak{g} \circ \mu_3 \circ \gamma \equiv 0$ over $\Delta$ and so also $d^{(n)}g/ds^n \equiv 0$ on $\Delta$. We let $\mathfrak{g}_i, \mathfrak{g}_{ij}, \mathfrak{g}_{ijk}$ be the partials of $\mathfrak{g}$ as in the proof of Prop. 1 and we let

$$\psi_i(s) = \sum_{j=1}^{3} \Omega_j^i(t_j(s))h_j.$$  

(REMARK 3.2. $\forall \mathfrak{g} \in I(W_3 \cap A)$ $\mathfrak{g}_i(0) = 0$ $i = 0, \ldots, g - 1$. In fact $\mathbb{P}TC_0 W_3$ by R.K.t. is the smooth rational ruled surface $R$ spanned by the trichords of $\tilde{C}$ in $\mathbb{P}^{g-1}$: in particular $R$ is not degenerate.

We evaluate the following derivatives at $s = 0$ for an arc $\gamma$ such that $\gamma(0) = (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in \Gamma$ (it is therefore understood that the $\mathfrak{g}$'s and their partials will be evaluated at $0 = \mu_3 \circ \gamma(0)$ and each $\Omega^j_i(t_j)$ will be evaluated at $\tilde{t}_j$):

$$\frac{d^2 g}{ds^2} \bigg|_{s=0} = \sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0)\psi_i(0)\psi_j(0) = 0$$

(3.3)

and if

$$s_1 = \sum_{i,j,k=0}^{g-1} \mathfrak{g}_{ijk}(0)\psi_i(0)\psi_j(0)\psi_k(0);$$

$$s_2 = 3 \sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0)\psi_i(0)\psi_j(0),$$

$$\frac{d^3 g}{ds^3} \bigg|_{s=0} = s_1 + s_2 = 0.$$  

(3.4)

We note that (3.3) and $s_1 i = 1, 2$ are homogeneous polynomials in $(h_1, h_2, h_3)$ of degree 2, 3 respectively. We evaluate (3.3) and (3.4) for $h_1 = 1$ and $h_2 = h_3 = 0$, thus getting:

$$\sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0)\Omega_j^1(\tilde{t}_j)\Omega_i^1(\tilde{t}_i) = 0$$

(3.5)

$$\sum_{i,j,k=0}^{g-1} \mathfrak{g}_{ijk}(0)\Omega_j^1(\tilde{t}_j)\Omega_k^1(\tilde{t}_k) = 3 \sum_{i,j=0}^{g-1} \mathfrak{g}_{ij}(0)\frac{d\Omega_j^1(\tilde{t}_j)}{dt_1} \Omega_i^1(\tilde{t}_i) = 0.$$  

(3.6)
(3.5) and (3.6) tell us that:

\[ \mathcal{C} \subset \mathbb{P} \tilde{\mathcal{C}}(4)_0(W_3) \subset \mathbb{P} \tilde{\mathcal{C}}(3)_0(W_3). \]  

(3.5) and (3.7) hold \( \forall \tilde{t} \in U_j \); by differentiating (3.5) one gets easily from (3.6)

\[
\sum_{\eta \kappa = 0}^{g-1} \partial_{\eta \kappa}(0) \Omega^\eta_{\eta'}(\tilde{t}_1) \Omega^\kappa_{\kappa'}(\tilde{t}_1) = 0
\]  

\[
(3.7)
\]

Since \( IT^C_0 W_3 \) is spanned by quadrics (by R.K.t.), we get from (1.7) that \( \mathbb{P} \tilde{\mathcal{C}}(3)_0(W_3) = \mathbb{P} T^C_0 W_3 = R \). So i) of Prop. 2 is proved. We want to prove that \( \mathbb{P} \tilde{\mathcal{C}}(4)_0(W_3) = \mathcal{C} \). For this it will suffice to show that:

For any trichord \( r \) of \( C, r \subseteq R \), there exists a \( e \in I(W_3 \cap A) \) such that the cubic polynomial

\[
c = \mathcal{g}_3 = \sum_{\eta \kappa = 0}^{g-1} \partial_{\eta \kappa}(0) z_\eta z_\kappa(3.9)
\]

is not identically zero on \( r \).

In fact if \( r \cap \mathcal{C} \) is a set of 3 distinct points \( c \) will be zero only at these points; if \( r \cap \mathcal{C} \) has multiple points of intersection (and this happens for finitely many trichords \( r \)), since \( \{c = 0\} \) cannot cut along \( R \) a divisor of the form \( \mathcal{C} + \sum_{i=1}^n r_i, plus \) a finite set of points (here \( r_i \) are some trichords of \( \mathcal{C} \subseteq R \)), \( c \mid_r \) will vanish exactly at the points of \( r \cap \mathcal{C} \). Moreover \( c \mid_r \) will vanish at each of these points with its corresponding multiplicity (this is easy to show by looking at what happens at a nearby trichord \( r' \)). In any case (3.9) implies by the above argument that locally over \( R \mathcal{C} \) is cut by a cubic hypersurface \( \{c = 0\} \) transversal to \( R \) and with \( c = \mathcal{g}_3 \) for a certain \( e \in I(W_3 \cap A) \); we get easily from this \( \mathcal{C} = \mathbb{P} \tilde{\mathcal{C}}(4)_0(W_3) \).

We now fix \( (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in \Gamma \) and the corresponding trichord \( r \) in \( \mathbb{P}^{g-1} \) in such a way that the set \( r \cap \mathcal{C} \) contains at least two distinct points, and note the following facts:

\[
z_i = \psi_i(0) \text{ is the } i-\text{th component of a vector } z \text{ in } T_{\eta} J(C) \simeq \mathbb{C}^g,
\]

whose representing point \( Z \in \mathbb{P}^{g-1} \) traces the line \( r \) as \( (h_1, h_2, h_3) \) vary in \( T_{(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3)} C^3 \), (this is the differential of \( \mu_3 \) at \( (\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \)).  

\[
(3.10)
\]

Let \( P_i \) be the point of \( C \) which is given in \( U_i \) by the value \( t_i = \tilde{t}_i \), \( \omega(P_i) \) be the vector \( (\Omega_0(\tilde{t}_1), \ldots, \Omega_{g-1}(\tilde{t}_i)) \), \( \omega'(P_i) \) be the vector \( ((d\Omega_i/dt)(\tilde{t}_i), \ldots, \).

(3.5) and (3.6) hold \( \forall \tilde{t} \in U_j \); by differentiating (3.5) one gets easily from (3.6)
\( (d\mathcal{O}_{x-y}/dt_1)(t_i) \), \( \Theta \) the matrix \( (\partial_i(0))_{i=0,\ldots,g-1} \).

\[
\begin{align*}
  s_2 &= (h_1^2, h_2^2, h_3^2)\cdot (\omega(P_1), \omega'(P_2), \omega'(P_3)) \cdot \Theta \\
  &= (\omega(P_1), \omega(P_2), \omega(P_3)) \cdot (h_1, h_2, h_3)
\end{align*}
\] (3.11)

\( \Theta \cdot z \) gives a linear form which is the equation of the tangent hyperplane to the quadric of equation \( z_i z_j = 0 \) at the point \( Z \) (and this holds also if the quadric is singular at \( Z \), in this case \( \Theta \cdot z = 0 \)). By varying \( \partial \) in \( I(W_3 \cap A) \), one gets a family \( \{ H_z \}_{z \in \Lambda} \) and one sees easily that \( \bigcap_{z \in \Lambda} H_z = \{ \text{the tangent plane to } R \text{ at } Z \} \).

It is well known that in the family of tangent planes to \( R \) at \( Z \), for \( Z \) varying in \( r \), any two tangent planes at distinct points of \( r \) are distinct.

An easy way to see (3.13) is to compute the tangent spaces to \( R \) from the parametrization \( (\lambda, t) \rightarrow f_h(t) + \lambda f_k(t) \) for the scroll \( R \), where \( f_h \) and \( f_k \) are parametrizations of degree \( h \), \( k \) rational normal curves which span disjoint linear spaces \( \mathbb{P}^k \) and \( \mathbb{P}^k \) in \( \mathbb{P}^{g-1} \) with \( h + k + 1 = g - 1 \).

In view of (3.4) and (3.10) to prove the statement (3.9), it will be enough to show that there exists a \( \lambda \in I(W_3 \cap A) \) such that \( s_2 \) is not identically 0 (on \( r \)). By assumption among \( P_1, P_2, P_3 \) at least two of the \( P' \)'s, let's say \( P_1 \) and \( P_2 \), are distinct and we may assume that \( P_2 \) is not a ramification point of the map \( C \rightarrow \mathbb{P}^1 \) given by the \( g_1^3 \). The coefficient of \( h_1 h_2^2 \) in \( s_2 \) is given by

\[
a_{21} = \omega'(P_2) \cdot \Theta \cdot \omega'(P_1).
\]

The product \( \omega(P_2) \cdot \Theta \cdot \omega'(P_1) \) is zero \( \forall \theta \in I(W_3 \cap A) \), because by taking all the linear forms \( \Theta \cdot \omega'(P_1) \) one gets the ideal of the tangent plane \( \pi_1 \) to \( R \) at \( \varphi_k(P_1) \) by (3.12), and \( \omega(P_2) \in r \subset \pi_1 \). The tangent line \( l_2 \) to \( C \) at \( \varphi_k(P_2) \) is given by parametric equations

\[
\lambda \omega(P_2) + \mu \omega'(P_2).
\]

Since \( l_2 \neq r \), \( l_2 \) cannot be contained in \( \pi_1 \) (otherwise \( \pi_1 = \) the tangent plane to \( R \) at \( \varphi_k(P_2) \) which is absurd by (3.13)), so we can choose \( \lambda \in I(W_3 \cap A) \) such that \( l_2 \notin \text{Ker } \{ \Theta \cdot \omega'(P_1) \} \). Then since \( l_2 = \{ \lambda \omega(P_2) + \mu \omega'(P_2) \} \) we get

\[
\tilde{a}_{21} = \omega'(P_2) \cdot \Theta \cdot \omega(P_1) \neq 0
\]

and (3.9) is proved in this case.
We are left to deal with the situation $P_1 = P_2 = P_3 = \bar{P}$; it is clear that we can take $U_1 = U_2 = U_3 = U$ and so $\forall i, j = 1, 2, 3$, $\Omega_i(t) = \Omega_j(t) = \Omega_k(t)$, since $t_i$ and $t_j$ are the same local coordinate on $U$. Here $t = 0$ corresponds to $\bar{P}$. By writing

$$\Omega_i(t) = \sum_{i=0}^{\infty} a_i^k t^i$$

and after setting

$$w_1 = \sum_{i=1}^{3} t_i, \quad w_2 = \sum_{i,j=1}^{3} t_i t_j, \quad w_3 = t_1 t_2 t_3,$$

and

$$\sum_{i=1}^{3} t_i = P_i(w_1, w_2, w_3),$$

one gets that the Abel–Jacobi map (with base point $\bar{P}$) $\mu_{(3)}: U^{(3)} \rightarrow J(c)$, where $U^{(3)}$ is the symmetric product of $U$ three times and $w_1, w_2, w_3$ are local coordinates on $U^{(3)}$, is given by:

$$(w_1, w_2, w_3) \mapsto \left( \ldots, \sum_{i=0}^{\infty} \frac{a_i^k}{l+1} P_{l+1}(w_1, w_2, w_3), \ldots \right).$$

We consider an analytic arc of curve $\gamma: \Delta \rightarrow U^{(3)}$ given by $w_i = h_i \cdot s$, with $\gamma(0) = (0, 0, 0) = (P_1, P_2, P_3) \in \Gamma$. $\forall \vartheta \in I(W_3 \cap A)$, $\tilde{\vartheta} = \vartheta \circ \mu_{(3)} \circ \gamma = 0$ on $\Delta$. We let

$$\tilde{\psi}_i(s) = \sum_{i=0}^{\infty} \frac{a_i^k}{l+1} \left( \sum_{r=1}^{3} \frac{\partial P_{l+1}}{\partial w_i} h_r \right),$$

$$s_1 = \sum_{i=0}^{\infty} \vartheta_{i}(0)\tilde{\psi}_i(0),$$

$$s_2 = \sum_{i=0}^{\infty} \vartheta_{i}(0)\tilde{\psi}_i'(0);$$

and so we get, (as before in (3.4)):

$$\left. \frac{d^3 g}{ds^3} \right|_{s=0} = s_1 + s_2 = 0. \quad (3.14)$$
REMARK 3.15. By using the identities

\[ P_1(w_1, w_2, w_3) = w_1, \quad P_2(w_1, w_2, w_3) = w_1^2 - 2w_2 \quad \text{and} \]
\[ P_n = w_1 P_{n-1} - w_2 P_{n-2} + w_3 P_{n-3} \]

one can compute easily all the isobaric polynomials \( P_n(w_1, w_2, w_3) \) and their partials at \((0, 0, 0)\). Here is a list of the ones we will use:

\[
\frac{\partial P_1}{\partial w_1} = 1, \quad \frac{\partial P_2}{\partial w_2} = -2, \quad \frac{\partial P_3}{\partial w_3} = 3, \quad \frac{\partial^2 P_2}{\partial w_1^2} = 2, \quad \frac{\partial^2 P_3}{\partial w_1 \partial w_2} = -3, \\
\frac{\partial^2 P_4}{\partial w_2^2} = 4, \quad \frac{\partial^2 P_4}{\partial w_1 \partial w_3} = 4, \quad \frac{\partial^2 P_5}{\partial w_2 \partial w_3} = -5, \quad \frac{\partial^2 P_6}{\partial w_3^2} = 6
\]

all the other 1st and 2nd order partials are zero at \((0, 0, 0)\).

By an easy computation and in view of (3.15), one gets

\[
\tilde{\psi}_i(0) = \sum_{r=0}^{2} (-1)^r \frac{\Omega^{(r)}_i(0)}{r!} h_{r+1}; \\
\tilde{\psi}'_i(0) = \sum_{r=0}^{5} (-1)^{r+1} \frac{\Omega^{(r)}_i(0)}{r!} h_r h_i,
\]

where

\[
\Omega^{(r)}_i(t) = \frac{d^r \Omega_i(t)}{dt^r}.
\]

\( s_2 \) in matrix notation can be written as:

\[
(h_1, h_2, h_3) \left( \begin{array}{c}
\omega(0), \\
-\omega'(0), \\
\frac{\omega''(0)}{2}
\end{array} \right) \cdot \Theta \\
\cdot \left( \begin{array}{c}
\frac{\omega''(0)}{2}, \\
\frac{\omega'''(0)}{3!}, \\
\frac{\omega''''(0)}{3!}, \\
\frac{\omega'''''(0)}{4!}, \\
\frac{\omega''''''(0)}{5!}
\end{array} \right) \\
\cdot (h_1^2, 2h_1 h_2, h_2^2, 2h_1 h_3, 2h_2 h_3, h_3^2)
\]

where \( \omega^{(r)}(t) \) is the vector \( (\Omega^{(r)}_i(t), \ldots, \Omega^{(r)}_{g-1}(t)) \) and \( \Theta = \{3 \partial_i(0)\} \).
The coefficient of $h_2^3$ in the cubic form $s_z$ is

$$a_{2,3} = -\omega'(0) \cdot \Theta \cdot \omega^{(3)}(0)/3!$$

The map $\varphi_K: C \to \mathbb{P}^{g-1}$ is given on $U$ by $\omega(t) = (\Omega_0(t), \ldots, \Omega_{g-1}(t))$. Since $\tilde{C} \subset R$, one has that $\omega(t) \cdot \Theta \cdot \omega(t) \equiv 0$ in $t \forall \vartheta \in I(W_3 \cap A)$, and by computing the fourth derivative of this identity, setting $t = 0$, and using the symmetry of $\Theta$ one gets:

$$\omega^{(4)}(0) \cdot \Theta \cdot \omega(0) + 4\omega'(0) \cdot \Theta \cdot \omega^{(3)}(0) + 3\omega''(0) \cdot \Theta \cdot \omega''(0) = 0.$$

(3.16)

If $\chi$ is the tangent plane to $R$ at $\tilde{P}$ one has $(\tilde{C} \cdot \chi)_{\tilde{P}} = \min_i (\tilde{C} \cdot H_i)_{\tilde{P}}$, where $\{H_i\}$ is the family of hyperplanes through $\chi$ (each $H_i$ will be defined by a linear form $LH_i$). $H_i \cap R = D_i$ is a divisor that on a suitable neighborhood of $\tilde{P} \subset R$ has the form $kr + \sigma$ where $k \geq 1$ and $\sigma$ is a local section of $R$ passing through 0: that is a curve section of the ruling of the scroll $R$ contained in this neighborhood of $\tilde{P}$. Therefore $(\tilde{C} \cdot \chi)_{\tilde{P}} = 4$. After writing

$$\omega(t) = \sum_{r=0}^{\infty} \omega^{(r)}(0) \frac{t^r}{r!},$$

we have that

$$LH_i(\omega(t)) = \sum_{r=0}^{\infty} LH_i(\omega^{(r)}(0)) \frac{t^r}{r!}$$

and therefore $LH_i(\omega^{(r)}(0)) = 0$ for $r = 0, 1, 2, 3$, but there exists $\tilde{r}$ such that $LH_i(\omega^{(\tilde{r})}(0)) \neq 0$. Since there exists a $\tilde{\vartheta} \in I(W_3 \cap A)$ such that $LH_{\tilde{r}} = \tilde{\Theta} \cdot \omega(0)$, we see that for this $\tilde{\vartheta}$

$$\omega^{(\tilde{r})}(0) \cdot \tilde{\Theta} \cdot \omega(0) \neq 0.$$

(3.17)

If $\{H_{\tilde{r}}\}$ is the family of hyperplanes through $\tilde{r}$, one sees easily that $LH_{\tilde{r}}(\omega^{(k)}(0)) = 0$ for $k = 0, 1, 2$ by the same argument applied above, and so in particular $\omega''(0) \in \tilde{r}$ from which, recalling that $r \subset R = \cap \{\text{quadrics of equation } z \cdot \Theta \cdot z = 0 \forall \vartheta \in I(W_3 \cap A)\}$, we deduce

$$\omega''(0) \cdot \Theta \cdot \omega''(0) = 0 \ \forall \vartheta \in I(W_3 \cap A).$$

(3.18)
By (3.17) and (3.18) we see that (3.16) gives
\[ a_{23} = -\omega'(0) \cdot \Theta \cdot \phi^{(3)}(0)/3! = (1/4!) \phi^{(4)}(0) \cdot \Theta \cdot \phi(0) \neq 0. \]
Therefore \( s_2 \) is not 0 for \( g = \tilde{g}, s_1 \) is not 0 too and (3.9) is proved, as well as ii) of Prop. 2.

**Remark 3.19.** From Prop. 2, ii) the Torelli theorem for the family of curves of genus 4 admitting two distinct \( g_1 \)'s follows immediately. This is a classical result (see [A.C.G.H.] and [K.2] for this result in char. \( p \neq 2 \)).

### 4. The plane quintic case

Let \( C \) be a smooth plane quintic. \( C \) has a unique \( g_5^2 \) and we let \( D \) be a divisor \( D = Q_1 + \cdots + Q_5 \), \( Q_i \neq Q_j \) for \( i \neq j \). We choose a basis \( \{ \sigma_0, \sigma_1, \sigma_2 \} \) for \( H^0(C, \mathcal{O}_C(D)) \) and homogeneous coordinates \( x_0, x_1, x_2 \) in \( \mathbb{P}^2 \) in such a way that the embedding \( \sigma: C \to \mathbb{P}^2 \) is given by \( \forall P \in C, x_i = \sigma_i(P) \) for \( i = 0, 1, 2 \).

Since by the adjunction formula we have \( \mathcal{O}_C(2D) = \mathcal{O}_C(K_C) \) we may let
\[
\{ \omega_0 = \sigma_0^2, \omega_1 = \sigma_0\sigma_1, \omega_2 = \sigma_0\sigma_2, \omega_3 = \sigma_1^2, \omega_4 = \sigma_1\sigma_2, \omega_5 = \sigma_2^2 \}
\]
be a basis for \( H^0(C, \mathcal{O}_C(K_C)) \).

It is clear that \( \varphi_K = v \circ \sigma \) where \( v \) is the Veronese embedding \( v: \mathbb{P}^2 \to \mathbb{P}^5 \) given by
\[
\begin{align*}
z_0 &= x_0^2, \\
z_1 &= x_0x_1, \\
z_2 &= x_0x_2, \\
z_3 &= x_1^2, \\
z_4 &= x_1x_2, \\
z_5 &= x_2^2
\end{align*}
\]
with \((z_0, \ldots, z_5)\) homogeneous coordinates in \( \mathbb{P}^5 \). \( v(\mathbb{P}^2) = S \) is the Veronese quartic surface in \( \mathbb{P}^5 \).

We let \( \text{Ch}(S) \) be the chordal variety of \( S \) and
\[
M = \begin{pmatrix}
z_0 & z_1 & z_2 \\
z_3 & z_4 & z_5 \\
z_2 & z_4 & z_5
\end{pmatrix}
\]
It is well known (see [Se. R.] pp. 128–130) that:

4.0. i) \( I(S) \), the homogeneous ideal of \( S \), is spanned by the six linearly independent \( 2 \times 2 \) minors of \( M \);

ii) \( \text{Ch}(S) \), the chordal variety of \( S \), is defined by the equation \( \det M = 0 \);

iii) \( \text{SingCh}(S) = S \).
Let $P_0 \in C$ be a base point. The Abel–Jacobi map $\mu_5: C^5 \to J(C)$ is given by

$$C^5 \ni (P^1, \ldots, P^5) \mapsto \left( \sum_{i=1}^{5} \int_{P_0}^{P_i} \omega_0, \ldots, \sum_{i=1}^{5} \int_{P_0}^{P_5} \omega_5 \right) \in J(C).$$

Let $\Gamma = \{(P_1, \ldots, P_5) \in C^5: P_1 + \ldots + P_5 \in g^2 \}$. By R.K.t. $\mu_5(\Gamma)$ is a triple point $T_5$ of the divisor $W_5 = \text{Im } \mu_5$ and, after modifying $\mu_5$ by a suitable translation, we will assume that $T_5 = 0 \in J(C)$.

**Proof of Proposition 3.** Over an open set $U_k \subset C$ with local coordinate $t_k$ we will denote by $\Omega_k(t_k)dt_k = \omega_{\|U_k}$. We let $\gamma: \Delta \to \Pi_{i=1}^{5} U_i \subset C^5$ be an analytic arc of curve in $C^5$ given by $t_i = h_i \cdot s + \tilde{t}_i$, where $s \in \Delta$, $h_i \in \mathbb{C}$ $\forall i = 1, \ldots, 5$, and $(\tilde{t}_1, \ldots, \tilde{t}_5) \in \Gamma \cap \Pi_{i=1}^{5} U_i$. If $\theta = 0$ is a local equation of $W_5$ in a neighborhood $A$ of $0 \in J(C)$, we have $g = \theta \circ \mu_5 \circ \gamma = 0$ over $\Delta$ and so $d^{3n}g/ds^n = 0$ on $\Delta$. We let $\theta_i, \theta_{ij}, \theta_{ijk}$ and so on be the partials of $\theta$ with respect to the variables carrying the lower indices and we also let

$$\psi_i(s) = \sum_{j=1}^{5} \Omega_j(t_j(s))h_j.$$ 

Since $0$ is a triple point of $W_5$ $\theta_i(0) = \theta_{ij}(0) = 0$ $i, j = 0, \ldots, g - 1$ and therefore, as in the derivation of (3.4), one gets the following derivatives:

$$\frac{d^3g}{ds^3}\bigg|_{s=0} = \sum_{ijk=0}^{5} \theta_{ijk}(0)\psi_i(0)\psi_j(0)\psi_k(0) = 0. \quad (4.1)$$

$$\frac{d^4g}{ds^4}\bigg|_{s=0} = \sum_{ijkl=0}^{5} \theta_{ijkl}(0)\psi_i(0)\psi_j(0)\psi_k(0)\psi_l(0) + 6 \sum_{ijk=0}^{5} \theta_{ijk}(0)\psi_i'(0)\psi_j(0)\psi_k(0) = 0. \quad (4.2)$$

$$\frac{d^5g}{ds^5}\bigg|_{s=0} = \sum_{ijklm=0}^{5} \theta_{ijklm}(0)\psi_i(0)\psi_j(0)\psi_k(0)\psi_l(0)\psi_m(0) + 10 \sum_{ijkl=0}^{5} \theta_{ijkl}(0)\psi_i'(0)\psi_j(0)\psi_k(0)\psi_l(0)$$

$$+ 10 \sum_{ijkl=0}^{5} \theta_{ijkl}(0)\psi_i'(0)\psi_j(0)\psi_k(0) + 15 \sum_{ijkl=0}^{5} \theta_{ijkl}(0)\psi_i''(0)\psi_j'(0)\psi_k(0) = 0. \quad (4.3)$$
(4.1), (4.2) and (4.3) are polynomials in \((h_1, \ldots, h_5)\) of degree 3, 4, 5 respectively. The coefficient of \(h_1^2 h_2\) in (4.1) is

\[
\sum_{ijk=0}^5 \theta_{ijk}(0) \Omega_i^j (\tilde{t}_1) \Omega_j^l (\tilde{t}_1) \Omega_k^m (\tilde{t}_2) = 0. \tag{4.4}
\]

(4.4) holds for any \((\tilde{t}_1, \tilde{t}_2) \in U_1 \times U_2\) (because one can always find \(\tilde{t}_3, \tilde{t}_4, \tilde{t}_5\) such that \((\tilde{t}_1, \ldots, \tilde{t}_5) \in \Gamma\)), therefore by the linear independence of the \(\Omega_k^m\)'s we find

\[
\sum_{ij=0}^5 \theta_{ijk}(0) \Omega_i^j \Omega_i^j \Omega_k^m = 0 \quad \forall \tilde{t}_1 \in U_1, \quad \forall k = 0, \ldots, 5. \tag{4.5}
\]

The coefficient of \(h_1^3\) in (4.1) is

\[
\sum_{ijk=0}^5 \theta_{ijk}(0) \Omega_i^j \Omega_i^j \Omega_k^m = 0 \quad \forall \tilde{t}_1 \in U_1. \tag{4.6}
\]

The first and second derivatives of (4.6) are

\[
\sum_{ijk=0}^5 \theta_{ijk}(0) \Omega_i^j \Omega_i^j \frac{d \Omega_k^m}{dt_1} = 0 \quad \forall \tilde{t}_1 \in U_1, \quad \text{and} \tag{4.7}
\]

\[
\sum_{ij=0}^5 \theta_{ijk}(0) \Omega_i^j \Omega_i^j \frac{d^2 \Omega_k^m}{dt_1^2} + 2 \sum_{ij=0}^5 \theta_{ijk}(0) \Omega_i^j \frac{d \Omega_i^j}{dt_1} \frac{d \Omega_k^m}{dt_1} = 0. \tag{4.8}
\]

We multiply (4.5) by \((d^2 \Omega_k^m)/(dt_1^2)\) and sum over \(k\): so we get

\[
\sum_{ijk=0}^5 \theta_{ijk}(0) \Omega_i^j \Omega_i^j \frac{d^2 \Omega_k^m}{dt_1^2} = 0 \tag{4.9}
\]

and, in view of (4.8) also:

\[
\sum_{ijk=0}^5 \theta_{ijk}(0) \Omega_i^j \frac{d \Omega_i^j}{dt_1} \frac{d \Omega_k^m}{dt_1} = 0. \tag{4.10}
\]

The coefficient of \(h_1^4\) in (4.2) is

\[
\sum_{ijkl=0}^5 \theta_{ijkl}(0) \Omega_i^j \Omega_i^j \Omega_i^j \Omega_i^j \Omega_k^m = 6 \sum_{ij=0}^5 \theta_{ijk}(0) \Omega_i^j \Omega_i^j \frac{d \Omega_k^m}{dt_1} + 0. \tag{4.11}
\]
From (4.7) and (4.11) we have

$$\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0) \Omega_{i}^{j} \Omega_{j}^{i} \Omega_{l}^{i} \Omega_{l}^{j} = 0$$  \hspace{1cm} (4.12)$$

and differentiating, one finds that

$$\sum_{jkl} \vartheta_{ijkl}(0) \frac{\partial \Omega_{i}^{j}}{\partial t_{l}} \Omega_{i}^{j} \Omega_{l}^{i} \Omega_{l}^{j} = 0.$$  \hspace{1cm} (4.13)$$

The coefficient of $h_{i}^{5}$ in (4.3) in view of (4.9), (4.10) and (4.13) is

$$\sum_{ijklm=0}^{5} \vartheta_{ijklm}(0) \Omega_{i}^{j} \Omega_{i}^{j} \Omega_{l}^{i} \Omega_{l}^{j} \Omega_{m}^{l} = 0.$$  \hspace{1cm} (4.14)$$

We now give a geometric interpretation of the relations found above.

By R.K.t. and the choices we made at the beginning \( \mathbb{P}TC_{0}W_{5} \) has equation

$$\det M = 0.$$

On the other hand the power series expansion of \( \vartheta \) at 0 gives for \( \mathbb{P}TC_{0}W_{5} \) the equation

$$\sum_{jk=0}^{5} \vartheta_{j}(0)z_{i}z_{j} = 0.$$  \hspace{1cm} (4.16)$$

and so (4.15) and (4.16) coincide (up to a scalar).

**Remark 4.17.** The equality of (4.15) and (4.16) gives, up to a constant multiplier, that

$$\vartheta_{035}(0) = 1, \quad \vartheta_{124}(0) = 2, \quad \vartheta_{223}(0) = \vartheta_{044}(0) = \vartheta_{115}(0) = -2$$

and all other $\vartheta_{ijk}(0) = 0$.

Then, by the coincidence of (4.15) and (4.16) and by 4.0.iii), one gets that $\text{Sing} \mathbb{P}TC_{0}W_{5} = S$, $S$ being defined either by

$$\text{rk} M = 1 \quad \text{or by}$$

$$\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0) z_{i}z_{j} = 0 \quad \forall k = 0, \ldots, 5.$$  \hspace{1cm} (4.18)$$

$$\sum_{ijkl=0}^{5} \vartheta_{ijkl}(0) z_{i}z_{j} = 0 \quad \forall k = 0, \ldots, 5.$$  \hspace{1cm} (4.19)$$
Then (4.5) shows that $\tilde{C} \subset S$ and (4.6), (4.12) and (4.14) allow to see that

$$\tilde{C} \subseteq \mathbb{P}OC(6)_0(W_5).$$

(4.20)

Since $\mathbb{P}\tilde{OC}(4)_0(W_5) = \mathbb{P}TC_0W_5$ and (4.15), the equation of $\mathbb{P}TC_0W_5$, defines $\text{Ch}(S)$, $i$) of Prop. 3 follows by applying Prop. 1.6 iii) page 232 in [A.C.G.H.], because the $g^2_3$ is semicanonical. Furthermore $\tilde{C} \subset S \subset \text{Ch}(S) = \mathbb{P}\tilde{OC}(4)_0(W_5) = \mathbb{P}\tilde{OC}(5)_0(W_5)$ so to prove ii) of Prop. 3 it is enough to prove that

$$S \cap Q = \text{twice } \tilde{C}$$

(4.21)

where $Q$ is the quintic hypersurface in $\mathbb{P}^5$ defined by the equation

$$q = \sum_{i,j,k,l,m=0}^5 \partial_{ijklm}(0)z_iz_jz_kz_lz_m = 0$$

We already know that $\tilde{C} \subset S \cap Q$ so to prove (4.21) it will suffice to show that for any line $l \subset \mathbb{P}^2$ with $l \cap \sigma(C) = R_1 + \cdots + R_5 \neq R_i$ for $i \neq j, Q$ intersects the conic $\nu(l)$ at each point $\nu(R_i)$ $i = 1, \ldots, 5$ exactly with multiplicity 2 (and therefore $Q \cap \nu(l) = 2(\nu(R_1) + \cdots + \nu(R_5))$), since then $\nu(Q) \subset \mathbb{P}^2$ has degree 10 and has a double point at every point of $C$, hence equals twice $C$. For this we assume that $R_1 + \cdots + R_5$ is the divisor $\sigma(D) = \Sigma_{i=1}^5 \sigma(Q_i)$, that $l$ is the line $\{x_2 = 0\}$, in particular that $R_1 = (1, 0, 0), R_2 = (0, 1, 0), R_3 = (1, 1, 0)$; and that $t$, the tangent line to $\sigma(C)$ at $R_1$, has equation $x_1 - x_2 = 0$. All of this can always be arranged by suitable change of coordinates of $\mathbb{P}^2$. We also let the 5-tuple $(\tilde{r}_1, \ldots, \tilde{r}_5)$ be local coordinates on $U^5$ for the 5-tuple $(Q_1, \ldots, Q_5)$. Since (4.19) vanishes on $S$ in particular it is clear that

$$\sum_{j,k=0}^5 \partial_{ijk}(0)\psi_j(0)\psi_k(0) = 0$$

(4.22)

for

$$\psi(0) = (\psi_0(0), \ldots, \psi_5(0)) \in \nu(l) \subset S \quad \forall i = 0, \ldots, 5$$

and also that

$$\sum_{j,k,l=0}^5 \partial_{ijkl}(0)\psi_j(0)\psi_k(0)\psi_l(0) = 0$$

(4.23)
for
\[ \psi(0) \in v(l) \quad \text{and} \quad i = 0, \ldots, 5. \]

In fact
\[
\sum_{ijkl=0}^5 \partial_{ijkl}(0) z_i z_j z_k z_l = 0 \quad i = 0, \ldots, 5
\]
define \( \text{Sing}K \), where \( K \) is the quartic hypersurface
\[
\sum_{ijkl=0}^5 \partial_{ijkl}(0) z_i z_j z_k z_l = 0,
\]
and so by Th.1.6 iii) p. 232 in [A.C.G.H.], since \( 2g_2^5 = |K_C| \), \( \text{Ch} (S) \) is a component of \( K \) and thus \( S = \text{Sing} \text{Ch}(S) \subset \text{Sing}K \). We multiply (4.22) by \( \psi''_i(0), (4.23) \) by \( \psi'_i(0) \) and we sum over \( i = 0, \ldots, 5 \) : thus we get that the second and the third summand of (4.3) are 0 for \( \psi(0) \in v(l) \) and so if
\[
s_1 = \sum_{ijkl=0}^5 \partial_{ijkl}(0) \psi_i(0) \psi_j(0) \psi_k(0) \psi_m(0),
\]
\[
s_2 = 15 \sum_{ijk=0}^5 \partial_{ijk}(0) \psi'_i(0) \psi'_j(0) \psi'_k(0)
\]
(4.3) becomes
\[
s_1 + s_2 = 0 \quad \text{for} \quad \psi(0) \in v(l). \quad (4.24)
\]
In order to make (4.24) explicit we write down the differential of \( \mu_5 \) at \( D \in C^5(D \in g_5^2) \) \( d\mu_5|_D : \mathbb{P}T\mathbb{C}_5 \to \mathbb{P}^5 = \mathbb{P}T_0J(C) \): this is given in our coordinate system by
\[
\begin{pmatrix}
z_0 \\
z_1 \\
z_5
\end{pmatrix}
= \begin{pmatrix}
\psi_0(0) \\
\psi_1(0) \\
\psi_5(0)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 1 & \omega_0(Q_4) & \omega_0(Q_5) & \begin{bmatrix} h_1 \end{bmatrix} \\
0 & 0 & 1 & \omega_1(Q_4) & \omega_1(Q_5) & \begin{bmatrix} h_2 \end{bmatrix} \\
0 & 0 & 0 & 0 & 0 & \begin{bmatrix} h_3 \end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
h_1 \\
h_2 \\
h_3
\end{pmatrix}
\end{pmatrix}
\]
The image of \( d \mu |_D \) (viewed in the projective space \( \mathbb{P}^5 \)) is the plane \( M \) containing the conic \( \nu(l) \). \( M \) has equations \( z_2 = z_4 = z_5 = 0 \) and \( \nu(l) \) has equation \( z_1^2 - z_0 z_3 = 0 \). If we restrict \( d \mu |_D \) to the subspace \( V = \{ h_4 = h_5 = 0 \} \subset TC^5_{Q_1, \ldots, Q_5} \) we get a specific linear projective isomorphism \( \alpha: \mathbb{P}(V) \to M \) given by \( z_0 = h_1 + h_3, z_1 = h_3, z_3 = h_2 + h_5, z_5 = z_1, \) and \( \nu(l) \) will be transformed by \( \alpha^{-1} \) into a conic \( \nu \subset \mathbb{P}(V) \) of equation \( h_1 h_2 + h_3 h_5 + h_2 h_3 = 0 \). \( \alpha^{-1}(\nu(R_1)) \) is the point \( (1, 0, 0) \in V \subset \mathbb{P}(V) \). We write the following parametrization \( \lambda: \mathbb{C} \to V \) of the conic \( V: h_1 = 1, h_2 = -u/(1 + u), h_3 = u, \lambda(0) = (1, 0, 0) \).

We observe that \( s_1 \) and \( s_2 \) are both homogeneous polynomials of degree 5 in the variables \( h_i \)'s and that \( s_1 |_{\nu(V)} = \alpha^* (q_M) \) and therefore by (4.24) \( s_2 |_{V} = \alpha^* (q_1_{\nu(l)}) \). Summing up we have

\[
\begin{array}{ccc}
\mathbb{P} TC^5_D & \mathbb{P}^5 & \mathbb{P} T_0 J(C) \\
\uparrow & \uparrow & \uparrow \\
\mathbb{P}(V) & \mathbb{M} & \\
\uparrow & \uparrow & \uparrow \\
\mathbb{C} & \mathbb{\nu} & \mathbb{\nu(l)} \\
\end{array}
\]

We now read (4.24) over \( V \) (or over \( \mathbb{C} \) by \( \lambda \)): for this we consider \( \lambda^* (s_2 |_{V}) \) and its first and second derivatives at \( u = 0 \). We state some facts we will need in the computation of these derivatives.

We have the following table:

\[
\begin{array}{ccc}
\ h_1(0) & = & 1 \\
\ h_1'(0) & = & 0 \\
\ h_1''(0) & = & 0 \\
\ h_2(0) & = & 0 \\
\ h_2'(0) & = & -1 \\
\ h_2''(0) & = & 2 \\
\ h_3(0) & = & 0 \\
\ h_3'(0) & = & 1 \\
\ h_3''(0) & = & 0 \\
\end{array}
\]

from which we see that every monomial in the \( h_i(u) \)'s and their derivatives containing factors of the form \( h_i(u) h_i'(u) \) or \( h_i(u) h_i''(u) \) vanishes at \( u = 0 \).

Since \( \sigma_2 \) vanishes at the five distinct points \( Q_1, \ldots, Q_5 \) it vanishes simply at each one of them so \( \sigma_2'(Q_1) \neq 0 \). (4.26)

We recall that \( \omega_0 = \sigma_0^2, \omega_1 = \sigma_0 \sigma_1 \) and so on. By computing derivatives and using \( \sigma_0(Q_1) = 1, \sigma_1(Q_1) = \sigma_2(Q_1) = 0 \) we get

\[
\begin{array}{ccc}
\Omega_0(\tilde{t}_1) = 2 \sigma_0(\tilde{Q}_1), \Omega_1(\tilde{t}_1) = \sigma_1(\tilde{Q}_1), \Omega_2(\tilde{t}_1) = \sigma_2(\tilde{Q}_1) \neq 0, \\
\Omega_3(\tilde{t}_1) = \Omega_4(\tilde{t}_1) = \Omega_5(\tilde{t}_1) = 0. \\
\end{array}
\]
Now

\[ \lambda^*(s_2|\psi) = 15 \sum_{ijklmn=0}^5 \partial_{ijk}(0)\Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_m)\Omega'_k(\tilde{t}_n) \cdot h_i^2(u)h_m^2(u)h_n(u), \quad (4.28) \]

so by (4.25) \( d/du \lambda^*(s_2|\psi)|_{u=0} \) reduces to

\[ 15 \sum_{ijklmn=0}^5 \partial_{ijk}(0)\Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_m)\Omega'_k(\tilde{t}_n)h_i^2(0)h_m^2(0)h_n(0). \quad (4.29) \]

To get non-zero summands in (4.29) one has to take \( l = m = 1 \) and \( n = 2, 3 \) thus getting

\[ 15 \sum_{ijkl=0}^5 \partial_{ijk}(0)\Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_1) \cdot (\Omega_k(\tilde{t}_3) - \Omega_k(\tilde{t}_2)). \quad (4.30) \]

After looking at the matrix of \( d\mu_{s_1|D} \) computed above one gets

\[ \Omega_k(\tilde{t}_3) - \Omega_k(\tilde{t}_2) = \begin{cases} 1 & k = 0, 1 \\ 0 & 2 \leq k \leq 5 \end{cases}, \]

so (4.30) actually is

\[ 15 \sum_{ijkl=0}^5 (\partial_{ij0}(0) + \partial_{ij1}(0)) \Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_1). \quad (4.31) \]

By (4.26) the products \( \Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_1) \) may be not 0 for \( 0 \leq i, j \leq 2 \). But by (4.17) all the \( \partial_{ij0}(0) \) and \( \partial_{ij1}(0) \) with \( 0 \leq i, j \leq 2 \) are zero, so (4.31) and therefore (4.29) is zero. Thus \( d/du \lambda^*(s_2|\psi)|_{u=0} = 0 \). \( d^2/du^2 \lambda^*(s_2|\psi)|_{u=0} \) reduces by (4.25) to

\[ 15 \left( \sum_{ijklmn=0}^5 \partial_{ijk}(0)\Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_m)\Omega'_k(\tilde{t}_n) \cdot 4h_i^2(0)h_m^2(0)^2h_n(0) \right) + \sum_{ijklmn=0}^5 \partial_{ijk}(0)\Omega'_i(\tilde{t}_i)\Omega'_j(\tilde{t}_m)\Omega'_k(\tilde{t}_n)h_i^2(0)h_m^2(0)^2h_n^2(0) \right). \quad (4.32) \]

The first summand may be not zero only for \( l = 2, 3 \) and \( m = n = 1 \); \( \Omega_k(\tilde{t}_1) \neq 0 \) only for \( k = 0 \) and \( \Omega_0(\tilde{t}_1) = 1 \), so it becomes

\[ 60 \left( \sum_{ij=0}^5 \partial_{ij0}(0)\Omega'_i(\tilde{t}_2)\Omega'_j(\tilde{t}_1) + \sum_{ij=0}^5 \partial_{ij0}(0)\Omega'_i(\tilde{t}_3)\Omega'_j(\tilde{t}_1) \right). \quad (4.33) \]
(4.33) is zero because $\Omega'_j(\tilde{t}) \neq 0$ only for $0 \leq j \leq 2$ (by (4.26)) and $\delta_{ij} \theta(0) = 0$ for $0 \leq j \leq 2$ by (4.17).

The second summand of (4.32) imposes $l = m = 1$ and $n = 2$; since $\Omega_k(\tilde{t}_2) \neq 0$ only for $k = 3$ (as it can be seen in the matrix of $d\mu_{s_{21}}$), we get

$$30 \sum_{ij=0}^{5} \delta_{ij3}(0)\Omega_j'(\tilde{t}_1)\Omega'_i(\tilde{t}_1) = 30\delta_{223}(0)\Omega'_2(\tilde{t}_1)^2 \neq 0$$

by (4.27) and (4.17). So $d/(d\nu^2)\lambda^*(s_{21}\nu)|_{\nu=0} \neq 0$ and the proof is complete.

**Remark 4.34.** From Prop. 3 ii) the Torelli theorem for smooth plane quintics follows immediately.

**Acknowledgements**

Research partially supported by M.P.I. funds.

We would like to thank Roy Smith and Robert Varley for some interesting conversations on the genus 3 case of Proposition 1: this seems to have been known classically (although we do not have any precise reference for this).

**References**


