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Topologically ∞-determined map germs are topologically cone-like

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Dedicated to Professor H. Noguchi on his 60th birthday

1. Introduction

Let $C_0^\infty(R^n, R^p)$ be the space of all C^∞ mappings of R^n into R^p with $f(0) = 0$. For a positive number γ, we set

$$S_\gamma^{m-1} = \{x \in R^m |\ \|x\| = \gamma\} \quad \text{and} \quad D_\gamma^m = \{x \in R^m |\ \|x\| \leqq \gamma\}.$$

For any C^∞ mapping f of $C_0^\infty(R^n, R^p)$, we write

$$f_{\gamma,} = f|_{f^{-1}(S_\gamma^{p-1})}, \quad f_{\gamma,0} = f|_{f^{-1}(D_\gamma^p - \{0\})}$$

$$f_{\gamma,\delta} = f|_{D_\gamma^n \cap f^{-1}(S_\delta^{p-1})} \quad \text{and} \quad f_{\gamma,\delta,0} = f|_{D_\gamma^n \cap f^{-1}(D_\delta^p - \{0\})}.$$

where expressions like $f|_{f^{-1}(S_\gamma^{p-1})}$ mean the restricted mappings and γ, δ are positive numbers.

Then T. Fukuda has proved the following Cone Structure Theorem (Theorem 1.1) in his papers ([2]). The purpose of this paper is to give a flat version of his Cone Structure Theorem.

THEOREM 1.1: *There exists an infinite codimensional subset Σ_∞ of $C_0^\infty(R^n, R^p)$ such that every C^∞ mapping f: $R^n \to R^p$ belonging to $C_0^\infty(R^n, R^p) - \Sigma_\infty$ has the following properties:*

(A) ($n \leqq p$) *there exists a positive number ε_0 such that for any number ε with $0 < \varepsilon \leqq \varepsilon_0$ we have*
(A-1) *the space $f^{-1}(S_\varepsilon^{p-1})$ is a compact C^∞ manifold without boundary,*

(A-2) *the restricted mapping* f_ε: $f^{-1}(S_\varepsilon^{p-1}) \to S_\varepsilon^{p-1}$ *is topologically stable* (C^∞ *stable if* (n, p) *is a nice pair of dimensions in J. Mather's sense*),

(A-3) *the restricted mapping* $f_{\varepsilon,0}$: $f^{-1}(D_\varepsilon^p - \{0\}) \to D_\varepsilon^p - \{0\}$ *is proper, topologically stable* (C^∞ *stable if* (n, p) *is nice*) *and topologically equivalent* (C^∞ *equivalent if* (n, p) *is nice*) *to the product mapping* $f_\varepsilon \times \mathrm{id.}_{(0,\varepsilon)}$: $f^{-1}(S_\varepsilon^{p-1}) \times (0, \varepsilon) \to S_\varepsilon^{p-1} \times (0, \varepsilon)$ *defined by* $(x, t) \to (f(x), t)$,

(B) $(n > p)$ *for any sufficiently small positive numbers* ε *and* δ, *the upper bound of* ε *depending on* f *and the upper bound of* δ *depending on* ε *and* f, *we have*

(B-1) $D_\varepsilon^n \cap f^{-1}(S_\delta^{p-1})$ *is a* C^∞ *manifold, in general with boundary,*

(B-2) *the restricted mapping* $f_{\varepsilon,\delta}$: $D_\varepsilon^n \cap f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ *is topologically stable* (C^∞ *stable if* (n, p) *is nice*),

(B-3) *the restricted mapping* $f_{\varepsilon,\delta,0}$: $D_\varepsilon^n \cap f^{-1}(D_\delta^p - \{0\}) \to D_\delta^p - \{0\}$ *is proper, topologically stable* (C^∞ *stable if* (n, p) *is nice*) *and topologically equivalent* (C^∞ *equivalent if* (n, p) *is nice*) *to the product mapping* $f_{\varepsilon,\delta} \times \mathrm{id.}$: $(D_\varepsilon^n \cap f^{-1}(S_\delta^{p-1}) \times (0, \delta) \to S_\delta^{p-1} \times (0, \delta)$ *defined by* $(x, t) \to (f(x), t)$.

For any C^∞ mapping f: $R^n \to R^p$, let A_f be the space of all C^∞ mappings of R^n into R^p with the same formal power series at 0 as f. The space A_f has the induced topology of the Whitney C^∞ topology. Then our main result can be stated as follows.

THEOREM 1.2: *For any* C^∞ *mapping* f *of* $C_0^\infty(R^n, R^p)$, *there exists a dense subset* A_f^0 *of* A_f *such that every* C^∞ *mapping* g: $R^n \to R^p$ *belonging to* A_f^0 *has the properties* (A), (B) *above.*

COROLLARY 1.3: *Topologically* ∞*-determined map germs are topologically cone-like.*

Our Corollary 1.3 answers the question proposed by C.T.C. Wall ([4], question 30).

2. Reduction of Theorem 1.2 to Theorem 2.1

The notations used here are essentially the same as those of Thom [7], Mather [5, 6], and Fukuda [1, 2]. For instance $J^r(n, p)$ is the set of the r-jets of C^∞ map germs: $(R^n, 0) \to (R^p, 0)$, $J^r(R^n, R^p)$ denotes the r-jet bundle of C^∞ mappings of R^n into R^p and ${}_mJ^r(R^n, R^p)$ is the m-fold r-jet bundle of C^∞ mappings of R^n into R^p:

${}_mJ^r(R^n, R^p) = \{(j^r g_1(q_1), \ldots, j^r g_m(q_m) \in (J^r(R^n, R^p))^m | (q_1, \ldots, q_m) \in (R^n)^{(m)}\}$, where for a set X, $X^{(m)}$ denotes the set $\{(q_1, \ldots, q_m) \in X^m | q_i \neq q_j$ if $i \neq j\}$ etc. Any points $x = (x_1, \ldots, x_m)$ of $(R^n)^m$ are called *multipoints.*

DEFINITION: Let X be a semi-algebraic submanifold of $J^k(R^n, R^p)$, U be a subset of $(R^n)^{(m)}$ and let μ: $(J^k(R^n, R^p))^m \to R$ be a polynomial function with the following properties (a) and (b):

Property (a): $C(\mu|_X) \cap (U \times R^p)^m \times (J^k(n, p))^m) = \phi$, where $C(\mu|_X)$ is the set of all critical points of the restricted function

$\mu|_X$: $X \to R$.

Property (b): μ depends only on the O-jet.

Then for any mapping g of $C_0^\infty(R^n, R^p)$, we say g *has the property* $T(X, \mu)$ *on* U if the following (1), (2) and (3) are satisfied:
(1) ${}_mj^k g$ *is transversal to* X *at every multipoint of* U,
(2) *if* codim $X = mn$, ${}_mj^k g(U) \cap X = \phi$,
(3) ${}_mj^k g$ *is transversal to* $X \cap \mu^{-1}(\varepsilon)$ *for all* $\varepsilon \in R$ *at every multipoint of* U.

THEOREM 2.1: *Let* X *be a semi-algebraic submanifold of* ${}_mJ^k(R^n, R^p)$, μ: $(J^k(R^n, R^p))^m \to R$ *be a polynomial function with the properties* (a) *and* (b) *above and let* $A_f^t(X, \mu)$ *be the space of all* C^∞ *mappings of* A_f *having the property* $T(X, \mu)$ *on* $(R^n - \{0\})^{(m)}$. *Then* $A_f^t(X, \mu)$ *is dense in* A_f *for any* f *of* $C_0^\infty(R^n, R^p)$.

Proof that Theorem 2.1 => Theorem 1.2

Recall Mather's various stability theorems ([3, 5, 6]).

LEMMA 2.2: *For a* C^∞ *proper stable mapping* F: $R^n \to R^p$, *there exists a unique pair* $(\mathscr{S}_F(R^n), \mathscr{S}_F(R^p))$ *of semi-algebraic stratifications of* R^n *and* R^p *satisfying the following properties*:
(1) F *is a stratified mapping with respect to* $(\mathscr{S}_F(R^n), \mathscr{S}_F(R^p))$,
(2) *any pair of strata of* $\mathscr{S}_F(R^n)$ *satisfies condition* a_F,
(3) *if* $(\mathscr{S}'(R^n), \mathscr{S}'(R^p))$ *is an another pair of stratifications which satisfies* (1) *and* (2), *then* $(\mathscr{S}'(R^n), \mathscr{S}'(R^p))$ *is a refinement of* $(\mathscr{S}_F(R^n), \mathscr{S}_F(R^p))$.

LEMMA 2.3: *For any pair of dimensions* (n, p), *there exist a positive number* k *and a semi-algebraic stratification* $\mathscr{S}(n, p)$ *of* $J^k(n, p)$ *invariant under the action of* $L^k(n) \times L^k(p)$ *such that*

(1) *two map germs f and g of $C_0^\infty(R^n, R^p)$ are topologically stable and topologically equivalent each other if their k-jets $j^k f(0)$ and $j^k g(0)$ belong to the same stratum X of $\mathscr{S}(n, p)$ with* codim $X \leqq n$.

(2) *if a proper C^∞ mapping f: $R^n \to R^p$ is multitransversal to the stratification $\mathscr{S}(J^k(R^n, R^p))$ of $J^k(R^n, R^p)$ canonically induced from $\mathscr{S}(n, p)$, then f is a stratified mapping with respect to the stratifications $(\mathscr{S}_f(R^n), \mathscr{S}_f(R^p))$ induced from $\mathscr{S}(J^k(R^n, R^p))$ by the multi-jet extension of f, any pair of strata of $\mathscr{S}_f(R^n)$ satisfies condition a_f and f is topologically stable,*

(3) *any other stratification of $J^k(n, p)$ satisfying* (1) *and* (2) *is a refinement of $\mathscr{S}(n, p)$.*

LEMMA 2.4: *For any pair of dimensions (n, p), there exists a closed semi-algebraic subset Σ of $J^k(n, p)$, where k is as in Lemma* 2.3, *invariant under the action of $L^k(n) \times L^k(p)$ having codimension $> n$ such that the canonically induced subset $\Sigma_{(R^n, R^p)}$ of $J^k(R^n, R^p)$ has the following properties*:

(1) *For any proper C^∞ mapping f: $R^n \to R^p$, the set $j^k f(R^n) \cap \Sigma_{(R^n, R^p)}$ is empty if and only if there exist an integer m and a proper C^∞ stable mapping F: $R^n \times R^m \to R^p \times R^m$ of the form $F(x, t) = (g(x, t), t)$, where $x \in R^n$, $t \in R^m$ and $g(x, 0) = f(x)$.*

(2) *Let f and F be as in* (1). *Then f is multitransversal to the stratification $\mathscr{S}(J^k(R^n, R^p))$ given by Lemma* 2.3 *if and only if the inclusion mappings i: $R^n \times \{0\} \to R^n \times R^m$ and j: $R^p \times \{0\} \to R^p \times R^m$ are both transversal to the canonical stratifications $(\mathscr{S}_F(R^n \times R^m), \mathscr{S}_F(R^p \times R^m))$ given by Lemma* 2.2.

(3) *Let F and f be as in* (2). *Then the pair of stratifications $(i^*(\mathscr{S}_F(R^n \times R^m)), j^*(\mathscr{S}_F(R^p \times R^m)))$ induced from $(\mathscr{S}_F(R^n \times R^m), \mathscr{S}_F(R^p \times R^m))$ by i and j coincides with the pair $(\mathscr{S}_f(R^n), \mathscr{S}_f(R^p))$, given by Lemma* 2.3(2).

Now we prove that Theorem 2.1 implies Theorem 1.2. Let k be the integer, $\mathscr{S}(J^k(R^n, R^p))$ be the semi-algebraic stratification and Σ be the semi-algebraic subset of $J^k(n, p)$ given by Lemma 2.3 and 2.4. Set

$$Q = \{j^k g(x) \in J^k(R^n, R^p - \{0\}) | \operatorname{grad}(\|g(x)\|) \neq 0\}.$$

Note that codim $(J^k(R^n, R^p - \{0\}) - Q)$ in $J^k(R^n, R^p - \{0\})$ is n. We recall Fukuda's construction of the semi-algebraic stratification $\mathscr{S}(Q)$ of Q ([2-II], pp. 506–507). For any stratum X of $\mathscr{S}(n - 1, p - 1)$, consider the subset $X(Q)$ such that a jet $j^k g(x)$ belongs to $X(Q)$ if and only if $j^k(g|_{U \cap g^{-1}(S_\delta^{p-1})})(x)$ is contained in $X(U \cap g^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$, where U is a sufficiently small neighborhood of x, $\delta = \|g(x)\|$ and $X(U \cap g^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ is the stratum of $J^k(U \cap g^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ corresponding to

$X \in \mathscr{S}(n-1, p-1)$. Then he proved $\mathscr{S}(Q) = \{X(Q) | X \in \mathscr{S}(n-1, p-1)\}$ was a semi-algebraic stratification of Q. Take a semi-algebraic stratification $\mathscr{S}(\Sigma)$ of Σ and set

$$S = \{j^k g(x) \in J^k(R^n, R^p) | g(x) = 0\}.$$

Let $\mathscr{A}$(resp. $\mathscr{B}$) be the set of all strata of the forms

$$X = (X_1 \times \cdots \times X_m) \cap \Delta_m \quad \text{with} \quad X_i \in \mathscr{S}(J^k(R^n, R^p))$$

$$\text{(resp. } X_i \in \mathscr{S}(Q)), \tag{$*$}$$

where $m \leqq p+1$ and $\Delta_m = \{j^k g_1(q_1), \ldots, j^k g_m(q_m) \in {}_m J^k(R^n, R^p) | g_1(q_1) = \cdots = g_m(q_m)\}$. Let μ_p: $(J^k(R^n, R^p))^m \to R$ be the function defined by $\mu_p(j^k g_1(x_1), \ldots, j^k g_m(x_m)) = \|g_1(x_1)\|^2$ and let μ_n: $J^k(R^n, R^p) \to R$ be the function defined by $\mu_n(j^k h(x)) = \|x\|^2$.

Then for any C^∞ mapping f of $C_0^\infty(R^n, R^p)$, set

$$A_f^t = \begin{cases} \left(\bigcap_{X \in \mathscr{A}} A_f^t(X, \mu_p)\right) \cap \left(\bigcap_{Y \in \mathscr{S}(\Sigma)} A_f^t(Y, \mu_n)\right) \cap (A_f^t(S, \mu_n)) & \\ & \text{(if } n \leqq p) \\ \left(\bigcap_{X \in \mathscr{B}} A_f^t(X, \mu_p)\right) \cap \left(\bigcap_{Y \in \mathscr{S}(q) \cup \mathscr{S}(\Sigma)} A_f^t(Y, \mu_n)\right) \cap (A_f^t(S, \mu_n)) & \\ & \text{(if } n > p). \end{cases}$$

Since A_f^t is a finite intersection of the sets of the forms $A_f^t(Z, \mu)$, this is dense in A_f by Theorem 2.1.

Take any mapping g of A_f^t. Then we have the following:

(A) $(n \leqq p)$

$$j^k g(R^n - \{0\}) \cap \Sigma = \phi, \tag{2.1}$$

$$j^k g(R^n - \{0\}) \cap S = \phi, \tag{2.2}$$

$$g|_{(R^n - \{0\})}: R^n - \{0\} \to R^p \textit{ is multitransversal to } \mathscr{S}(J^k(R^n, R^p)), \tag{2.3}$$

for any integer $m \leqq p+1$ and any manifold X of $\mathscr{A}$,
${}_m j^k(g|_{(R^n - \{0\})})$ *is transversal to $X \cap \mu_p^{-1}(\varepsilon)$ for every $\varepsilon > 0$.* (2.4)

(B) $(n > p)$

$$j^k g(R^n - \{0\}) \cap \Sigma = \phi, \tag{2.1}$$

for any integer $m \leqq p + 1$ *and any manifold* X *of* $\mathscr{B}$,
$${}_m j^k (g|_{(R^n - g^{-1}(0))}) \text{ is transversal to } X \cap \mu_p^{-1}(\varepsilon) \text{ for every } \varepsilon > 0, \tag{2.4}$$

$$j^k g(R^n - g^{-1}(0)) \subset Q, \tag{2.5}$$

$$j^k (g|_{(R^n - \{0\})}) \text{ is transversal to } S \cap \mu_n^{-1}(\varepsilon) \text{ for every } \varepsilon > 0, \tag{2.6}$$

for any stratum X *of* $\mathscr{S}(Q)$, $j^k(g|_{(R^n - g^{-1}(0))})$ *is transversal to*
$$X \cap \mu_n^{-1}(\varepsilon) \text{ for every } \varepsilon > 0. \tag{2.7}$$

Then Fukuda proved that any C^∞ mapping g satisfying (2.1), (2.2.), (2.3) and (2.4) (resp. (2.1), (2.4), (2.5), (2.6) and (2.7)) had the property (A) (resp. (B)) in Theorem 1.1 using only Lemma 2.3, Lemma 2.4, Second Isotopy Lemma, Mather's results on C^∞ stability ([6]) and (2.1)–(2.7) (see [2-I], pp. 242–249 and [2-II], pp. 510–514).

Therefore exactly as in his proof, we have our conclusion. Q.E.D.

3. Lemmas

For any jet $j^r f(x) \in J^r(R^n, R^p)$, we denote $\Phi_0(j^r f(x))$ by $j_0^r f(x)$, where Φ_0: $J^r(R^n, R^p) = R^n \times R^p \times J^r(n, p) \to R^n \times J^r(n, p)$ is the canonical projection.

LEMMA 3.1 [2-I]: *Let* $g_1, \ldots, g_m$ *and* f *be* C^∞ *functions defined on* R^n *and let* $p_1, \ldots, p_m$ *be m distinct points of* R^n *all different from a point* x *of* R^n *and let* r *and* k *be positive integers. Then there exists a polynomial* $H(x) = \Sigma a_\omega x^\omega$ *of* n *variables of degree* $r + m(k + 1)$ *satisfying the following conditions*:

$$j^r H(x) = j_0^r f(x) \times \{0\}, \tag{3.1}$$

$$j^k H(p_i) = j^k g_i(p_i), \quad i = 1, \ldots, m. \tag{3.2}$$

Moreover the polynomial $H(x)$ *can be chosen so that it depends only on* $(j_0^r f(x), j^k g_1(p_1), \ldots, j^k g_m(p_m))$. *We thus denote it by*

$$H_{(j_0^r f(x), j^k g_1(p_1), \ldots, j^k g_m(p_m))}(x). \tag{3.3}$$

For any positive integers r and s with $s > r$, π_r^s: $(J^s(R^n, R^p))^m \to (J^r(R^n, R^p))^m$ denotes the canonical projection: $\pi_r^s(j^s g_1(q_1), \ldots, j^s g_m(q_m)) = (j^r g_1(q_1), \ldots, j^r g_m(q_m))$ and for a multi point $x = (x_1, \ldots, x_m)$ of $(R^n)^m$, set

$$(R^n)_x^m = \{(p_1, \ldots, p_m) \in (R^n)^m | p_i \neq x_i\} \quad \text{and}$$

$$\begin{aligned}(R^n)_x^{(m)} &= (R^n)_x^m \cap (R^n)^{(m)} \\ &= \{(p_1, \ldots, p_m) \in (R^n)^m | p_i \neq x_i, p_i \neq p_j\}.\end{aligned}$$

For a C^∞ function f: $R^n \to R$ and a multi point $x = (x_1, \ldots, x_m)$ of $(R^n)^m$, consider the following mappings:

$$j_{f,x}: (R^n)_x^m \to (R^n)^m \times (J^{r+k+1}(n, 1))^m$$

$$j_{\Omega,x}: (R^n)^{(m)} \times (\pi_r^{r+k+1})^{-1}(j_0^r f(x_1), \ldots, j_0^r f(x_m)) \to {}_mJ^k(R^n, R)$$

defined by

$$\begin{aligned} j_{f,x}(p_1, \ldots, p_m) &= (j_0^{r+k+1}(H_{(j_0^r f(x_1), j^k f(p_1))}(x))(x_1), \ldots, j_0^{r+k+1} \\ &\quad \times (H_{(j_0^r f(x)_m), j^k f(p_m))}(x))(x_m)) \end{aligned} \tag{3.4}$$

$$j_{\Omega,x}(p_1, \ldots p_m, z_1, \ldots, z_m) = (j^k h_{z_1}(p_1), \ldots, j^k h_{z_m}(p_m)), \tag{3.5}$$

where for an s-jet z, h_z is its unique polynomial representative of degree s.

LEMMA 3.2:

(1) $j_{f,x}$: $(R^n)_x^m \to (R^n) \times (J^{r+k+1}(n, 1))^m$ *can be uniquely extended to a* C^∞ *mapping, denoted by the same symbol*

$$j_{f,x}: (R^n)^m \to (R^n)^m \times (J^{r+k+1}(n, 1))^m,$$

(2) $j_{f,x}(x_1, \ldots, x_m) = (j_0^{r+k+1} f(x_1), \ldots, j_0^{r+k+1} f(x_m))$,

(3) $j_{f,x}((R^n)^m) \subset (\pi_r^{r+k+1})^{-1}(j_0^r f(x_1), \ldots, j_0^r f(x_m))$,

(4) $j_{\Omega,x}$ *maps* $(R^n)_x^{(m)} \times (\pi_r^{r+k+1})^{-1}(j_0^r f(x_1), \ldots, j_0^r f(x_m))$ *submersively onto* $(R^n)_x^{(m)} \times R^m \times (J^k(n, 1))^m$.

Lemma 3.2 follows easily from Corollary 4.3 of [2-I].

LEMMA 3.3: *Let W be a semi-algebraic subset of* $(R^n)^m \times (J^r(n, p))^m$, *$X$ be a semi-algebraic submanifold of* ${}_mJ^k(R^n, R^p)$ *and let* μ: $(J^k(R^n, R^p))^m \to R$ *be a polynomial function with the properties* (a) *and* (b) *in the definition. Then there exist an integer s with $s > r$ and a closed semi-algebraic subset* Σ_W *of* $(\pi^s_r)^{-1}(W)$ *having codimension* $\geqq 1$ *such that for any multipoint* $x = (x_1, \ldots, x_m)$ *and any mapping* f: $R^n \to R^p$ *with* $(j^s_0 f(x_1), \ldots, j^s_0 f(x_m)) \in (\pi^s_r)^{-1}(W) - \Sigma_W$, *there exists a neighborhood* $U_{f,x}$ *of x in* $(R^n)^m$ *such that f has the property* $T(X, \mu)$ *on* $U_{f,x} \cap (R^n)^{(m)}_x$.

This Lemma 3.3 plays an essential role for our proof of Theorem 2.1. Our proof of Lemma 3.3 is a slight elaboration of Fukuda's proof of his Transversality Theorem (see pp. 236–238 of [2-I]).

PROOF OF LEMMA 3.3. Let Φ_1: $(R^n)^m \times (J^r(n, p))^m \to (R^n)^m$ be the canonical projection. There exists a semi-algebraic stratification $\mathcal{S}(W)$ of W such that the restricted mapping $\Phi_1|_W$: $W \to \Phi_1(W)$ is a stratified mapping (Theorem 2 of [1]). Let W_0 be any stratum of $\mathcal{S}(W)$ with maximal dimension. It suffices to construct Σ for W_0 since $\mathcal{S}(W)$ has finitely many strata. We identify the i-jet $z \in J^i(R^n, R^p)$ with its polynomial representative $h_z(x)$. Set $s = r + k + 1$, $\Omega = (R^n)^m \times (\pi^s_r)^{-1}(W_0)$ and $Q = \Psi((\pi^s_r)^{-1}(W_0))$. Here Ψ: $(R^n)^m \times (J^s(n, p))^m \to (R^n)^m \times (R^n)^m \times (J^s(n, p))^m$ is the polynomial mapping defined by

$$\Psi(x, z) = (x, x, z). \tag{3.6}$$

Remark that Q is a semi-algebraic submanifold of Ω. For any multipoint x of $(R^n)^m$, set $\Omega_x = \Omega \cap ((R^n)^m \times \{x\} \times (J^s(n, p))^m)$ and $Q_x = Q \cap ((R^n)^m \times \{x\} \times (J^s(n, p))^m)$. Since the restriction $\Phi_1|_{W_0}$: $W_0 \to \Phi_1(W_0)$ is a submersion, Ω_x (resp. Q_x) is a semi-algebraic submanifold of Ω (resp. Q). Define a mapping j_Ω: $\Omega \to (J^k(R^n, R^p))^m$ by

$$\begin{aligned} &j_\Omega(q_1, \ldots, q_m, j^s_0 g_1(p_1), \ldots, j^s_0 g_m(p_m)) \\ &\quad = (j^k h_{j^s_0 g_1(p_1)}(q_1), \ldots, j^k h_{j^s_0 g_m(p_m)}(q_m)). \end{aligned} \tag{3.7}$$

Put $j_{\Omega,x} = j_\Omega|_{\Omega_x}$ for any multipoint x of $(R^n)^m$. For any C^∞ mapping $g = (g_1, \ldots, g_p)$: $R^n \to R^p$ and any multipoint x of $(R^n)^m$ with $j^r_0 g(x) \in W_0$, define a mapping $j_{g,x} = (R^n)^m \to \Omega_x$ by

$$j_{g,x}(q_1, \ldots, q_m) = (q_1, \ldots q_m, j_{g_1,x}(q_1, \ldots, q_m), \ldots, j_{g_p,x}(q_1, \ldots q_m)) \tag{3.8}$$

where $j_{g_i,x}$ is the mapping defined in (3.4) for g_i.

Then from (3.7) and (3.8), we have

for any multipoint x of $(R^n)^m$, the following diagram is commutative;
$${}_m j^k g \;=\; j_{\Omega,x} \circ (j_{g,x}|_{(R^n)(m)}). \tag{3.9}$$

$$\begin{tikzcd}
& \Omega \arrow[d, "j_{\Omega,x}"] \\
(R^n)^{(m)} \arrow[ur, "j_{g,x}"] \arrow[r, "{}_m j^k g"'] & (J^k(R^n, R^p))^m
\end{tikzcd}$$

From the definition of Q_x and (3.8), we have

$$j_{g,x} \textit{ is transversal to } Q_x \textit{ for any multipoint } x \textit{ of } (R^n)^m. \tag{3.10}$$

From Lemma 3.2, we see

the restriction of $j_{\Omega,x}$ to

$$((R^n)^{(m)}_x \times (R^n)^m \times (J^s(n, p))^m) \cap \Omega_x \tag{3.11}$$

submerses it into ${}_mJ^k(R^n, R^p)$ for any multipoint x of $(R^n)^m$.

Since $j_{\Omega,x}$ is a polynomial mapping and X is a semi-algebraic submanifold of ${}_mJ^k(R^n, R^p)$,

$$X^*_x \;=\; (j_{\Omega,x})^{-1}(X) \cap ((R^n)^{(m)}_x \times (R^n)^m \times (J^s(n, p))^m)$$

is a semi-algebraic subset of Ω_x. Set $X^* = \bigcup_{x \in (R^n)^m} X_x^*$. The set X^* is a semi-algebraic submanifold of Ω, since j_Ω is a polynomial mapping which maps

$$\bigcup_{x \in (R^n)^m} (((R^n)_x^{(m)} \times (R^n)^m \times (J^s(n, p))^m) \cap \Omega_x)$$

submersively into ${}_mJ^k(R^n, R^p)$.

Let Σ_1 be the topological closure of the set of points of Q at which the pair (X^*, Q) does not satisfy the Whitney conditions (a) and (b). Let

$$\Phi_2 \colon (R^n)^m \times (R^n)^m \times (J^s(n, p))^m \to (R^n)^m \times (J^s(n, p))^m$$

be the canonical projection

$$\Phi_2(x_1, x_2, z) = (x_1, z). \tag{3.12}$$

Remark that $\Phi_2|_Q \colon \to \Phi_2(Q)$ is a diffeomorphism and $(\Phi_2|_Q)^{-1} = \Psi|_{\Phi_2(Q)}$ from (3.6) and (3.12).

Let f be a C^∞ mapping and $x = (x_1, \ldots, x_m)$ be a multipoint with $(j_0^s f(x_1), \ldots, j_0^s f(x_m)) \in ((\pi_r^s)^{-1}(W_0) - \Psi(\Sigma_1))$. Then from (3.8), (3.10), the Whitney conditions and the fact $((x_1, \ldots, x_m), (j_0^s f(x_1), \ldots, j_0^s f(x_m))) \in Q - \Sigma_1$, we have

$$j_{f,x} \textit{ is transversal to } X_x^* \textit{ at every multipoint of } (R^n)_x^{(m)} \textit{ near } x. \tag{3.13}$$

From (3.9), (3.11) and (3.13), we have

$${}_mj^k f \textit{ is transversal to } X \textit{ at every multipoint of } (R^n)_x^{(m)} \textit{ near } x. \tag{3.14}$$

From the Whitney condition (a), we have

$$\begin{aligned}&\textit{if } \operatorname{codim} X^* = \operatorname{codim} X = mn, \textit{ then } {}_mj^k f(y) \cap X = \phi \\ &\textit{at every multipoint } y \textit{ of } (R^n)_x^{(m)} \textit{ near } x.\end{aligned} \tag{3.15}$$

Next from (3.11) and the property (a) of μ,

$$\begin{aligned}&\textit{for any multipoint } x \textit{ of } (R^n)^m,\ \mu \circ j_{\Omega,x} \colon X_x^* \to R \\ &\textit{has no critical points on } (R^n)_x^{(m)} \textit{ near } x.\end{aligned} \tag{3.16}$$

From the property (b) of μ, we see $\mu \circ j_{\Omega,x}(Q_x) = \{\text{one point}\}$.

Let Σ_x be the set of points of Q_x at which the pair $(X_x^*, Q_x \cap (Q - \Sigma_1))$ does not satisfy the Thom condition $a_{\mu \circ j_{\Omega,x}}$.

Now we consider the following mapping:

$$\Phi_1 \circ \Phi_2|_\Omega : \Omega = (R^n)^m \times (\pi_r^s)^{-1}(W_0) \to \Phi_1(W_0).$$

Here recall that $\Phi_1: (R^n)^m \times (J^s(n, p))^m \to (R^n)^m$ is the canonical projection and $\Phi_2: (R^n)^m \times (R^n)^m \times (J^s(n, p))^m \to (R^n)^m \times (J^s(n, p))^m$ is defined by $\Phi_2(x_1, x_2, z) = (x_2, z)$. From the construction of W_0, we see the restriction $\Phi_1 \circ \Phi_2|_{Q=(\pi_r^s)^{1}(W_0)}: Q \to \Phi_1(W_0)$ is a submersion. From (3.11), we see the restriction $\Phi_1 \circ \Phi_2|_{X^* = \bigcup_{x \in (R^n)^m} X_x^*}: X^* \to \Phi_1(W_0)$ is also a submersion. Furthermore $\tilde{\mu}(Q) = 0$, where $\tilde{\mu}: (R^n)^m \times (R^n)^m \times (J^s(n, p))^m \to R$ is a polynomial function defined by $\tilde{\mu}(x_1, x_2, z) = \mu \cdot j_\Omega(x_1, x_2, z) - \mu \circ j_\Omega(x_2, x_2, z)$. Therefore from Lemma 6.3.1 of [1], we see $\bigcup_{x \in (R^n)^m} \Sigma_x$ is a semi-algebraic subset of Q having codimension $\geqq 1$. We denote its closure by Σ_2. Set $\Sigma = \Sigma_1 \cup \Sigma_2$. From the construction of Σ and the remark just below (3.12), we have

the set $\Phi_2(\Sigma)$ is a closed semi-algebraic subset of $(\pi_r^s)^{-1}(W_0)$ having codimension $\geqq 1$. (3.17)

Let f be a C^∞ mapping and $x = (x_1, \ldots x_m)$ be a multipoint with $(j_0^s f(x_1), \ldots, j_0^s f(x_m)) \in ((\pi_r^s)^{-1}(W_0) - \Psi(\Sigma)$. From (3.10), (3.16), the Thom condition $a_{\mu \circ j_{\Omega,x}}$ and the fact that $((x_1, \ldots, x_m), (j_0^s f(x_1), \ldots, j_0^s f(x_m))) \in Q - \Sigma$, we have

for every ε and every multipoint x of $(R^n)^m$, $j_{f,x}$ is transversal to $X_x^* \cap (\mu \circ j_\Omega)^{-1}(\varepsilon)$ at every multipoint of $(R^n)_x^{(m)}$ near x. (3.18)

From (3.9), (3.11) and (3 18), we have

for every ε and every multipoint x of $(R^n)^m$, ${}_m j^k f$ is transversal to $X \cap \mu^{-1}(\varepsilon)$ at every multipoint of $(R^n)_x^{(m)}$ near x.

Now assertions (3.14), (3.15), (3.17) and (3.19) complete the proof.

Q.E.D.

4. Proof of Theorem 2.1

Set $W = (R^n)_0^{(m)} \times (J^r(m, p))^m$. We apply Lemma 3.3 $(nm + 1)$ times to obtain an integer s and a closed semi-algebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$

having codimension $\geqq nm + 1$ such that for any multipoint $x = (x_1, \ldots, x_m)$ of $(R^n)^m$ and any mapping $g: R^n \to R^p$ with $(j_0^s g(x_1), \ldots, j_0^s g(x_m)) \in (\pi_r^s)^{-1}(W) - \Sigma_W$, there exists a neighborhood $U_{g,x}$ of x in $(R^n)^m$ such that

$$\textit{the mapping } g \textit{ has the property } T(X, \mu) \textit{ on } U_{g,x} \cap (R^n)_x^{(m)}. \tag{4.1}$$

We want to show that

$$\textit{the mapping } g \textit{ has the property } T(X, \mu) \textit{ on } (R^n)_0^{(m)}. \tag{4.2}$$

And we also want to show that

the intersection of A_f and the set of such mappings is a dense subset of A_f (4.3)

Since (4.2) is much stronger than (4.1), we can not conclude (4.2) and (4.3) from (4.1) by the direct application of Mather's Multitransversality Theorem.

By this reason, we continue our proof as follows, which is slightly different from the one of Wilson's Multijet Transversal Extention Theorem ([8], page 677), to obtain our conclusion.

$(R^n)_0^{(m)}$ can be covered by a countable collection of compact sets $U_1^i \times \cdots \times U_m^i$, where for each i $U_1^i, \ldots, U_m^i$ are compact, mutually disjoint semi-algebraic coordinate patches of $R^n - \{0\}$. Set

$$\begin{aligned}\Sigma_W^i &= \pi_1^{-1}(U_1^i \times \cdots \times U_m^i) \cap (\Sigma_W \times (R^p)^m) \\ &= ((U_1^i \times \cdots \times U_m^i) \times (R^p)^m \times (J^s(n, p))^m) \cap (\Sigma_w \times (R^p)^m)\end{aligned}$$

where $\pi_1: (J^s(R^n, R^p))^m \to (R^n)^m$ denotes the s-jet bundle projection $\pi_1(j^s g_1(x_1), \ldots, j^s g_m(x_m)) = (x_1, \ldots, x_m)$. If the set Σ_W^i is not empty, Σ_W^i is a semi-algebraic subset of $(J^s(R^n, R^p))^m$. Σ_W^i can be covered by a countable collection of compact submanifolds M_j^i (with boundary), furthermore the M_j^i may be chosen so that, for any i, j, M_j^i is a submanifold of a stratum of $\mathscr{S}(\Sigma_W^i)$, where $\mathscr{S}(\Sigma_W^i)$ is a semi-algebraic stratification of Σ_W^i. Set

$$B_j^i = \{g \in C^\infty(R^n, R^p)|_m j^s g \pitchfork \Sigma_W^i \quad \text{on} \quad M_j^i\} \quad \text{and} \quad A_j^i = A_f \cap B_j^i.$$

By the proof of Wilson's Multijet Transversal Extension Theorem ([8], p. 677), A_j^i is an open dense subset of A_f for any i, j. Set $A^i = \bigcap_j A_j^i$. Since the codimension of M_j^i in $(J^s(R^n, R^p))^m$ is greater than nm for any i, j, we see

$$A^i = \{g \in A_f|_m j^s g(U_1^i \times \cdots \times U_m^i) \cap (\Sigma_W \times (R^p)^m) = \phi\}.$$

Since $U_1^i \times \cdots \times U_m^i$ is compact, for any mapping g: $R^n \to R^p$ with $g \in A^i$ we can choose finite multipoints $x^1, \ldots, x^q$ of $U_1^i \times \cdots \times U_m^i$ such that for any multipoint x of $U_1^i \times \cdots \times U_m^i$ $(\subset (R^n)_0^{(m)})$ there exists $x^k \in \{x^1, \ldots, x^q\}$ with $x \in U_{g,xk} \cap (R^n)_{xk}^{(m)}$. Hence by (4.1) we have:
for any mapping g: $R^n \to R^p$ with $g \in A^i$,

the mapping g has the property $T(X, \mu)$ *on* $U_1^i \times \cdots \times U_m^i$. (4.4)

Therefore for any mapping g: $R^n \to R^p$ with $g \in \bigcap_i A^i$, we have (4.2).

Since $A_f = \{g \in C^\infty(R^n, R^p) | j^\infty f(0) = j^\infty g(0)\}$ is a Baire space (see p. 677 of [8]) and $\bigcap_i A^i = \bigcap_{i,j} A_j^i$ is a residual subset of A_f, $\bigcap_i A^i$ is dense in A_f. Hence $A_f^t(X, \mu))$ $(\supset \bigcap_i A^i)$ is dense in A_f. Q.E.D.

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