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## Resolution of quasi-homogeneous singularities and plurigenera

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**Abstract.** First given a Cohen-Macaulay isolated quasi-homogeneous singularity I describe the resolution process based on Demazure's work and generalize some results of Orlik and Wagreich in dimension two. After using this resolution of singularities I calculate explicitly the plurigenera for a Gorenstein quasi-homogeneous singularity in terms of an invariant called index of regularity of the Hilbert function. In a second part similar calculations are given for plurigenera of complete intersection singularities which are generic in the Newton polyhedron sense. Proofs need the explicit resolution process developed by the author in [7].

### Introduction

The calculation of numerical invariants associated to isolated singularities such as geometric genus, plurigenera gives a way to understand singularities: for example rational singularities are those with geometric genus zero, surface rational double points are those with all the Knöller plurigenera equal to zero, and surface quotient singularities are those with all  $L^2$ -plurigenera equal to zero.

In Section 0 I recall the definition of *dualizing module* and that of the *plurigenera* ( $L^2$ , Log or Knöller plurigenera). In the first section we introduce an important invariant: *the index of regularity of the Hilbert function* for a graded module over the ring of polynomials  $B = k[X_1, \dots, X_n]$   $\deg x_i = e_i \geq 1$ ; this invariant appears in the literature in particular cases and different representations (*cf.* Lazard [12], Schenzel [13], Goto-Watanabe [11]). In the fourth section we'll express the  $L^2$ - and Log plurigenera for a Gorenstein quasi-homogeneous isolated singularity in terms of this invariant. The second section recalls some results of Demazure about normal graded rings and we apply this in the third section to prove a resolution theorem for isolated quasi-homogeneous singularities. The fifth section is devoted to the calculation of the three plurigenera for complete intersections generic in the Newton polyhedron sense.

The author was inspired by the calculation of the geometric genus in papers of Laufer [3], Merle-Teissier [5] and the author [7]. We prove that effective calculation can be done for  $L^2$  and Log plurigenera; calculation of Knöller plurigenera is more complicated. The proof shows in particular that the three plurigenera depend only on the Newton polyhedron.

### 0. Dualizing modules, plurigenera

0.1. Let  $B = k[x_1, \dots, x_n]$  and  $U$  an ideal of  $B$ , we put  $A = B/U$ ,  $n-r = \dim A$ , and we assume that  $A$  is a domain. Denote by  $\Omega_{B/k}^1$  (resp.  $\Omega_{A/k}^1$ ) the  $B$ -module of differentials. (Sometimes we also work from the geometric point of view and we will utilise the corresponding notations such  $\text{Spec}(A)$  or the sheaf associated to a  $A$ -module.) Then we have the following fundamental sequence:

$$U/U^2 \xrightarrow{\delta} \Omega_{B/k}^1 \otimes_B A \longrightarrow \Omega_{A/k}^1 \longrightarrow 0.$$

Recall that  $\Omega_{B/k}^1$  is a free  $B$ -module generated by  $dx_1, \dots, dx_n$  and for  $P \in U$ ,  $\delta(P)$  is the class of  $\sum_{i=1}^n \partial P / \partial x_i dx_i$  in  $\Omega_{B/k}^1 \otimes_B A$ . Then this last  $A$ -module is free and generated by  $\varepsilon_1, \dots, \varepsilon_n$  where  $\varepsilon_i$  is the class of  $dx_i$ .

0.2. DEFINITION: Put  $\Upsilon = \delta(U/U^2)$ . We have a  $\mathbf{A}$ -bilinear map

$$[\ , \ ]: \bigwedge^r \Upsilon \times \bigwedge^{n-r} \Omega_{A/k}^1 \rightarrow \bigwedge^n (\Omega_{B/k}^1 \otimes_B A) \cong A(\varepsilon_1 \wedge \dots \wedge \varepsilon_n)$$

in the following way:

Let  $v \in \wedge^r \Upsilon$  and  $\alpha \in \wedge^{n-r} \Omega_{A/k}^1$ ,  $\alpha = \sum_i \alpha_{i,1} \wedge \dots \wedge \alpha_{i,n-r}$  and  $z \in \wedge^{n-r} (\Omega_{B/k}^1 \otimes_B A)$ ,  $z = \sum z_{i,1} \wedge \dots \wedge z_{i,n-r}$  such that  $z_{i,j}$  is a preimage of  $\alpha_{i,j}$ . With these notations let  $[v, \alpha]$  the element in  $A$  such that

$$v \wedge z = [v, \alpha] \varepsilon_1 \wedge \dots \wedge \varepsilon_n.$$

Remark that because  $\mathbf{A}$  is a domain, the singular locus of  $\text{Spec}(A)$  is of codimension  $\geq 1$  and the generic point of  $\text{Spec}(A)$  is nonsingular, so by tensoring with the fraction field  $K(A)$  of  $\mathbf{A}$  this product becomes a duality product.

0.3. DEFINITION: The dualizing module of  $A$  is the  $A$ -module  $\omega_A$ :

$$\omega_A = \left\{ \alpha \in K(A) \otimes_A \bigwedge^{n-r} \Omega_{A/k}^1 \left[ \bigwedge^r Y, \alpha \right] \subset A \right\}$$

We'll call also  $\omega_A$  the dualizing module of  $X = \text{Spec}(A)$  and the sheaf associated to this module will be noted  $\omega_X$ .

0.4. LEMMA: Suppose that  $U$  is generated by a regular sequence  $(f_1, \dots, f_k)$ ,  $B$  and  $A$  are localized in the maximal ideal  $\mathfrak{m} = (x_1, \dots, x_n)$  and the subscripts are arranged so that  $\delta\bar{x}_{k+1}, \dots, \delta\bar{x}_n$  is a basis of  $K(A) \otimes \Omega_{A/k}^1$ . Then  $\omega_A$  is generated by

$$\varpi = \frac{\delta\bar{x}_{k+1} \wedge \dots \wedge \delta\bar{x}_n}{\left( \frac{\partial(f_1, \dots, f_k)}{\partial(x_1, \dots, x_k)} \right)}$$

where the denominator is the class in  $A$  of the jacobian determinant. In others words  $\varpi \in \omega_A$  becomes from  $z \in K(A) \otimes_A \bigwedge^{n-k} (\Omega_{B/k}^1 \otimes_B A)$  such that

$$\delta\bar{f}_1 \wedge \dots \wedge \delta\bar{f}_k \wedge z = \delta\bar{x}_1 \wedge \dots \wedge \delta\bar{x}_n.$$

0.5. Plurigenera. Let  $(X, \mathfrak{m})$  be a germ of a normal Cohen-Macaulay singularity,  $\dim X = d$  and  $X_{\text{reg}}$  the set of regular points of  $X$ . Let  $\omega_{X_{\text{reg}}}$  the sheaf of  $d$ -regular forms on  $X_{\text{reg}}$  (remark that this is also the dualizing module on  $X_{\text{reg}}$ ). Let  $i: X_{\text{reg}} \hookrightarrow X$  and put  $\omega_X^{[m]} = i_*(\omega_{X_{\text{reg}}}^{\otimes m})$ . Then  $\omega_X^{[m]}$  is a reflexive rank one sheaf for all integers  $m \geq 1$  and to  $\omega_X$  corresponds a Weil divisor  $K_X$  such that  $\omega_X^{[m]} = \mathcal{O}_X(mK_X)$ .

0.5.1. If  $k = \mathbb{C}$  we can define the sheaf  $L^{2/m}\Omega_X$  of locally  $2/m$ -integrable  $m$ -fold  $d$ -forms as follows.

Let  $V \subset X$  be an open relatively compact set in the usual topology of  $X$  and say

$$\omega \in L_m^2 \Omega_X(V) \subset \omega_X^{[m]} \quad \text{if} \quad \int_{V \cap X_{\text{reg}}} (\omega \wedge \bar{\omega})^{1/m} < \infty$$

where  $(\omega \wedge \bar{\omega})^{1/m}$  is defined in local coordinates  $(z_1, \dots, z_d)$ : if  $\omega = \phi(z)(dz_1 \wedge \dots \wedge dz_d)^m$  then

$$(\omega \wedge \bar{\omega})^{1/m} = |\phi(z)|^{2/m} \left( \frac{i}{2\pi} \right)^d dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_d \wedge d\bar{z}_d$$

and the integral means  $\lim_{V'} \int_{V \setminus V'} (\omega \wedge \bar{\omega})^{l/m}$  where  $V'$  moves in the neighbourhoods of  $X_{\text{sing}} = X \setminus X_{\text{reg}}$ .

We give the following results of Burns [1]:

0.5.2. LEMMA: *Suppose that  $X$  has only quotient singularities, i.e., locally  $X$  is the quotient of an open ball  $B(0, \varepsilon) \subset \mathbf{C}^n$  by a finite group of linear unitary transformations, where no element fixes an hyperplane in  $\mathbf{C}^n$ . Then for all  $V$ :*

$$L^{2/m} \Omega_X(V) = \omega_X^{[m]}(V).$$

*The following is a generalisation of Picard's lemma:*

0.5.3. LEMMA [9]: *Let  $\pi: \tilde{X} \rightarrow X$  be a good resolution of singularities,  $E$  the reduced exceptional divisor. Then*

$$L^{2/m} \Omega_X(V) = \pi_* \mathcal{O}(mK_{\tilde{X}} + (m - 1)E)$$

0.5.4. DEFINITION: Suppose that  $\mathbf{m}$  is an isolated singularity of  $X$ . Let  $\pi: \tilde{X} \rightarrow X$  be a good resolution of singularities, let  $E$  be the reduced exceptional divisor. Then we define the plurigenera:

1) The Knöller plurigenera

$$\gamma_m = \dim_k \omega_X^{[m]} / \pi_*(\omega_{\tilde{X}}^{\otimes m});$$

2) The  $L^2$ -plurigenera

$$\delta_m = \dim_k \omega_X^{[m]} / \pi_*(\omega_{\tilde{X}}^{\otimes m}((m - 1)E));$$

3) The Log-plurigenera

$$\lambda_m = \dim_k \omega_X^{[m]} / \pi_*(\omega_{\tilde{X}}^{\otimes m}(mE)).$$

REMARK:

- 1) The original definition of  $L^2$ -plurigenera is given in terms of the sheaf  $L^{2/m} \omega_X$  and here we use the lemma 0.5.3.
- 2) In dimension two one can find an exact formula for Knöller plurigenera in [6].

**1. Graded rings, Hilbert function (cf. [8])**

1.1. Let  $A$  be a  $\mathbf{Z}$ -graded ring such that  $A_m = 0$  for  $m < 0$ ,  $A_0 = k$  a field. We assume that  $A$  is a  $k$ -algebra of finite type. This means that  $A$  is a quotient of a polynomial ring  $B = k[X_1, \dots, X_n]$ ,  $\deg X_i = e_i \geq 1$ . Let  $M, N$  be two  $A$ -graded modules of finite type. We denote by  $\text{Hom}_A(M, N)$  the graded module of  $A$ -homomorphisms; by  $\text{Ext}'_A(M, N)$  the left derived module of  $\text{Hom}_A(M, N)$  with its natural graduation, by  $M[m]$  the shift of the module  $M$ , i.e.  $M[m]_i = M_{i+m}$  for all  $i \in \mathbf{Z}$ .

In what follows we will work in the category of  $B$ -graded modules of finite type.

1.2. DEFINITION: The Hilbert function of  $M$  is given by  $H(M, m) = \dim M_m$ .

The Poincare series of  $M$  is given by  $F(M, \lambda) = \sum_m H(M, m)\lambda^m$ .

1.3. LEMMA:

1. For  $M$  as before, there exists  $h$  polynomials  $Q_0, \dots, A_{h-1}$  with rational coefficients such that

$$H(M, mh + i) = Q_i(m) \quad \forall i = 0 \dots h - 1, \quad m \geq 0$$

2.  $\forall M, \exists s$  such that there are  $s$  polynomials  $S_1, \dots, S_s \in \mathbf{Q}[X]$  with rational coefficients and  $\alpha_1, \dots, \alpha_s$   $s$ -roots of unity such that

$$H(M, m) = \sum_{i=1}^s S_i(m)\alpha_i^m \quad m \geq 0.$$

The proof of this lemma is first obtained for  $M = B = k[X_1, \dots, X_n]$  and after we use the Hilbert syzygy theorem.

1.4. DEFINITION: The index of regularity of the Hilbert function is the integer  $a(M)$  such that:

$$H(M, mh + i) = Q_i(m) \quad \forall i = 0 \dots h - 1, \quad m \text{ with } mh + i > a(M)$$

but

$$H(M, mh_0 + i_0) \neq Q_{i_0}(m_0) \quad \text{for } i_0 \in [0 \dots h - 1] \text{ with } m_0h + i_0 = a(M).$$

1.5. PROPOSITION:

1.  $a(M)$  doesn't depend on the polynomials  $Q_i$  which appear in the lemma. In fact  $a(M)$  is the degree of the rational function  $F(M, \lambda)$
2.  $a(M)$  is the minimum of the integers such that

$$H(M, m) = \sum_{i=1}^s S_i(m)\alpha_i^m \text{ for } m > \alpha(M)$$

3. Let  $\omega_B = B[-e_1 \dots -e_n]$ , put  $E^i = \text{Ext}_B^i(M, \omega_B)$ . Then:

$$a(M) = -\min \left\{ m \mid \text{the coefficient of } \lambda^m \text{ in } \sum_{i=0}^n (-1)^i F(E^i, \lambda) \text{ is } \neq 0 \right\}.$$

REMARK: If  $M$  is a Cohen-Macaulay  $B$ -module of dimension  $d$ , we know that  $E^i = 0 \forall i \neq n - d$  and  $\omega_M = \text{Ext}_B^{n-d}(M, \omega_B)$  is the dualizing module of  $M$ .

1.6. COROLLARY: *If  $M$  is Cohen Macaulay then*

$$a(M) = -\min \{m / (\omega_M)_m \neq 0\}.$$

1.7. LEMMA: *If  $f \in B$  is a homogeneous regular element for  $M$  (i.e., a non zero divisor in  $M$ ) then  $a(M/fM) = a(M) + \text{deg } f$ . In particular if  $f_1, \dots, f_k$  is a regular sequence in  $B$  then*

$$a(B/(f_1, \dots, f_k)) = \sum_{i=1}^k \text{deg } f_i - \sum_{i=1}^n \text{deg } X_i.$$

EXAMPLE: Let  $B = k[X]$  with  $\text{deg } X = e \geq 1$ ; then the Hilbert function of  $B$  is given by

$$H(m) = \begin{cases} 1 & \text{if } m \in e\mathbf{N}; \\ 0 & \text{otherwise.} \end{cases}$$

and the polynomials  $Q_i$  are  $Q_0 \equiv 1, Q_i \equiv 0$  for  $i = 1 \dots e - 1$ ; the Hilbert function coincides with one of this polynomials for  $m > -e$ .

**2. Normal graded rings**

Let  $Z$  be a normal variety. We denote by  $\text{Wdiv}(Z, \mathbf{Z})$  the group of Weil divisors on  $Z$ , i.e., the free abelian group generated by the codimension one subvarieties of  $Z$  and  $\text{Wdiv}(Z, \mathbf{Q})$  the Weil divisors with rational coefficients. For  $D = \sum_i r_i W_i$  we set  $\lfloor D \rfloor = \sum_i \lfloor r_i \rfloor W_i$  where  $\lfloor r_i \rfloor$  is the integral part of  $r_i$ .

We associate to  $D$  the sheaf  $\mathcal{O}_Z(D)$  given by

$$\mathcal{O}_Z(D)(U) = \{f \in K(Z) / \text{div}(f) + D|_U \geq 0\}.$$

Then  $\mathcal{O}_Z(D)$  is a reflexive rank one sheaf and we have  $\mathcal{O}_Z(D) = \mathcal{O}_Z(\lfloor D \rfloor)$ ,  $\text{Hom}(\mathcal{O}_Z(D), \mathcal{O}_Z) = \mathcal{O}_Z(-\lfloor D \rfloor)$ . We say that  $D \in \text{Wdiv}(Z, \mathbf{Q})$  is a Cartier divisor if  $D$  is a Weil divisor with integer coefficients and the sheaf  $\mathcal{O}_Z(D)$  is a free  $\mathcal{O}_Z$ -module. Let  $D$  be a Cartier divisor and  $D'$  a Weil divisor with rational coefficients then the canonical morphism given by multiplication in the fraction field  $K(Z)$  of  $Z$ :

$$\mathcal{O}_Z(D) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(D') \rightarrow \mathcal{O}_Z(D + D')$$

is an isomorphism.

The following two theorems are due to Demazure (cf. [2]):

**2.2. THEOREM:** *Let  $A = \bigoplus_{n \geq 0} A_n$  be a normal graded algebra of finite type over a field  $k$ ,  $T$  a homogeneous element of degree one in the field of fractions  $K(A)$  of  $A$ . Consider the normal- $k$  scheme  $X = \text{Proj}(A)$ . Then there exists one and only one Weil divisor with rational coefficients such that*

$$A_n = H^0(X, \mathcal{O}_X(nD))T^n \quad \text{for } n \geq 0$$

in  $K(A)$  and

$$\mathcal{O}(n) = \mathcal{O}_X(nD)T^n \quad \text{for } n \in \mathbf{Z}.$$

**2.3.** Demazure also describes a modification of the singularity of the cone  $\text{Spec}(A)$  in the vertex  $\{\mathfrak{m}\}$ , where  $\mathfrak{m}$  is the maximal graded ideal of  $A$ . Let

$$C^+ = \text{Spec} \left( \bigoplus_{n \geq 0} \mathcal{O}_X(nD)T^n \right),$$

$$U(X, D) = \text{Spec} \left( \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(nD)T^n \right)$$

and

$$S = C^+ - U(X, D).$$

2.4. THEOREM ([2]):  $C^+$  is normal and we have the following cartesian diagram with  $\Psi$  birational

$$\begin{array}{ccccc}
 & U(X, D) \xrightarrow{\simeq} & U = Y - \{\mathfrak{m}\} & & \\
 & \downarrow & \downarrow & & \\
 X \xleftarrow{\pi} & C^+ & \xrightarrow{\Psi} & Y = \text{Spec}(A) & \\
 & \uparrow & & \uparrow & \\
 & S & \longrightarrow & \{\mathfrak{m}\} & 
 \end{array}$$

and  $X$  is the geometric quotient of  $C^+$  (resp  $U(X, D)$ ) by the  $k^*$ -action. The map  $\pi$  induces an isomorphism between  $S$  and  $X$ .

For the proofs of these theorems we refer to [2].

2.5. Moreover we have a one to one correspondence between codimension one irreducible subvarieties of  $X$  and codimension one  $k^*$ -invariant irreducible subvarieties in  $C^+$ . This correspondance is given by

$$V \mapsto F_V = \pi^{-1}(V)_{\text{red}} \in C^+$$

also, if

$$D = \sum_V \{p_V/q_V\}V$$

with  $(p_V, q_V) = 1$ , Demazure defines a homomorphism

$$\begin{array}{ccc}
 \pi^*: \text{Wdiv}(X, \mathbf{Q}) & \rightarrow & \text{Wdiv}(C^+, \mathbf{Q}) \\
 V & \mapsto & q_V F_V
 \end{array}$$

and he proves that

- 1)  $\text{div}(T) = \pi^*(D) + S \in \text{Wdiv}(C^+, \mathbf{Z})$ ;
- 2)  $\mathcal{O}_{C^+}(-rS) = \bigoplus_{n \geq r} \mathcal{O}_X(nD)T^n$  as graded  $\mathcal{O}_{C^+}$ -modules and
- 3) for  $B \in \text{Wdiv}(X, \mathbf{Z})$  we have

$$\mathcal{O}_{C^+}(\pi^*(B)) \simeq \bigoplus_{n \geq 0} \mathcal{O}_X(B + nD)T^n.$$

The following lemma is found in [10]

2.6. LEMMA:

1) Let  $D$  be as above and put

$$D' = \sum_V \{q_V - 1\}/q_V V$$

Then

$$\omega_{C^+} = \mathcal{O}_{C^+}(\pi^*(K_X + D') - S)$$

where  $K_X$  is a canonical divisor on  $X$ . Moreover  $A$  is a Gorenstein ring if and only if  $K_X + D'$  is linearly equivalent to the divisor  $a(A)D$  where  $a(A)$  is the index of regularity of the Hilbert function of  $A$ .

2) If  $A$  is normal and Gorenstein,  $Y = \text{Spec}(A)$ ,  $\Psi: C^+ \rightarrow Y$  as before then

$$\Psi^*(\omega_Y^{\otimes m}) = \mathcal{O}_{C^+}(mK_{C^+} + (ma(A) + m)S)$$

for  $m \geq 1$ .

**3. Resolution of singularities process**

3.1. *Local description of maps  $\pi$  and  $\Psi$  in Demazure's construction*

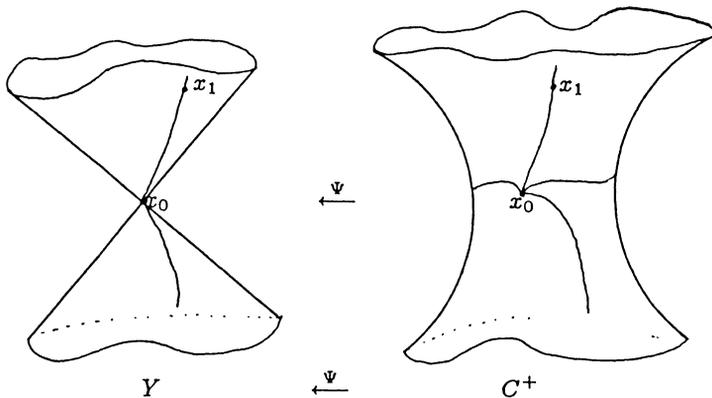
The map  $\pi: C^+ \rightarrow X$  is affine, so let  $V$  an affine open set in  $X$ ,  $V = \text{Spec}(H^0(V, \mathcal{O}_X))$  and  $\pi^{-1}(V) = \text{Spec}(\bigoplus_{n \geq 0} H^0(V, \mathcal{O}_X(nD)))$ , the map  $\pi$  is then given by the inclusion of rings and  $\psi: C^+ \rightarrow Y$  is given by the restriction map

$$A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) \rightarrow \bigoplus_{n \geq 0} H^0(V, \mathcal{O}_X(nD)).$$

we have the following theorem obtained in collaboration with F. Knop. During my stay in Chapel Hill J. Wahl showed me another proof.

3.2. THEOREM: *With the above notations assume that  $Y = \text{Spec}(A)$  has only an isolated singularity in the vertex  $\{\mathfrak{m}\}$ ; then  $C^+$  and  $X$  have only cyclic quotients singularities.*

*Proof.* The statement is local and by 3.1 we can suppose that  $C^+, X, U(X, D)$  are affine varieties by the  $k^*$ -action. We identify  $S$  with  $X$ . Let  $x_1 \in C^+ - S$  and  $x_0 \in X$  the unique point in  $X$  on the closure of  $k^* \cdot x_1$



Let  $F$  be the isotropy group of  $x_1$ . By hypothesis  $U(X, D) \simeq Y - \{\mathfrak{m}\}$  is non singular so we can apply the Slice theorem who says that there exists a slice  $\Sigma$  (non singular affine variety) invariant by the action of  $F$  such that  $U(X, D)$  is the geometric quotient of  $k^* \times \Sigma$  by  $F$ , where the action of  $F$  is given by

$$\forall t \in F, a \in k^*, b \in \Sigma \quad t \cdot (a, b) = (at^{-1}, tb).$$

In other words we have *locally* a  $k^*$ -isomorphism

$$(k^* \times \Sigma)/F \simeq U(X, D).$$

In particular  $S$  is locally isomorphic to  $\Sigma/F$  and this shows that  $S$  has only cyclic quotient singularities. We'll prove that in fact we have a geometrical quotient

$$(\mathbb{A}_k^1 \times \Sigma)/F \simeq C^+$$

Consider the following diagram

$$\begin{array}{ccc} (k^* \times \Sigma)/F \hookrightarrow & \Gamma & \xrightarrow{\psi_2} & C^+ \\ & \downarrow \psi_1 & & \downarrow \pi \\ & (\mathbb{A}_k^1 \times \Sigma)/F & \longrightarrow & X = \Sigma/F \end{array}$$

where  $\Gamma$  is the closure of the diagonal embedding of  $(k^* \times \Sigma)/F$  in the product of affine spaces  $(\mathbf{A}_k^1 \times \Sigma)/F$  and  $C^+$ . Both  $F$  and  $k^*$  acts on  $\mathbf{A}_k^1 \times \Sigma$  in a natural way. Now it is clear that any point in  $\mathbf{A}_k^1 \times \Sigma/F$  has a preimage by  $\Psi_1$ , i.e.,  $\Psi_1$  is surjective and the same is true for  $\Psi_2$ , also  $(\mathbf{A}_k^1 \times \Sigma)/F$  and  $C^+$  are normal and affine so  $\Psi_1$  and  $\Psi_2$  are isomorphisms (cf. [4] Lemma 1.8).

**4. Computation of  $L^2$ - and Log-plurigenera for Gorenstein quasi-homogeneous isolated singularities**

4.1. THEOREM: *Let  $(Y = \text{Spec}(A), \mathfrak{m})$  be a quasi homogeneous Gorenstein isolated singularity then*

1) *the  $L^2$ - plurigenus  $\delta_m$  of order  $m$  is given by*

$$\delta_m = \dim_{\mathbb{C}} \left( \bigoplus_{i \leq ma(A)} A_i \right)$$

2) *The Log-plurigenus  $\lambda_m$  is given by*

$$\lambda_m = \dim_{\mathbb{C}} \left( \bigoplus_{i < ma(A)} A_i \right)$$

where  $a(A)$  is the index of regularity of the Hilbert function of  $A$ .

4.2. COROLLARY: *Let  $A = k[X_1, \dots, X_n]/(f_1, \dots, f_k)$  be a quasi homogeneous complete intersection isolated singularity; then the  $L^2$ -plurigenera and the Log-plurigenera can be effectively computed from the Koszul graded complex:*

$$0 \rightarrow B[-\deg f_1 - \dots - \deg f_k] \rightarrow \dots \rightarrow \bigoplus_{i=1}^k B[-\deg f_i] \rightarrow B \rightarrow A \rightarrow 0$$

*this turns out to be equivalent to finding the number of integer points in some polyhedra.*

The proof of the corollary will follow from that of the theorem

*Proof of the Theorem.* With the above notation let  $\phi: \tilde{C}^+ \rightarrow C^+$  be a good resolution of singularities and  $E'$  the reduced exceptional divisor. Then because  $C^+$  has only cyclic quotient singularities we have for any integer  $m > 0$ :

$$\phi_* \mathcal{O}_{\tilde{C}^+}(mK_{\tilde{C}^+} + (m - 1)E') = \mathcal{O}_{C^+}(mK_{C^+})$$

(In fact we have

$$\phi_* \mathcal{O}_{\tilde{C}^+}(mK_{\tilde{C}^+} + jE') = \mathcal{O}_{C^+}(mK_{C^+})$$

for any integer  $j \geq m - 1$ ) and:

$$\phi_* \mathcal{O}_{\tilde{C}^+}(mK_{\tilde{C}^+} + (m - 1)E' + (m - 1)\tilde{S}) = \mathcal{O}_{C^+}(mK_{C^+} + (m - 1)S)$$

where  $\tilde{S}$  is the proper transform of  $S$ .

**REMARK:** In the Gorenstein case, according to 2.6,  $K_{C^+}$  is linearly equivalent to  $(a(A) + 1)\pi^*(D)$  so  $S$  is not a fixed component of  $K_{C^+}$ .

Now using 4.2, 2.6 and the projection formula we obtain:

$$\begin{aligned} & (\Psi \circ \phi)_* \mathcal{O}_{\tilde{C}^+}(mK_{\tilde{C}^+} + (m - 1)(E' + \tilde{S})) \\ &= \omega_Y^{\otimes m} \otimes \Psi_* \mathcal{O}_{C^+}(-(ma(A) + 1)S) \end{aligned}$$

and the  $\mathbf{A}$ -module associated to this sheaf on  $Y$  is

$$\bigoplus_{n \geq 1} (A[ma(A)])_n$$

so

$$\delta_m = \dim_{\mathbf{C}} \left( \bigoplus_{n \leq 0} (A[ma(A)])_n \right) = \dim_{\mathbf{C}} \bigoplus_{n \leq ma(A)} A_n.$$

### 5. Newton generic complete intersections

5.1. DEFINITION: Let  $f \in \mathbf{C}[X_1, \dots, X_n]$  be,  $f(X) = \sum c_\alpha X^\alpha$ .

a) The Newton polyhedron of  $f$ :  $\Gamma^+(f) = \text{Convex hull in } \mathbf{R}^n \text{ of } \bigcap_{c_\alpha \neq 0} \{\alpha\} + \mathbf{R}_+^n$ ;

- b) The Newton boundary of  $f$ :  $\Gamma(f)$  is the union of the compact faces of  $\Gamma^+(f)$ ;
- c) We say that  $f$  is convenient if for all  $i \in [1 \dots n]$  there exists  $m_i$  such that  $c_{m_i} X_i^{m_i}$  appears in  $f$ .

5.2. DEFINITION: Let  $f_l \in \mathbf{C}[X_1, \dots, X_n]$ ,  $l = 1 \dots k$  a sequence of convenient polynomials,  $\Gamma_l^+$  their Newton polyhedra; for  $a \in (\mathbf{R}_+)^n$  we denote

$$m^l(a) = \min \{ \langle a, \alpha \rangle \mid \alpha \in \Gamma_l^+ \}$$

and

$$f_l^a = \sum_{\substack{\alpha \in \Gamma_l^+ \\ \langle a, \alpha \rangle = m^l(a)}} A_{l,\alpha} X^\alpha$$

The sequence  $\{f_1, \dots, f_k\}$  is said to be non-degenerate (with respect to their Newton polyhedron) if for all  $1 \leq j \leq k$  and all  $a \in (\mathbf{R}_+ \setminus \{0\})^n$  the following condition is satisfied:

In each point  $q \in (\mathbf{C} \setminus \{0\})^n$  such that  $f_1^a(q) = \dots = f_j^a(q) = 0$  the differentials  $df_1^a(q), \dots, df_j^a(q)$  are linearly independent in the tangent space of  $\mathbf{C}^n$  in  $q$ .

In this case we also say that the germ  $H_k$  defined near the origin by

$$H_k = \{x \in \mathbf{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}$$

is nondegenerate with respect to  $\Gamma_1^+, \dots, \Gamma_k^+$ .

REMARK:

- 1) This is a generic notion in the Zariski topology sense;
- 2) We have only a finite number of conditions in  $a \in (\mathbf{R}_+ \setminus \{0\})^n$ ; in fact in general the set  $\{\alpha \in \Gamma_l^+ \mid \langle a, \alpha \rangle = m^l(a)\}$  will have only one element and a monomial is never zero for  $q \in (\mathbf{C} \setminus \{0\})^n$ .

5.3. THEOREM: Let  $f_1, \dots, f_k \in \mathbf{C}[X_1, \dots, X_n]$  be a nondegenerate sequence (with respect to their Newton polyhedron), let  $H_k = \{x \in \mathbf{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}$  be the isolated singularity defined near the

origin by  $f_1, \dots, f_k$  then the  $L^2$ -plurigenera are given by

$$\begin{aligned} \delta_m(H_k) &= R(m(\Gamma_1^+ + \dots + \Gamma_k^+)) \\ &\quad - \sum_{i=1}^k R(m(\Gamma_1^+ + \dots + \hat{\Gamma}_i^+ + \dots + \Gamma_k^+)) \\ &\quad + \dots + (-1)^{k-1} \sum_{i=1}^k R(m\Gamma_i^+), \end{aligned}$$

where  $R(\Delta)$  designs the number of integral points  $\beta$  with all coordinates positive such that  $m(1, \dots, 1) + \beta$  is not in the interior of  $\Delta$ . ( $\hat{\Gamma}_i^+$  means that the term  $\Gamma_i^+$  is not in the sum.)

The same formula can be used to calculate the Log-plurigenera but the meaning of the function  $R$  must be changed by:  $R(\Delta)$  designs the number of integer points  $\beta$  with all coordinates positive such that  $m(1, \dots, 1) + \beta$  is not in  $\Delta$ .

Let me recall the following:

5.4. PROPOSITION: (Resolution of singularities) [7]. Let  $f_1, \dots, f_k$  be a non degenerate sequence with respect to their Newton polyhedra. Let  $H_j = \{x \in \mathbf{C}^n \mid f_1(x) = \dots = f_j(x) = 0\}$ . There exists a simplicial polyhedra decomposition  $\Sigma$  of  $(\mathbf{R}_+)^n$  and  $\pi: X_\Sigma \rightarrow \mathbf{C}^n$  where we note by  $X_\Sigma$  the toric variety defined by  $\Sigma$  such that:

- 1)  $X_\Sigma$  is non singular and the restriction  $\pi: X_\Sigma \setminus \pi^{-1}(0) \rightarrow X \setminus \{0\}$  is an isomorphism.
- 2) For all  $j = 1 \dots k$  the proper transform  $\tilde{H}_j$  of  $H_j$  is non singular near the exceptional divisor and cuts it transversely.
- 3) Locally the map  $\pi$  is described as follows:

Let  $\sigma$  be a simplex in  $\Sigma$ , i.e.,  $\sigma = \langle a^1, \dots, a^n \rangle$  where  $a^1, \dots, a^n$  is a basis of  $\mathbf{Z}^n$ . Then  $\pi: X_\sigma = \mathbf{C}^n \rightarrow \mathbf{C}^n$  is given by

$$x_i = y_1^{a_i^1} \dots y_n^{a_i^n}$$

The proper transform  $\tilde{H}_j$  of  $H_j(j = 1 \dots k)$  is locally described by the polynomials  $\tilde{f}_j(y)$  defined by the relation

$$f_j \circ \pi = y_1^{m^j(a^1)} \dots y_n^{m^j(a^n)} \tilde{f}_j(y)$$

and  $\tilde{f}_j(0) \neq 0$ .

5.5. LEMMA: Let  $\pi: \tilde{H}_k \rightarrow H_k$  be the resolution map of 5.4 and  $\omega_{H_k}$  (resp.  $\omega_{\tilde{H}_k}$ ) the dualizing module of  $H_k$  (rep.  $\tilde{H}_k$ ) Let

$$\varpi = \frac{\delta \bar{x}_{k+1} \wedge \dots \wedge \delta \bar{x}_n}{\left( \frac{\partial(f_1, \dots, f_k)}{\partial(x_1, \dots, x_k)} \right)}$$

by a generator of  $\omega_{H_k}$ . then

$$\pi^*(\varpi) = \frac{x_1 \dots x_n \tilde{\varpi}}{(y_1 \dots y_n) y_1^{\sum_{i=1}^k m^i(a^1)} \dots y_n^{\sum_{i=1}^k m^i(a^n)}}$$

where  $\tilde{\varpi}$  is a generator of  $\omega_{\tilde{H}_k}$ .

*Proof.* This follows from 0.4 and the formulas

$$dx_i = \sum_j a_i^j \frac{x_i}{y_j} dy_j$$

$$f_i \circ \pi = y_1^{m_i(a^1)} \dots y_n^{m_i(a^n)} \tilde{f}_i(y)$$

with  $\tilde{f}_i(0) \neq 0$ .

From this lemma we obtain immediately the following:

5.6. LEMMA: Let  $A = k[X_1, \dots, X_n]/(f_1, \dots, f_k)$ ,  $p \geq 0$ ,  $m > 0$ .

For  $a^i \in \Sigma$ ,  $a^i \in (\mathbf{R} \setminus \{0\})^n$  and  $P \in k[X_1, \dots, X_n]$

we note:

$$\langle a^i, P \rangle = \min \{ \langle a^i, \alpha \rangle / \alpha \text{ a monomial in } P \}$$

and let  $\bar{P}$  be the class of  $P$  in  $A$ .

Then:

$$\begin{aligned} \pi_* (\mathcal{O}_{\tilde{H}_k}(mK_{\tilde{H}_k} + pE)) &= \{ \bar{P} \in A / \langle a^i, P \cdot (x_1 \dots x_n)^m \rangle \\ &\geq \langle a^i, (f_1 \dots f_k)^m \rangle + m - p \}. \end{aligned}$$

We have three particular cases:

1st particular case. –  $p = m$  (Log plurigenera)

$$\pi_*(\mathcal{O}_{\tilde{H}_k}(mK_{\tilde{H}_k} + pE)) = \{P \in A / \langle a^j, P \cdot (x_1 \dots x_n)^m \rangle \geq \langle a^j, (f_1 \dots f_k)^m \rangle\}$$

This means that for the monomials  $\alpha$  in  $P \alpha + m(1, \dots, 1)$  is in the Newton polyhedron  $\Gamma^+((f_1, \dots, f_k)^m)$

2nd particular case. –  $p = m - 1$  ( $L^2$ -Plurigenera)

$$\pi_*(\mathcal{O}_{\tilde{H}_k}(mK_{\tilde{H}_k} + pE)) = \{P \in A / \langle a^j, P \cdot (x_1 \dots x_n)^m \rangle > \langle a^j, (f_1 \dots f_k)^m \rangle\}.$$

This means that for the monomials  $\alpha$  in  $P \alpha + m(1, \dots, 1)$  is in the interior of the Newton polyhedron  $\Gamma^+((f_1, \dots, f_k)^m)$ .

3rd particular case. –  $p = 0$  (Knöller Plurigenera)

$$\begin{aligned} \pi_*(\mathcal{O}_{\tilde{H}_k}(mK_{\tilde{H}_k})) &= \{P \in A / \langle a^j, P \cdot (x_1 \dots x_n)^m \rangle \\ &\geq \langle a^j, (f_1 \dots f_k)^m \rangle + m\}. \end{aligned}$$

Remark that in this case we can't give a direct interpretation only in function of the Newton polyhedron  $\Gamma^+(f_1, \dots, f_k)$  but also on the  $a^j \in \Sigma$ . In fact we need a precise description of the polyhedral decomposition  $\Sigma$  and this is not known in dimension  $\geq 3$ .

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