

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 65, n° 2 (1988), p. 155-170

http://www.numdam.org/item?id=CM_1988__65_2_155_0

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Non-abelian extensions have nonsimple spectrum

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Received 11 February 1986; accepted in revised form 8 October 1987

Abstract. Let T be a G extension of a measure preserving transformation T_0 , where G is a compact non-abelian group. The maximal spectral multiplicity of T is greater than or equal to the supremum of the dimensions of the irreducible representations of G ; i.e., non-abelian extensions have nonsimple spectrum. As a partial converse we show that generically (in an appropriate topology) abelian extensions have simple spectrum. Generalizations, corollaries and applications are discussed.

1. Introduction

One important method for producing examples in ergodic theory is the compact group extension construction. Given a measure preserving transformation T_0 of a Lebesgue probability space (X_0, μ_0) , a compact metrizable group G , and a measurable function $\varphi: X_0 \rightarrow G$, this construction yields a new transformation T with the property that it commutes with a free measure preserving action of G . Conversely, any transformation T which commutes with such an action is the G extension of some transformation T_0 .

Along with the many specific examples which have been constructed by this method (cf. e.g., [BFK], [Fu], [J], [MwN]), the general properties of compact group extensions have also been studied (cf. e.g., [P], [Ru1], [Ru2], [R4], [N]). In this paper we will consider the general question of what spectral multiplicity properties a compact group extension can have.

For a measure preserving transformation T of (X, μ) , the spectral multiplicity of T is defined to be the spectral multiplicity of the corresponding induced unitary operator U_T on $L^2(X, \mu)$. In the context of operator theory, it follows from the spectral theory that a unitary operator U has nonsimple spectrum if and only if it has a non-abelian commutant. If T commutes with a free measure-preserving G action, then U_T commutes with the unitary

representation of G induced on $L^2(X, \mu)$ by this G action. When G is non-abelian we get that U_T has nonsimple spectrum, which implies (by definition) the same for T . This shows that non-abelian group extensions have nonsimple spectrum.

Actually, we are able to obtain better estimates of the multiplicity than just this. In Proposition 1 we show that lower bounds on the multiplicity function are given by the dimensions of the irreducible representations of G . In particular (Theorem 1) the maximal spectral multiplicity M_T satisfies the estimate $M_T \geq D_G$, where D_G denotes the supremum of the dimensions of the irreducible representations of G .

For groups G with a finite upper bound on the dimensions of the irreducible representations, this inequality gives a finite lower bound on the multiplicity. In particular, such an estimate always holds for finite non-abelian group extensions. This fact is interesting because it is one of the very few general conditions known to imply finite lower bounds on the spectral multiplicity of a transformation T . In fact, for many years the question of the existence of any transformations with nonsimple spectrum of finite multiplicity remained open. Although there are now several constructions for such transformations (cf. [O], [R1], [R2], [R3], [K], [G] (multiplicity 2 only) and [MwN]), with the exception of the construction of Katok, [K],* lower bounds on the multiplicity have usually been obtained by ad hoc methods. In contrast, there are several well known general conditions which imply a finite upper bound on the multiplicity (cf. Lemma 1 below).

Complementing the results for non-abelian extensions, we also study the spectral multiplicity of G extensions where the group G is abelian. Even if T_0 has simple spectrum it is possible for an abelian extension T of T_0 to have nonsimple spectrum (or even infinite spectral multiplicity). However, typically this does not happen. In the natural topology on the space of all abelian G extensions T , the set of those T with simple spectrum is generic (i.e., has a dense G_δ subset).

2. Definitions and other preliminaries

Let T be a measure preserving transformation of a Lebesgue probability space (X, μ) and let U_T be corresponding the induced unitary operator on $L^2(X, \mu)$, (defined $U_T f(x) = f(T^{-1}x)$). By the Spectral Theorem (cf. [K]), U_T is determined up to equivalence by a spectral measure class σ and a

* A. Katok [K] has shown that in general if T if the Cartesian k 'th power of a transformation T_0 then $M_T \geq k!$.

$\{+\infty, 1, 2, \dots\}$ valued multiplicity function m on the circle. The set of *essential spectral multiplicities* of T (cf. [K]), is the set of all σ essential values of m . We denote this set by \mathcal{M}_T . The *maximal spectral multiplicity*, M_T , is defined $M_T = \sup \mathcal{M}_T$ (cf. [K]). The transformation T has *simple spectrum* if $M_T = 1$, or equivalently, if $\mathcal{M}_T = \{1\}$. Otherwise, T has *nonsimple spectrum*. When T is not ergodic it automatically has nonsimple spectrum for a trivial reason, so we will usually only be concerned with the ergodic case.

Throughout this paper G will denote a compact metrizable group. In particular, G may be finite. Suppose that $\mathcal{L} = \{L_g\}_{g \in G}$ is a measurable measure preserving left action of G on (X, μ) . The action \mathcal{L} is called (μ -almost) *free* if

$$\mu\{x: L_g x = x \text{ for some } g \neq id\} = 0. \tag{1}$$

The transformation T is said to *commute* with \mathcal{L} if

$$L_g T = T L_g \tag{2}$$

for all $g \in G$ and for μ almost all x . If (1) and (2) hold, G is said to be in the commutant of T .

Given a transformation T and an action \mathcal{L} of G satisfying (1) and (2), let (X_0, μ_0) be the Lebesgue space generated by the partition of (X, μ) into G orbits. Let T_0 be the factor transformation induced on (X_0, μ_0) by T . Up to sets of measure 0, we can write $(X, \mu) = (X \times G, \mu_0 \times \gamma)$, where γ is Haar measure on G . It follows from (1) and (2) that there exists a measurable function $\varphi: X_0 \rightarrow G$ so that

$$T(x, y) = (T_0 x, \varphi(x)y) \tag{3}$$

and

$$L_g(x, y) = (x, yg^{-1}). \tag{4}$$

In general, a transformation T obtained from another transformation T_0 by (3) is called a *group extension* (or *G extension*) of T_0 . The function φ which appears in (3) is called the *cocycle* of the extension (cf. [K] for an explanation of this terminology). Clearly, every G extension T commutes with the G action $\{L_g\}$. This shows G is in the commutant of T if and only if T is a G extension of some transformation T_0 .

Now let us consider the following generalization of (3). Let T_0 be a transformation of (X_0, μ_0) . Let K be a closed subgroup of G . We define

$(X, \mu) = (X_0 \times G/K, \mu_0 \times \gamma_K)$, where γ_K denotes Haar measure projected to K . Let $\varphi: X_0 \rightarrow G$ be measurable and define a measure preserving transformation T on (X, μ) by

$$T(x, yK) = (T_0x, \varphi(x)yK). \tag{5}$$

Because T acts isometrically on the compact homogeneous “fibers” of the extension (5), it is called an *isometric extension* of T_0 .

We note: if T_0 is ergodic then there always exist cocycles φ so that the G extension (3) and isometric extension (5) are ergodic.

3. Lower bounds on multiplicity

We begin by recalling some basic facts from the representation theory of compact groups (cf. [M1]). The left action of G on G/K by translation gives rise to a *quasi-regular representation* of G on $L^2(G/K, \gamma_K)$, namely: $W'_g f(yK) = f(gyK)$. When K is the trivial subgroup that is the *regular representation* of G , which we denote by W_g . It follows from the Peter–Weyl Theorem (cf. [M1]), that the regular representation W_g decomposes into an orthogonal direct sum of finite dimensional irreducible unitary representations:

$$W_g = \bigoplus_{j=1}^t \bigoplus_{k=1}^{d_j} W_g^{j,k}, \tag{6}$$

where $W_g^{j,k} = W_g|_{H_{j,k}}$, and

$$L^2(G, \gamma) = \bigoplus_{j=1}^t \bigoplus_{k=1}^{d_j} H_{j,k}. \tag{7}$$

In addition, $t \leq +\infty$ (with $t < +\infty$ if and only if G is finite), $d_j = \dim H_{j,k} \leq d_{j+1}$, and $W_g^{j,k}$ is equivalent to $W_g^{j',k'}$ if and only if $j = j'$. Furthermore, every irreducible unitary representation of G appears in this decomposition, its multiplicity being equal to its dimension.

For K nontrivial, by the theory of induced representations (cf. [M1]), the quasi-regular representation W'_g is the representation obtained by inducing the one dimensional identity representation Id_K of K to G . Applying Peter–Weyl Theorem again, we obtain a decomposition:

$$W'_g = \bigoplus_{j=1}^t \bigoplus_{k=1}^{m_j} W_g^{j,k}. \tag{8}$$

From the Frobenius Reciprocity Theorem (cf. [M1]), we have that $0 \leq m_j \leq d_j = \dim W_g^{i,k} < +\infty$. Thus W'_g is a subrepresentation of W_g .

The following proposition is a generalization of a lemma which is well known for compact abelian extension transformations. It is the main technical ingredient for all of the results of this paper.

PROPOSITION 1. *Suppose a transformation T of (X, μ) is an isometric extension of a transformation T_0 of (X_0, μ_0) by G/K , where K is a closed subgroup of a compact metrizable group G . Let t and m_j be as in (8). Then there exists a U_T invariant orthogonal decomposition*

$$L^2(X, \mu) = \bigoplus_{j=1}^t \bigoplus_{k=1}^{m_j} \mathcal{H}_{j,k} \tag{9}$$

where $U_T|_{\mathcal{H}_{j,k}}$ and $U_T|_{\mathcal{H}_{i,k}}$ are equivalent under conjugation by a unitary isomorphism.

Proof. For each j, k , let $S_{j,k}: H_{j,k} \rightarrow H_{j,1}$ be the unitary intertwining operator for (8) satisfying

$$S_{j,k} \circ W_g^{i,k} = W_g^{1,k} \circ S_{j,k}.$$

For each j , we choose an arbitrary orthonormal basis $\{e_{j,1}^1, \dots, e_{j,1}^{d_j}\}$ for $H_{j,1}$, and define

$$e_{j,k}^r = S_{j,k} e_{j,1}^r. \tag{10}$$

Then

$$\mathcal{B} = \{e_{j,k}^r : j = 1, \dots, t, k = 1, \dots, m_j, r = 1, \dots, d_j\}$$

forms an orthonormal basis for $L^2(G/K, \gamma_K)$.

We define $\mathcal{H}_{j,k}$ to be the set of all $f \in L^2(X, \mu)$ of the form

$$f(x, y) = \sum_{r=1}^{m_j} b_r(x) e_{j,k}^r(y) \tag{11}$$

where $b_r(x) \in L^2(X_0, \mu_0)$ for $r = 1, \dots, m_j$, and $y \in G/K$. Note that $f \in \mathcal{H}_{j,k}$ if and only if for μ_0 a.e. $x \in X_0, f(x, \cdot) \in H_{j,k}$. It follows that the subspaces $\mathcal{H}_{j,k}$ are orthogonal and span $L^2(X, \mu)$.

For $f \in \mathcal{H}_{j,k}$ we have

$$\begin{aligned} (U_T f)(x, y) &= \sum_{r=1}^{m_j} (U_T b_r e'_{j,k})(x, y) \\ &= \sum_{r=1}^{m_j} b_r(T_0 x) W_{\varphi(x)} e'_{j,k}(y), \end{aligned}$$

and since $H_{j,k}$ is $W_{\varphi(x)}$ invariant,

$$W_{\varphi(x)} e'_{j,k} = \sum_{s=1}^{m_j} w_{s,r}(\varphi(x)) e^s_{j,k},$$

where the functions $w_{s,r}(\varphi(x))$ are matrix elements for $W_{\varphi(x)}^{j,k}$. Thus,

$$(U_T f)(x, y) = \sum_{s=1}^{m_j} c_s(x) e^s_{j,k}(y),$$

with

$$c_s(x) = \sum_{r=1}^{m_j} b_r(T_0 x) w_{s,r}(\varphi(x)). \tag{12}$$

The fact that $c_s \in L^2(X_0, \mu_0)$ follows from the fact that the matrix elements $w_{r,s}$ are continuous, and therefore bounded on G .

To finish the proof, we define a unitary operator $R: \mathcal{H}_{j,k_1} \rightarrow \mathcal{H}_{j,k_2}$ by

$$\begin{aligned} Rf(x, y) &= \sum_{r=1}^{m_j} b_r(x) S_{j,k_2} S_{j,k_1}^{-1} e'_{j,k_1}(y), \\ &= \sum_{r=1}^{m_j} b_r(x) e'_{j,k_2}(y). \end{aligned} \tag{13}$$

Thus,

$$\begin{aligned} R \circ U_T f(x, y) &= \sum_{r=1}^{m_j} b_r(T_0 x) S_{j,k_2} S_{j,k_1}^{-1} W_{\varphi(x)} e'_{j,k_1}(y) \\ &= \sum_{r=1}^{m_j} b_r(T_0 x) W_{\varphi(x)} S_{j,k_2} S_{j,k_1}^{-1} e'_{j,k_1}(y) \\ &= U_T \circ Rf(x, y). \end{aligned}$$

■

Let us denote by $\mathcal{D}_{G/K}$ the set of multiplicities m_j of the irreducible representations occurring in the quasi-regular representation associated with G/K . Let $D_{G/K} = \sup \mathcal{D}_{G/K}$. For the regular representation of G we use the notations \mathcal{D}_G and D_G .

THEOREM 1. *Let G be a compact group. Suppose that T is a G extension, or equivalently, that T commutes with a free measure preserving G action. If G is non-abelian then $M_T \geq D_G > 1$. In particular, T has nonsimple spectrum.*

Proof. Since $K = \{id\}$, (9) holds with $m_j = d_j$ for all j . Since G is non-abelian, not every irreducible unitary representation is one dimensional, so $d_j \geq 1$ for some j . Letting $\mathcal{H}_j = \bigoplus_{k=1}^{d_j} \mathcal{H}_{j,k}$, we note that the spectral multiplicities of $U_T|_{\mathcal{H}_j}$ is a multiple of d_j . ■

In the general case, Proposition 1 and the proof of Theorem 1 really gives a bit more information about the multiplicities.

COROLLARY 1. *For an isometric extension T , the essential values of the spectral multiplicity function of $U_T|_{\mathcal{H}_j}$ are multiples of m_j .*

We conclude this section by describing sufficient conditions for an isometric extension (which is not in general a group extension), to have nonsimple spectrum.

PROPOSITION 2. *Let G be a compact metrizable group and K be a closed subgroup. Suppose there exist closed subgroups K_1 and K_2 , with $K \leq K_1 \leq K_2 \leq G$, where K_1 is normal in K_2 , and K_2/K_1 is non-abelian. Then there is at least one irreducible representation of G which appears with multiplicity greater than 1 in the quasi-regular representation associated with G/K .*

Proof. The quasi-regular representation has a subrepresentation equivalent to the regular representation of K_2/K_1 . ■

We call an isometric extension by G/K which satisfies the hypotheses of Proposition 2 *properly non-abelian*. Note that since we do not require K to be normal in K_1 , or K_2 to be normal in G , this condition is rather weak. We do not know whether it is also a necessary condition for the existence of multiplicity in the quasi-regular representation.

PROPOSITION 3. *If T is a properly non-abelian isometric extension of some transformation T_0 , then $M_T \geq D_{K_1/K_2} > 1$.*

Note. Any properly non-abelian isometric extension T of T_0 has a factor T_2 which is itself a K_2/K_1 extension of some other isometric extension T_1 of T_2 .

4. Upper bounds and the abelian case

In this section we show how additional assumptions on T_0 and G can sometimes be used to eliminate the (real) possibility that $M_T = +\infty$. The following lemma lists some well known conditions which imply finite upper bounds for M_T .

LEMMA 1. *If T admits a good r -cycle approximation (cf. [R1]), or has rank $\leq r$ [C], or may be realized as an interval exchange transformation involving $\leq r + 1$ intervals [O], then $M_T \leq r$.*

For convenience, in any of the three cases above (or in any other situation where we have an *a priori* upper bound r on M_T), we will denote r by R_T . Note that $R_T = 1$ implies that T is ergodic.

A combination of Lemma 1 with the results of Section 3 yields the following.

COROLLARY 2. *Suppose T is a properly non-abelian isometric extension. Then $R_T \geq D_{K_1/K_2} > 1$. In particular, if T commutes with a free action of a compact metrizable non-abelian group G , then $R_T \geq D_G > 1$, (i.e., T is not rank 1).*

Recently J. King [Ki] has shown that if T is rank 1 then the comutant $C(T)$ of T is abelian. As we noted in the introduction, a non-abelian comutant is already enough to imply nonsimple spectrum. Thus, applying Lemma 1, we obtain another proof of King's result.* Notice that the compactness assumption in Corollary 2 was not needed. An interpretation of Corollary 2 in terms of $C(T)$ is that the rank R_T imposes restrictions on what compact subgroups $C(T)$ can have. As the next result shows, even finite rank (i.e., $R_T < +\infty$) has implications for $C(T)$.

For many infinite compact groups G there are irreducible unitary representations of G of arbitrarily high dimension, i.e., $D_G = +\infty$. We will call such a group *large*. An example of a large compact group is the group $SU(2)$ of complex 2×2 unitary matrices with determinant 1.

COROLLARY 3. *Suppose T commutes with the action of a large compact group. Then $R_T = +\infty$. In particular, T does not have finite rank.*

* We wish to thank the referee for pointing out the connection between our results and King's.

C. Moore [Mo] has shown that a locally compact group G satisfies a finite upper bound on the dimensions of its irreducible representations if and only if G contains an open subgroup of finite index. It follows that a compact metrizable group is large unless it satisfies this condition.

Next, we discuss a simple example of a situation where these results can be applied to construct an interesting class of transformations. Let T_0 be an irrational rotation on the circle, viewed as an interval exchange transformation on $[0, 1]$ involving 2 intervals. Let α denote the point of a discontinuity (i.e., T_0 has rotation number $2\pi(1 - \alpha)$). Let G be a finite non-abelian group with $\text{card}(G) = s$ and $D_G = r > 1$. Let $\varphi: [0, 1] \rightarrow G$ be a piece-wise continuous function with $t - 2$ discontinuities, all of which occur at rational points in $[0, 1]$. Also suppose that $\text{Im}(\varphi)$ generates G . Let T be the G extension of T_0 with cocycle φ . This situation has been studied by Veech [V], who has shown that if α is poorly enough approximated by rationals (i.e., if it has bounded partial sums in its continued fraction expansion), then T is ergodic. Now clearly T is an interval exchange transformation with $R_T < st$. Thus we have:

COROLLARY 4. *The transformation T constructed above is an ergodic interval exchange transformation with $1 < r \leq M_T < st < +\infty$.*

Corollary 4 provides a recipe for constructing many examples of ergodic interval exchange transformations with nonsimple spectrum of finite multiplicity. Specific examples showing that there exist interval exchange transformations arbitrary finite M_T appear in [R1].

The next theorem concerns the theory of the typical properties of abelian group extensions.

DEFINITION 1. Let Φ denote the set of measurable functions $\varphi: X_0 \rightarrow G$, and let d_G be the translation invariant metric on G of diameter 1. The L^1 -topology on Φ is the topology given by the metric

$$d(\varphi_1, \varphi_2) = \int_{X_0} d_G(\varphi_1(x), \varphi_2(x)) \, d\mu_0.$$

A property of $\varphi \in \Phi$ is called *generic* if it holds for all φ belonging to a dense G_δ subset Φ_0 of Φ . A *partition* ξ of (X_0, μ_0) is a finite collection of disjoint subsets of (X_0, μ_0) with equal measure and total measure 1. A transformation T preserves ξ if $TE \in \xi$ whenever $E \in \xi$. A sequence ξ_n of partitions is said to be *generating*, denoted $\xi_n \rightarrow \varepsilon$, if for any set E , there exists a union E_n of elements of ξ_n with $\mu_0(E \Delta E_n) \rightarrow 0$ as $n \rightarrow \infty$.

DEFINITION 2 (cf. [K]). A transformation T_0 admits a *good cyclic approximation* $(T_{0,n}, \xi_{0,n})$, where $T_{0,n}$ is a transformation preserving the partition $\xi_{0,n}$, if (i) $\xi_{0,n} \rightarrow \varepsilon$, and (ii)

$$\lim_{n \rightarrow \infty} q_n \sum_{k=0}^{q_n-1} \mu_0(T_0^k E \Delta T_{0,n}^k E) = 0,$$

where $E \in \xi_{0,n}$, and $q_n = \text{card } \xi_{0,n}$.

THEOREM 2. *If G is abelian and T_0 admits a good cyclic approximation, then for a generic set of $\varphi \in \Phi$ the corresponding G extension T has simple spectrum, and in particular T is ergodic.*

Note. Since the condition that T_0 admits a good cyclic approximation is generic in the weak topology on the set of all measure preserving transformations (cf. [K]), an alternative statement would be that the generic compact abelian group extension has simple spectrum. Furthermore, the theorem can be strengthened to say that the generic compact abelian group extension is, in fact, weakly mixing but not mixing.

Proof. Since G is abelian, all of its irreducible representations are 1 dimensional characters, each of which occurs with multiplicity 1 in the regular representation. We denote these characters by $x_j(y)$, $y \in G$, $1 \leq j \leq t \leq +\infty$. The decomposition (9) becomes

$$L^2(X, \mu) = \bigoplus_{j=1}^t \mathcal{H}_j, \tag{14}$$

where $\mathcal{H}_j = \{x_j f : f \in L^2(X_0, \mu_0)\}$.

Now T is approximated by $T_{0,n}$ which cyclically permutes the partition $\xi_{0,n}$ of (X_0, μ_0) . Let Φ_n denote the set of functions in Φ which are constant on the elements of $\xi_{0,n}$. For $\varphi \in \Phi$, $\varphi_n \in \Phi_n$, $k \geq 0$, we define

$$\varphi'(k, x) = \varphi(T^{k-1}x) \dots \varphi(Tx)\varphi(x),$$

and

$$\varphi'_n(k, x) = \varphi_n(T_{0,n}^{k-1}x) \dots \varphi_n(T_{0,n}x)\varphi_n(x). \tag{15}$$

By the cyclicity of $T_{0,n}$, $\varphi'_n(q_n, x) = \varphi_n'' = \text{constant}$. Given any $g \in G$, we can modify φ_n , within Φ_n , by changing its value on just one element of $\xi_{0,n}$

to make $\varphi_n'' = g$. From the fact that $\xi_{0,n} \rightarrow \varepsilon$, it follows that for any $\varphi \in \Phi$, $g \in G, m > 0$, there exists n sufficiently large and φ_n , depending on φ, g , and m , with $d(\varphi_n, \varphi) < 1/m$ and $\varphi_n'' = g$.

For any pair $0 \leq j \leq j' \leq t$ there exists $g(j, j') \in G$ so that $x_j(g(j, j')) \neq x_{j'}(g(j, j'))$. Let $\delta = \delta(j, j', q_n)$ be such that $|x_j(y) - 1| < 1/q_n^2$ and $|x_{j'}(y) - 1| < 1/q_n^2$ whenever $d_G(y, id) < \delta$.

We denote $N_\theta(\varphi) = \{\bar{\varphi} \in \Phi: d(\varphi, \bar{\varphi}) < \theta\}$ and define

$$\Phi_0 = \bigcap_{0 \leq j < j' \leq n} \bigcap_{M=1}^\infty \bigcap_{N=1}^\infty \bigcup_{n=M}^\infty \bigcup_{m=M}^\infty \bigcup_{\varphi \in \Phi} N_{\delta(j,j',q_n)/q_n^3}(\varphi_n(\varphi, g(j, j'), m)).$$

Clearly Φ_0 is dense G_δ in Φ .

Now $\varphi \in \Phi_0$ if and only if for each pair j, j' there exists a sequence $n(k) \rightarrow \infty$ as $k \rightarrow \infty$ and $\varphi_{n(k)} \in \Phi_{n(k)}$ such that

$$\lim_{k \rightarrow \infty} \frac{q_{n(k)}^2}{\delta(j, j', q_{n(k)})} d(\varphi_{n(k)}, \varphi) = 0, \tag{16}$$

and

$$\varphi_{n(k)}'' = g(j, j'). \tag{17}$$

In particular, (16) is already enough to show that for each $j, U_T|_{\mathcal{H}_j}$ has simple spectrum. The argument, which follows, is similar to Theorem 5.1 in [KS]. We pass to a subsequence $n(k)$ satisfying (16) (which for convenience we just denote by n). Then we fix an element $C_{0,n}$ of $\xi_{0,n}$ and let $C_{r,n} = T_{0,n}^r C_{0,n}$, $r = 0, \dots, q_n - 1$. Given $0 < \varepsilon < 1$, it follows that for n sufficiently large, there exist subsets $B_{r,n}$ of $C_{r,n}$ with

$$TB_{r,n} = B_{r+1,n}, \quad r = 0, \dots, q_n - 2, \tag{18}$$

$$\mu_0(C_{r,n} \setminus B_{r,n}) < \varepsilon/q_n, \tag{19}$$

and

$$d_G(\varphi_n'(r, x), \varphi_n'(s, x)) < \varepsilon/q_n, \tag{20}$$

for all $x \in B_{s,n}, r, s = 0, \dots, q_n - 1$ (cf. e.g., [KS] or [R3]).

Let $h = x_j f \in \mathcal{H}_j$ with $\|h\|_2 = 1$. Since $E_{0,n} \rightarrow \varepsilon$, for n sufficiently large, there exists $h' = x_j f' \in \mathcal{H}_j$, with f' constant on each element of $\xi_{0,n}$,

$\|h'\|_2 = 1$, and $\|h - h'\| < \varepsilon$. Let $T_{1,n}$ denote the extension of $T_{0,n}$ obtained by substituting φ_n for φ in (3). Then

$$h' = \sum_{k=0}^{q_n-1} \alpha_k U_{T_{1,n}}^k x_j 1_{C_{0,n}},$$

where $1_{C_{0,n}}$ is the characteristic function of $C_{0,n}$. Let

$$h'' = \sum_{k=0}^{q_n-1} \alpha_k U_T^k x_j 1_{B_{0,n}}.$$

We have by (18), (19), and (20),

$$\begin{aligned} & \|h' - h''\|_2 \\ &= \left\| \sum_{k=0}^{q_n-1} \alpha_k (x_j(\varphi'_n(k, \cdot))1_{C_{n,k}} - x_j(\varphi'(k, \cdot))1_{B_{n,k}}) \right\|_2 \\ &= \left[\sum_{k=0}^{q_n-1} |\alpha_k|^2 \int_{X_0} |x_j(\varphi'_n(k, x))1_{C_{n,k}}(x) - x_j(\varphi'(k, x))1_{B_{n,k}}(x)|^2 d\mu_0 \right]^{1/2} \\ &\leq \left[\sum_{k=0}^{q_n-1} |\alpha_k|^2 \left[\int_{B_{n,k}} |x_j(\varphi'_n(k, x)\varphi'^{-1}(k, x))|^2 d\mu_0 + 2\mu_0(C_{n,k} \setminus B_{n,k}) \right] \right]^{1/2} \\ &\leq \left[\sum_{k=0}^{q_n-1} 3|\alpha_k|^2 \varepsilon / q_n \right]^{1/2} = (3\varepsilon)^{1/2} \|h'\|_2 = (3\varepsilon)^{1/2}. \end{aligned}$$

So $\|h - h''\|_2 < \varepsilon + (3\varepsilon)^{1/2}$, and h'' belongs to the cyclic subspace generated by $x_j 1_{B_{0,n}}$. To complete the proof that the spectrum is simple, we assume that it is not and apply [KS] Lemma 3.1 to obtain a contradiction.

To show that $U_T|_{\mathcal{H}_j}$ and $U_T|_{\mathcal{H}'_j}$ have mutually singular spectral types, let $h_n = x_j f_n \in \mathcal{H}_j$ and $h'_n = x_j f'_n$, with f_n and f'_n constant on the elements of $\xi_{0,n}$. Choose a subsequence $n(k) \rightarrow \infty$, satisfying (16) and (17) (again for convenience denoted by n). Then it follows from (3) and the cyclicity of $T_{0,n}$ that

$$U_{T_{1,n}}^{q_n} h_n = \lambda' h_n,$$

and

$$U_{T_{1,n}}^{q_n} h'_n = \lambda' h'_n,$$

where $\lambda = x_j(g(j, j')) \neq \lambda' = x_{j'}(g(j, j'))$. As in the proof of [KS] Theorems 3.3 and 3.4, we obtain a further refinement $n(m)$ and disjoint sets $\Gamma_{n(m)}^j$ and $\Gamma_{n(m)}^{j'}$ consisting of small intervals around the $q_{n(m)}$ 'th roots of λ and λ' respectively. Letting ϱ and ϱ' denote, correspondingly, measures of maximal spectral type for U_T on \mathcal{H}_j and $\mathcal{H}_{j'}$, we have $\lim_{m \rightarrow \infty} \varrho(\Gamma_{n(m)}^j) = 1$ and $\lim_{m \rightarrow \infty} \varrho'(\Gamma_{n(m)}^{j'}) = 1$. This implies $\varrho \perp \varrho'$.

Clearly, for an extension T of T_0 to have simple spectrum it is necessary for T_0 to have simple spectrum. We will now show that even in this case there may exist some abelian extension T of T_0 with nonsimple spectrum. One (well known) example with this property is the skew shift transformation on the 2-torus $[0, 1]^2$ defined by $T(x, y) = (x + \alpha, x + y) \bmod 1$, where α is irrational. Here T_0 is just an irrational rotation on the circle. The spectral multiplicity, which is infinite in this example, comes from a non-abelian commutant on the operator theoretic level rather than from non-commuting point transformations in the commutant of T .

Another example more closely related to the theme of this paper is the following. Let T_0 be an arbitrary measure preserving transformation of (X_0, μ_0) , let \mathbb{Z}/n be a finite cyclic ring, and let β be a unit of \mathbb{Z}/n with multiplicative order m . For an arbitrary cocycle $\psi_1: X_0 \rightarrow \mathbb{Z}/m$ we construct the \mathbb{Z}/m extension T_1 of T_0 defined by

$$T_1(x, y) = (T_0x, \varphi(x) + y),$$

(in additive notation). Then we construct the following \mathbb{Z}/n extensions T of T_1 ,

$$T(x, y, z) = (T_0x, \psi_1(x) + y, \theta(y)\psi_2(x) + z), \tag{21}$$

where $\psi_2: X_0 \rightarrow \mathbb{Z}/n$ is arbitrary and $\theta(y) = \beta^y$. An easy computation (cf. [R1]) shows that for any ψ_1 and ψ_2 , T has nonsimple spectrum. This fact was first noticed in the case $m = 2, n = 3, \psi_3 = 1$ by Oseledec [O], who used it to construct the first example of an ergodic transformation with nonsimple spectrum of finite multiplicity. The general case was later employed by the author in [R1], [R2] and [R3] to obtain other examples (cf. below).

Now suppose that T_0 admits a good cyclic approximation. Then it is not hard to show that for the generic cocycle ψ_1 the transformation T_1 also admits a good cyclic approximation (cf. [R1]), and it follows that T_1 has simple spectrum. Thus T is an abelian extension of T_1 that has nonsimple spectrum. It is fairly easy to explain where the multiplicity comes from in this example. The transformation T_1 already has a nontrivial commutant

(it contains \mathbb{Z}/m), and the extension by \mathbb{Z}/n is special; the cocycle $\varphi: X_0 \times \mathbb{Z}/m \rightarrow \mathbb{Z}/n$ given by $\varphi(x, y) = \theta(y)\psi_2(x)$ is “anti-symmetric” with respect to \mathbb{Z}/m . Thus the commutant for T_1 does not commute with the extension to T .

A closer look at this example is even more revealing. It turns out that T is really an extension of T_0 by a certain non-abelian group, namely the semi-direct product group $G = \mathbb{Z}/m \times_0 \mathbb{Z}/n$. (Recall that this is the group of pairs $(y, z) \in \mathbb{Z}/m \times \mathbb{Z}/n$ with the multiplication $(y, z)(y', z') = (y + y', \theta(y')z + z')$.) The cocycle $\varphi: X_0 \rightarrow G$ for the extension (21) is given by $\varphi(x) = (\psi_1(x), \psi_2(x))$. This shows that T is the general G extension of T_0 .

Now let $0_1, \dots, 0_k$ denote the orbits of θ acting on \mathbb{Z}/n by iteration. We claim $\mathcal{D}_G = \{d_j: d_j = \text{card } 0_j\}$. Indeed, it follows from Mackey, [M2] section 2.2, Theorems A and B, that corresponding to each orbit 0_j there are m/d_j inequivalent irreducible representations, of G , each with dimension d_j . These exhaust the irreducibles.

In [R3] we show how to construct examples of such groups where \mathcal{D}_G is subject only to the following conditions: (i) $1 \in \mathcal{D}_G$, (ii) \mathcal{D}_G is finite, and (iii) \mathcal{D}_G is closed under the operation of taking least common multiples. Furthermore, for such groups G one can prove generalization of Theorem 2 (cf. Theorem 3), which shows that the generic G extension T has $\mathcal{M}_T = \mathcal{D}_G$. This provides (in [R3], but by a slightly different proof), many new examples of the possibilities for \mathcal{M}_T . In particular, this class of transformations includes the examples with arbitrary finite multiplicity constructed in [R1].

We conclude by stating more general conditions for the generic isometric extension by G/K to have $\mathcal{M}_T = \mathcal{D}_{G/K}$. For a linear transformation M , $\text{ev}(M)$ will denote its set of eigenvalues.

THEOREM 3. *Suppose G/K has the following properties:*

- (i) *For any two irreducible representations W_g^1 and W_g^2 which occur in the corresponding quasi-regular representation, there exists $g_0 \in G$ such that $\text{ev}(W_{g_0}^1) \cap \text{ev}(W_{g_0}^2) = \emptyset$, and*
- (ii) *For each irreducible representation W_g^1 occurring in the corresponding quasi-regular representation, there exists $g_0 \in G$ so that $W_{g_0}^1$ has only simple eigenvalues.*

Also, suppose that T_0 admits a good cyclic approximation.

Then for a generic set of $\varphi \in \Phi$, the G/K isometric extension T constructed from T_0 using φ has $\mathcal{M}_T = \mathcal{D}_{G/K}$.

The proof of Theorem 3 is a generalization of the proof of Theorem 2 (cf. also the proof of Lemma 3.5 in [R3]). The idea is that the spectrum is simple

in each of the subspaces $\mathcal{H}_{j,k}$ of $L^2(X, \mu)$ (in the notation of Proposition 1), using (ii), and that the spectral types for these subspaces when $j \neq j'$ are pairwise disjoint, using (i).

So far we have been unable to find groups, other than semi-direct products of finite cyclic groups, where conditions (i) and (ii) hold. For example they fail for the alternating group A_5 . Thus the examples in [R3] remain the only examples with finite \mathcal{M}_T which can be computed.

Acknowledgement

Partially supported by NSF grant DMS 85-04025.

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