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Topologically $\infty$-determined map germs are topologically cone-like

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1. Introduction

Let $C_{0}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{p})$ be the space of all $C^{\infty}$ mappings of $\mathbb{R}^{n}$ into $\mathbb{R}^{p}$ with $f(0) = 0$. For a positive number $\gamma$, we set

$$S_{\gamma}^{n-1} = \{x \in \mathbb{R}^{n} | \|x\| = \gamma\} \quad \text{and} \quad D_{\gamma}^{n} = \{x \in \mathbb{R}^{n} | \|x\| \leq \gamma\}. $$

For any $C^{\infty}$ mapping $f$ of $C_{0}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{p})$, we write

$$ f_{\gamma, \delta} = f|_{f^{-1}(S_{\delta}^{n-1})}, \quad f_{\gamma, 0} = f|_{f^{-1}(D_{\gamma}^{n}-\{0\})}, \quad f_{\gamma, \delta, 0} = f|_{D_{\gamma}^{n} \cap f^{-1}(D_{\delta}^{n}-\{0\})}. $$

where expressions like $f|_{f^{-1}(S_{\delta}^{n-1})}$ mean the restricted mappings and $\gamma$, $\delta$ are positive numbers.

Then T. Fukuda has proved the following Cone Structure Theorem (Theorem 1.1) in his papers ([2]). The purpose of this paper is to give a flat version of his Cone Structure Theorem.

**Theorem 1.1:** There exists an infinite codimensional subset $\Sigma_{\infty}$ of $C_{0}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{p})$ such that every $C^{\infty}$ mapping $f: \mathbb{R}^{n} \to \mathbb{R}^{p}$ belonging to $C_{0}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{p}) - \Sigma_{\infty}$ has the following properties:

(A) $(n \leq p)$ there exists a positive number $\varepsilon_{0}$ such that for any number $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_{0}$ we have

(A-1) the space $f^{-1}(S_{\varepsilon}^{p-1})$ is a compact $C^{\infty}$ manifold without boundary,
(A-2) The restricted mapping \( f : f^{-1}(S^{p-1}_x) \to S^{p-1}_x \) is topologically stable (\( C^\infty \) stable if \( (n, p) \) is a nice pair of dimensions in J. Mather’s sense),

(A-3) The restricted mapping \( f_{\epsilon, \delta} : f^{-1}(D^p_\delta - \{0\}) \to D^p_\delta - \{0\} \) is proper, topologically stable (\( C^\infty \) stable if \( (n, p) \) is nice) and topologically equivalent (\( C^\infty \) equivalent if \( (n, p) \) is nice) to the product mapping \( f_{\epsilon, \delta} \times \text{id}_{(0, \delta)} : f^{-1}(S^{p-1}_x) \times (0, \delta) \to S^{p-1}_x \times (0, \delta) \) defined by \( (x, t) \to (f(x), t) \).

(B) For any sufficiently small positive numbers \( \epsilon \) and \( \delta \), the upper bound of \( \epsilon \) depending on \( f \) and the upper bound of \( \delta \) depending on \( \epsilon \) and \( f \), we have

(B-1) \( D^p_\delta \cap f^{-1}(S^{p-1}_x) \) is a \( C^\infty \) manifold, in general with boundary,

(B-2) The restricted mapping \( f_{\epsilon, \delta} : D^p_\delta \cap f^{-1}(S^{p-1}_x) \to S^{p-1}_x \) is topologically stable (\( C^\infty \) stable if \( (n, p) \) is nice),

(B-3) The restricted mapping \( f_{\epsilon, \delta} : D^p_\delta \cap f^{-1}(D^p_\delta - \{0\}) \to D^p_\delta - \{0\} \) is proper, topologically stable (\( C^\infty \) stable if \( (n, p) \) is nice) and topologically equivalent (\( C^\infty \) equivalent if \( (n, p) \) is nice) to the product mapping \( f_{\epsilon, \delta} \times \text{id}_{(0, \delta)} : D^p_\delta \cap f^{-1}(S^{p-1}_x) \times (0, \delta) \to S^{p-1}_x \times (0, \delta) \) defined by \( (x, t) \to (f(x), t) \).

For any \( C^\infty \) mapping \( f : R^n \to R^p \), let \( A_f \) be the space of all \( C^\infty \) mappings of \( R^n \) into \( R^p \) with the same formal power series at 0 as \( f \). The space \( A_f \) has the induced topology of the Whitney \( C^\infty \) topology. Then our main result can be stated as follows.

**Theorem 1.2:** For any \( C^\infty \) mapping \( f \) of \( C^\infty_0(R^n, R^p) \), there exists a dense subset \( A_f^0 \) of \( A_f \) such that every \( C^\infty \) mapping \( g : R^n \to R^p \) belonging to \( A_f^0 \) has the properties (A), (B) above.

**Corollary 1.3:** Topologically \( \infty \)-determined map germs are topologically cone-like.

Our Corollary 1.3 answers the question proposed by C.T.C. Wall ([4], question 30).

**2. Reduction of Theorem 1.2 to Theorem 2.1**

The notations used here are essentially the same as those of Thom [7], Mather [5, 6], and Fukuda [1, 2]. For instance \( J^r(n, p) \) is the set of the \( r \)-jets of \( C^\infty \) map germs: \( (R^n, 0) \to (R^p, 0) \), \( J^r(R^n, R^p) \) denotes the \( r \)-jet bundle of \( C^\infty \) mappings of \( R^n \) into \( R^p \) and \( mJ^r(R^n, R^p) \) is the \( m \)-fold \( r \)-jet bundle of \( C^\infty \) mappings of \( R^n \) into \( R^p \).
$mJ'(R^n, R^p) = \{(j'g_1(q_1), \ldots, j'g_m(q_m) \in (J'(R^n, R^p))^m| (q_1, \ldots, q_m) \in (R^n)^m)\}$, where for a set $X$, $X^{(m)}$ denotes the set $\{(q_1, \ldots, q_m) \in X^m| q_i \neq q_j \text{ if } i \neq j \}$ etc. Any points $x = (x_1, \ldots, x_m)$ of $(R^n)^m$ are called multipoints.

**Definition:** Let $X$ be a semi-algebraic submanifold of $J^k(R^n, R^p)$, $U$ be a subset of $(R^n)^{(m)}$ and let $\mu: (J^k(R^n, R^p))^m \rightarrow R$ be a polynomial function with the following properties (a) and (b):

Property (a): $C(\mu|_X) \cap (U \times R^n)^m \times (J^k(n, p))^m) = \phi$, where $C(\mu|_X)$ is the set of all critical points of the restricted function $\mu|_X: X \rightarrow R$.

Property (b): $\mu$ depends only on the O-jet.

Then for any mapping $g$ of $C^\infty(R^n, R^p)$, we say $g$ has the property $T(X, \mu)$ on $U$ if the following (1), (2) and (3) are satisfied:

1. $mJ^k g$ is transversal to $X$ at every multipoint of $U$,
2. if $\text{codim } X = mn$, $mJ^k g(U) \cap X = \phi$,
3. $mJ^k g$ is transversal to $X \cap \mu^{-1}(\varepsilon)$ for all $\varepsilon \in R$ at every multipoint of $U$.

**Theorem 2.1:** Let $X$ be a semi-algebraic submanifold of $mJ^k(R^n, R^p)$, $\mu: (J^k(R^n, R^p))^m \rightarrow R$ be a polynomial function with the properties (a) and (b) above and let $A_f(X, \mu)$ be the space of all $C^\infty$ mappings of $A_f$ having the property $T(X, \mu)$ on $(R^n - \{0\})^{(m)}$. Then $A_f(X, \mu)$ is dense in $A_f$ for any $f$ of $C^\infty(R^n, R^p)$.

*Proof that Theorem 2.1 = > Theorem 1.2*

Recall Mather’s various stability theorems ([3, 5, 6]).

**Lemma 2.2:** For a $C^\infty$ proper stable mapping $F: R^n \rightarrow R^p$, there exists a unique pair $(\mathcal{S}_F(R^n), \mathcal{S}_F(R^p))$ of semi-algebraic stratifications of $R^n$ and $R^p$ satisfying the following properties:

1. $F$ is a stratified mapping with respect to $(\mathcal{S}_F(R^n), \mathcal{S}_F(R^p))$,
2. any pair of strata of $\mathcal{S}_F(R^n)$ satisfies condition $a_F$,
3. if $(\mathcal{S}'(R^n), \mathcal{S}'(R^p))$ is another pair of stratifications which satisfies (1) and (2), then $(\mathcal{S}'(R^n), \mathcal{S}'(R^p))$ is a refinement of $(\mathcal{S}_F(R^n), \mathcal{S}_F(R^p))$.

**Lemma 2.3:** For any pair of dimensions $(n, p)$, there exists a positive number $k$ and a semi-algebraic stratification $\mathcal{S}(n, p)$ of $J^k(n, p)$ invariant under the action of $L^k(n) \times L^k(p)$ such that
(1) two map germs \( f \) and \( g \) of \( C_0^\infty(\mathbb{R}^n, \mathbb{R}^p) \) are topologically stable and topologically equivalent each other if their \( k \)-jets \( j^k f(0) \) and \( j^k g(0) \) belong to the same stratum \( X \) of \( \mathcal{S}(n, p) \) with codim \( X \leq n \).

(2) if a proper \( C^\omega \) mapping \( f: \mathbb{R}^n \to \mathbb{R}^p \) is multitransversal to the stratification \( \mathcal{S}(J^k(\mathbb{R}^n, \mathbb{R}^p)) \) of \( J^k(\mathbb{R}^n, \mathbb{R}^p) \) canonically induced from \( \mathcal{S}(n, p) \), then \( f \) is a stratified mapping with respect to the stratifications \( (\mathcal{S}_f(\mathbb{R}^n), \mathcal{S}_f(\mathbb{R}^p)) \) induced from \( \mathcal{S}(J^k(\mathbb{R}^n, \mathbb{R}^p)) \) by the multi-jet extension of \( f \), any pair of strata of \( \mathcal{S}_f(\mathbb{R}^n) \) satisfies condition \( a_f \) and \( f \) is topologically stable,

(3) any other stratification of \( J^k(n, p) \) satisfying (1) and (2) is a refinement of \( \mathcal{S}(n, p) \).

**Lemma 2.4:** For any pair of dimensions \( (n, p) \), there exists a closed semi-algebraic subset \( \Sigma \) of \( J^k(n, p) \), where \( k \) is as in Lemma 2.3, invariant under the action of \( L^k(n) \times L^k(p) \) having codimension > \( n \) such that the canonically induced subset \( \Sigma_{(\mathbb{R}^n, \mathbb{R}^p)} \) of \( J^k(\mathbb{R}^n, \mathbb{R}^p) \) has the following properties:

(1) For any proper \( C^\omega \) mapping \( f: \mathbb{R}^n \to \mathbb{R}^p \), the set \( j^k f(\mathbb{R}^n) \cap \Sigma_{(\mathbb{R}^n, \mathbb{R}^p)} \) is empty if and only if there exist an integer \( m \) and a proper \( C^\omega \) stable mapping \( F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \times \mathbb{R}^m \) of the form \( F(x, t) = (g(x, t), t) \), where \( x \in \mathbb{R}^n, t \in \mathbb{R}^m \) and \( g(x, 0) = f(x) \).

(2) Let \( f \) and \( F \) be as in (1). Then \( f \) is multitransversal to the stratification \( \mathcal{S}(J^k(\mathbb{R}^n, \mathbb{R}^p)) \) given by Lemma 2.3 if and only if the inclusion mappings \( i: \mathbb{R}^n \times \{0\} \to \mathbb{R}^n \times \mathbb{R}^n \) and \( j: \mathbb{R}^p \times \{0\} \to \mathbb{R}^p \times \mathbb{R}^p \) are both transversal to the canonical stratifications \( (\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m), \mathcal{S}_F(\mathbb{R}^p \times \mathbb{R}^m)) \) given by Lemma 2.2.

(3) Let \( F \) and \( f \) be as in (2). Then the pair of stratifications \( (i^*(\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)), j^*(\mathcal{S}_F(\mathbb{R}^p \times \mathbb{R}^m))) \) induced from \( (\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m), \mathcal{S}_F(\mathbb{R}^p \times \mathbb{R}^m)) \) by \( i \) and \( j \) coincides with the pair \( (\mathcal{S}_f(\mathbb{R}^n), \mathcal{S}_f(\mathbb{R}^p)) \), given by Lemma 2.3(2).

Now we prove that Theorem 2.1 implies Theorem 1.2. Let \( k \) be the integer, \( \mathcal{S}(J^k(\mathbb{R}^n, \mathbb{R}^p)) \) be the semi-algebraic stratification and \( \Sigma \) be the semi-algebraic subset of \( J^k(n, p) \) given by Lemma 2.3 and 2.4. Set

\[
Q = \{ j^k g(x) \in J^k(\mathbb{R}^n, \mathbb{R}^p - \{0\}) \mid \text{grad}(\|g(x)\|) \neq 0 \}.
\]

Note that codim \( (J^k(\mathbb{R}^n, \mathbb{R}^p - \{0\}) - Q) \) in \( J^k(\mathbb{R}^n, \mathbb{R}^p - \{0\}) \) is \( n \). We recall Fukuda’s construction of the semi-algebraic stratification \( \mathcal{S}(Q) \) of \( Q \) ([2-II], pp. 506–507). For any stratum \( X \) of \( \mathcal{S}(n - 1, p - 1) \), consider the subset \( X(Q) \) such that a jet \( j^k g(x) \) belongs to \( X(Q) \) if and only if \( j^k(g|_{U \cap g^{-1}(S_{\delta}^{p-1})})(x) \) is contained in \( X(U \cap g^{-1}(S_{\delta}^{p-1}), S_{\delta}^{p-1}) \), where \( U \) is a sufficiently small neighborhood of \( x, \delta = \|g(x)\| \) and \( X(U \cap g^{-1}(S_{\delta}^{p-1}), S_{\delta}^{p-1}) \) is the stratum of \( J^k(U \cap g^{-1}(S_{\delta}^{p-1}), S_{\delta}^{p-1}) \) corresponding to...
$X \in \mathcal{S}(n - 1, p - 1)$. Then he proved $\mathcal{S}(Q) = \{X(Q) | X \in \mathcal{S}(n - 1, p - 1)\}$ was a semi-algebraic stratification of $Q$. Take a semi-algebraic stratification $\mathcal{S}(\Sigma)$ of $\Sigma$ and set

$$S = \{ j^k g(x) \in J^k(R^n, R^p) | g(x) = 0 \}.$$

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of all strata of the forms

$$X = (X_1 \times \cdots \times X_m) \cap \Delta_m \quad \text{with} \quad X_i \in \mathcal{S}(J^k(R^n, R^p))$$

(resp. $X_i \in \mathcal{S}(Q)$),

\begin{equation}
(*)
\end{equation}

where $m \leq p + 1$ and $\Delta_m = \{ j^k g_1(q_1), \ldots, j^k g_m(q_m) \in J^k(R^n, R^p) | g_1(q_1) = \cdots = g_m(q_m) \}$. Let $\mu_p : (J^k(R^n, R^p))^m \to R$ be the function defined by $\mu_p(j^k g_1(x_1), \ldots, j^k g_m(x_m)) = \|g_1(x_1)\|^2$ and let $\mu_n : J^k(R^n, R^p) \to R$ be the function defined by $\mu_n(j^k h(x)) = \|x\|^2$.

Then for any $C^\infty$ mapping $f$ of $C^\infty_\mu (R^n, R^p)$, set

$$A^p_j = \begin{cases} 
\left( \bigcap_{X \in \mathcal{A}} A^p_j(X, \mu_p) \right) \cap \left( \bigcap_{Y \in \mathcal{S}(\Sigma)} A^p_j(Y, \mu_n) \right) \cap (A^p_j(S, \mu_n)) & \text{(if } n \leq p) \\
\left( \bigcap_{X \in \mathcal{B}} A^p_j(X, \mu_p) \right) \cap \left( \bigcap_{Y \in \mathcal{S}(Q) \cup \mathcal{S}(\Sigma)} A^p_j(Y, \mu_n) \right) \cap (A^p_j(S, \mu_n)) & \text{(if } n > p). 
\end{cases}$$

Since $A^p_j$ is a finite intersection of the sets of the forms $A^p_j(Z, \mu)$, this is dense in $A^p_j$ by Theorem 2.1.

Take any mapping $g$ of $A^p_j$. Then we have the following:

(A) ($n \leq p$)

$$j^k g(R^n - \{0\}) \cap \Sigma = \phi,$$

(2.1)

$$j^k g(R^n - \{0\}) \cap S = \phi,$$

(2.2)

$$g|_{(R^n - \{0\}) : R^n - \{0\} \to R^p \text{ is multitransversal to } \mathcal{S}(J^k(R^n, R^p))},$$

(2.3)

for any integer $m \leq p + 1$ and any manifold $X$ of $\mathcal{A}$,

$$m j^k(g|_{(R^n - \{0\})}) \text{is transversal to } X \cap \mu_p^{-1}(\varepsilon) \text{ for every } \varepsilon > 0.$$

(2.4)
Then Fukuda proved that any COO mapping $g$ satisfying (2.1), (2.2), (2.3) and (2.4) (resp. (2.1), (2.4), (2.5), (2.6) and (2.7)) had the property (A) (resp. (B)) in Theorem 1.1 using only Lemma 2.3, Lemma 2.4, Second Isotopy Lemma, Mather's results on COO stability ([6]) and (2.1)-(2.7) (see [2-1], pp. 242-249 and [2-11], pp. 510-514).

Therefore exactly as in his proof, we have our conclusion. Q.E.D.

3. Lemmas

For any jet $j^r f(x) \in J^r(\mathbb{R}^n, \mathbb{R}^p)$, we denote $\Phi_0(j^r f(x))$ by $j^r_0 f(x)$, where $\Phi_0 : J^r(\mathbb{R}^n, \mathbb{R}^p) = \mathbb{R}^n \times \mathbb{R}^p \times J^r(n, p) \to \mathbb{R}^n \times J^r(n, p)$ is the canonical projection.

**Lemma 3.1** [2-1]: Let $g_1, \ldots, g_m$ and $f$ be $C^\infty$ functions defined on $\mathbb{R}^n$ and let $p_1, \ldots, p_m$ be $m$ distinct points of $\mathbb{R}^n$ all different from a point $x$ of $\mathbb{R}^n$ and let $r$ and $k$ be positive integers. Then there exists a polynomial $H(x) = \Sigma a_\alpha x^\alpha$ of $n$ variables of degree $r + m(k + 1)$ satisfying the following conditions:

$$j^r H(x) = j^r_0 f(x) \times \{0\}, \quad (3.1)$$

$$j^k H(p_i) = j^k_0 g_i(p_i), \quad i = 1, \ldots, m. \quad (3.2)$$

Moreover the polynomial $H(x)$ can be chosen so that it depends only on $(j^r_0 f(x), j^k_0 g_1(p_1), \ldots, j^k_0 g_m(p_m))$. We thus denote it by

$$H_{(j^r_0 f(x), j^k_0 g_1(p_1), \ldots, j^k_0 g_m(p_m))}(x). \quad (3.3)$$
For any positive integers \( r \) and \( s \) with \( s > r \), \( \pi_r: (J^s(R^n, R^r))^m \to (J^r(R^n, R^r))^m \) denotes the canonical projection: \( \pi_r(j^s g_1(q_1), \ldots, j^s g_m(q_m)) = (j^r g_1(q_1), \ldots, j^r g_m(q_m)) \) and for a multi point \( x = (x_1, \ldots, x_m) \) of \((R^n)^m\), set

\[
(R^n)_x^m = \{(p_1, \ldots, p_m) \in (R^n)^m | p_i \neq x_i \} \quad \text{and}
\]

\[
(R^n)^{(m)}_x = (R^n)^m_x \cap (R^n)^{(m)}
\]

\[
= \{(p_1, \ldots, p_m) \in (R^n)^m | p_i \neq x_i, p_i \neq p_j \}.
\]

For a \( C^\infty \) function \( f: R^n \to R \) and a multi point \( x = (x_1, \ldots, x_m) \) of \((R^n)^m\), consider the following mappings:

\[
j_{f,x}: (R^n)^m_x \to (R^n)^m \times (J^{r+k+1}(n, 1))^m
\]

\[
j_{0,x}: (R^n)^{(m)}_x \times (\pi_r^{r+k+1})^{-1}(j_0^r f(x_1), \ldots, j_0^r f(x_m)) \to mJ^k(R^n, R)
\]

defined by

\[
j_{f,x}(p_1, \ldots, p_m) = (j_0^{r+k+1}(H_{j_0^r f(x_1), j_0^r f(p_1)}(x_1), \ldots, j_0^{r+k+1})
\]

\[
\times (H_{j_0^r f(x_m), j_0^r f(p_m)}(x_m))(x_m))
\]

(3.4)

\[
j_{0,x}(p_1, \ldots, p_m, z_1, \ldots, z_m) = (j^k h_{z_1}(p_1), \ldots, j^k h_{z_m}(p_m)),
\]

(3.5)

where for an \( s \)-jet \( z \), \( h_z \) is its unique polynomial representative of degree \( s \).

**Lemma 3.2:**

1. \( j_{f,x}: (R^n)^m_x \to (R^n)^m \times (J^{r+k+1}(n, 1))^m \) can be uniquely extended to a \( C^\infty \) mapping, denoted by the same symbol

\[
j_{f,x}: (R^n)^m \to (R^n)^m \times (J^{r+k+1}(n, 1))^m,
\]

2. \( j_{f,x}(x_1, \ldots, x_m) = (j_0^{r+k+1} f(x_1), \ldots, j_0^{r+k+1} f(x_m)) \),
3. \( j_{f,x}((R^n)^m) \subset (\pi_r^{r+k+1})^{-1}(j_0^r f(x_1), \ldots, j_0^r f(x_m)) \),
4. \( j_{0,x} \) maps \((R^n)^{(m)}_x \times (\pi_r^{r+k+1})^{-1}(j_0^r f(x_1), \ldots, j_0^r f(x_m)) \) submersively onto \((R^n)^{(m)}_x \times R^m \times (J^k(n, 1))^m\).

Lemma 3.2 follows easily from Corollary 4.3 of [2-1].
LEMMA 3.3: Let $W$ be a semi-algebraic subset of $(\mathbb{R}^n)^m \times (J^r(n, p))^m$, $X$ be a semi-algebraic submanifold of $J^r(n, p)$ and let $\mu: (J^r(n, p))^m \to \mathbb{R}$ be a polynomial function with the properties (a) and (b) in the definition. Then there exist an integer $s$ with $s > r$ and a closed semi-algebraic subset $\Sigma_w$ of $(\mathbb{R}^n)^m \times (J^r(n, p))^m$ having codimension $\geq 1$ such that for any multipoint $x = (x_1, \ldots, x_m)$ and any mapping $f: \mathbb{R}^n \to \mathbb{R}^p$ with $(j_0^r f(x_1), \ldots, j_0^r f(x_m)) \in (\mathbb{R}^n)^m \times (J^r(n, p))^m - \Sigma_w$, there exists a neighborhood $U_{f,x}$ of $x$ in $(\mathbb{R}^n)^m$ such that $f$ has the property $T(X, \mu)$ on $U_{f,x} \cap (\mathbb{R}^n)^m$.

This Lemma 3.3 plays an essential role for our proof of Theorem 2.1. Our proof of Lemma 3.3 is a slight elaboration of Fukuda’s proof of his Transversality Theorem (see pp. 236–238 of [2-I]).

PROOF OF LEMMA 3.3. Let $\Phi_i: (\mathbb{R}^n)^m \times (J^r(n, p))^m \to (\mathbb{R}^n)^m$ be the canonical projection. There exists a semi-algebraic stratification $\mathcal{S}(W)$ of $W$ such that the restricted mapping $\Phi_i|_W: W \to \Phi_i(W)$ is a stratified mapping (Theorem 2 of [1]). Let $W_0$ be any stratum of $\mathcal{S}(W)$ with maximal dimension. It suffices to construct $\Sigma$ for $W_0$ since $\mathcal{S}(W)$ has finitely many strata. We identify the $i$-jet $z \in J^r(\mathbb{R}^n, \mathbb{R}^p)$ with its polynomial representative $h_z(x)$. Set $s = r + k + 1$, $\Omega = (\mathbb{R}^n)^m \times (\mathbb{R}^n)^m$ and $Q = \Psi((\mathbb{R}^n)^m \times (\mathbb{R}^n)^m \times (J^r(n, p))^m)$. Here $\Psi: (\mathbb{R}^n)^m \times (J^r(n, p))^m \to (\mathbb{R}^n)^m \times (\mathbb{R}^n)^m \times (J^r(n, p))^m$ is the polynomial mapping defined by

$$\Psi(x, z) = (x, x, z).$$ (3.6)

Remark that $Q$ is a semi-algebraic submanifold of $\Omega$. For any multipoint $x$ of $(\mathbb{R}^n)^m$, set $\Omega_x = \Omega \cap ((\mathbb{R}^n)^m \times \{x\} \times (J^r(n, p))^m)$ and $Q_x = Q \cap ((\mathbb{R}^n)^m \times \{x\} \times (J^r(n, p))^m)$. Since the restriction $\Phi_i|_{W_0}: W_0 \to \Phi_i(W_0)$ is a submersion, $\Omega_x$ (resp. $Q_x$) is a semi-algebraic submanifold of $\Omega$ (resp. $Q$).

Define a mapping $j_0: \Omega \to (J^r(\mathbb{R}^n, \mathbb{R}^p))^m$ by

$$j_0(q_1, \ldots, q_m, j_0^r g_1(p_1), \ldots, j_0^r g_m(p_m))$$

$$= (j^k h_{j_0^r g_1(p_1)}(q_1), \ldots, j^k h_{j_0^r g_m(p_m)}(q_m)).$$ (3.7)

Put $j_{0,x} = j_0|_{\Omega_x}$ for any multipoint $x$ of $(\mathbb{R}^n)^m$. For any $C^\infty$ mapping $g = (g_1, \ldots, g_p): \mathbb{R}^n \to \mathbb{R}^p$ and any multipoint $x$ of $(\mathbb{R}^n)^m$ with $j_0^r g(x) \in W_0$, define a mapping $j_{g,x} = (\mathbb{R}^n)^m \to \Omega_x$ by

$$j_{g,x}(q_1, \ldots, q_m) = (q_1, \ldots, q_m, j_{g_1,x}(q_1), \ldots, j_{g_m,x}(q_1, \ldots, q_m)).$$ (3.8)

where $j_{g_i,x}$ is the mapping defined in (3.4) for $g_i$. 
Then from (3.7) and (3.8), we have

\[ mJ^k g = j_{\Omega,x} \circ (j_{g,x}|_{(R^n)^m}). \]  

(3.9)

From the definition of \( Q_x \) and (3.8), we have

\[ j_{g,x} \text{ is transversal to } Q_x \text{ for any multipoint } x \text{ of } (R^n)^m. \]  

(3.10)

From Lemma 3.2, we see

the restriction of \( j_{\Omega,x} \) to

\[ \left( (R^n)^{(m)}_x \times (R^n)^m \times (J^x(n, p))^m \right) \cap \Omega_x \]  

(3.11)

submerges it into \( mJ^k(R^n, R^p) \) for any multipoint \( x \) of \((R^n)^m\).

Since \( j_{\Omega,x} \) is a polynomial mapping and \( X \) is a semi-algebraic submanifold of \( mJ^k(R^n, R^p) \),

\[ X_x^* = (j_{\Omega,x})^{-1}(X) \cap \left( (R^n)^{(m)}_x \times (R^n)^m \times (J^x(n, p))^m \right) \]
is a semi-algebraic subset of $\Omega_x$. Set $X^* = \bigcup_{x \in (\mathbb{R}^n)^m} X_x^*$. The set $X^*$ is a semi-algebraic submanifold of $\Omega$, since $j_{\Omega}$ is a polynomial mapping which maps

$$
\bigcup_{x \in (\mathbb{R}^n)^m} (((\mathbb{R}^n)^m_x \times (\mathbb{R}^n)^m \times (J^s(n, p))^m) \cap \Omega_x)
$$

submersively into $mJ^k(\mathbb{R}^n, \mathbb{R}^p)$.

Let $\Sigma_1$ be the topological closure of the set of points of $Q$ at which the pair $(X^*, Q)$ does not satisfy the Whitney conditions (a) and (b). Let

$$
\Phi_2: (\mathbb{R}^n)^m \times (\mathbb{R}^n)^m \times (J^s(n, p))^m \rightarrow (\mathbb{R}^n)^m \times (J^s(n, p))^m
$$

be the canonical projection

$$
\Phi_2(x_1, x_2, z) = (x_1, z).
$$

(3.12)

Remark that $\Phi_2|_Q: \rightarrow \Phi_2(Q)$ is a diffeomorphism and $(\Phi_2|_Q)^{-1} = \Psi|_{\Phi_2(Q)}$ from (3.6) and (3.12).

Let $f$ be a $C^\infty$ mapping and $x = (x_1, \ldots, x_m)$ be a multipoint with $(j^i_0f(x_1), \ldots, j^i_0f(x_m)) \in ((\pi^2_0)^{-1}(W_0) - \Psi(\Sigma_1))$. Then from (3.8), (3.10), the Whitney conditions and the fact $((x_1, \ldots, x_m), (j_0^if(x_1), \ldots, j_0^if(x_m))) \in Q - \Sigma_1$, we have

$$
j_0f, is transversal to $X_x^*$ at every multipoint of $(\mathbb{R}^n)^m_x$ near $x$. (3.13)
$$

From (3.9), (3.11) and (3.13), we have

$$
mj^k is transversal to $X$ at every multipoint of $(\mathbb{R}^n)^m_x$ near $x$. (3.14)
$$

From the Whitney condition (a), we have

$$
it codim X^* = codim X = mn, then mj^k f(y) \cap X = \phi
$$
at every multipoint $y$ of $(\mathbb{R}^n)^m_x$ near $x$. (3.15)

Next from (3.11) and the property (a) of $\mu$,

$$
for any multipoint $x$ of $(\mathbb{R}^n)^m$, $\mu \circ j_{\Omega,x}: X_x^* \rightarrow R$
has no critical points on $(\mathbb{R}^n)^m_x$ near $x$. (3.16)
$$

From the property (b) of $\mu$, we see $\mu \circ j_{\Omega,x}(Q_x) = \{\text{one point}\}$. 

Let $\Sigma$ be the set of points of $Q$ at which the pair $(X^*_x, Q_x \cap (Q - \Sigma))$ does not satisfy the Thom condition $a_{\mu_j, j^m}$.

Now we consider the following mapping:

$$\Phi_1 \circ \Phi_2|_\Omega: \Omega = (R^n)^m \times (\pi^*_x)^{-1}(W_0) \to \Phi_1(W_0).$$

Here recall that $\Phi_1: (R^n)^m \times (J^*(n, p))^m \to (R^n)^m$ is the canonical projection and $\Phi_2: (R^n)^m \times (R^n)^m \times (J^*(n, p))^m \to (R^n)^m \times (J^*(n, p))^m$ is defined by $\Phi_2(x_1, x_2, z) = (x_2, z)$. From the construction of $W_0$, we see the restriction $\Phi_1 \circ \Phi_2|_{Q = (0, \sigma)^1(W_0)} : Q \to \Phi_1(W_0)$ is a submersion. From (3.11), we see the restriction $\Phi_1 \circ \Phi_2|_{x^* = (0, \sigma)^1(W_0)} : X^* \to \Phi_1(W_0)$ is also a submersion. Furthermore $\Phi_1 \circ \Phi_2|_{x^* = (0, \sigma)^1(W_0)} : X^* \to \Phi_1(W_0)$ is a polynomial function defined by $\mu(x_1, x_2, z) = \mu \cdot j_0(x_1, x_2, z) - \mu \circ j_0(x_2, x_2, z)$. Therefore from Lemma 6.3.1 of [1], we see $\cup_{x \in (R^n)^m} \Sigma_x$ is a semi-algebraic subset of $Q$ having codimension $\geq 1$. We denote its closure by $\Sigma_2$. We denote its closure by $\Sigma_2$. From the construction of $\Sigma$ and the remark just below (3.12), we have

$$\text{the set } \Phi_2(\Sigma) \text{ is a closed semi-algebraic subset of } (\pi^*_x)^{-1}(W_0) \text{ having codimension } \geq 1. \quad (3.17)$$

Let $f$ be a $C^\infty$ mapping and $x = (x_1, \ldots, x_m)$ be a multipoint with $(j^m_0f(x_1), \ldots, j^m_0f(x_m)) \in ((\pi^*_x)^{-1}(W_0) - \Psi(\Sigma))$. From (3.10), (3.16), the Thom condition $a_{\mu_j, j^m}$ and the fact that $(x_1, \ldots, x_m), (j^m_0f(x_1), \ldots, j^m_0f(x_m)) \in Q - \Sigma$, we have

$$\text{for every } \varepsilon \text{ and every multipoint } x \text{ of } (R^n)^m, j^m f \text{ is transversal to } X^*_x \cap (\mu_j)^{-1}(\varepsilon) \text{ at every multipoint of } (R^n)^m_x \text{ near } x. \quad (3.18)$$

From (3.9), (3.11) and (3.18), we have

$$\text{for every } \varepsilon \text{ and every multipoint } x \text{ of } (R^n)^m, m^k f \text{ is transversal to } X \cap \mu^{-1}(\varepsilon) \text{ at every multipoint of } (R^n)^m_x \text{ near } x. \quad (3.19)$$

Now assertions (3.14), (3.15), (3.17) and (3.19) complete the proof.

Q.E.D.

4. Proof of Theorem 2.1

Set $W = (R^n)^m \times (J^*(m, p))^m$. We apply Lemma 3.3 $(nm + 1)$ times to obtain an integer $s$ and a closed semi-algebraic subset $\Sigma_W$ of $(\pi^*_x)^{-1}(W)$.
having codimension $\geq nm + 1$ such that for any multipoint $x = (x_1, \ldots, x_m)$ of $(R^n)_m$ and any mapping $g: R^n \rightarrow R^p$ with 
$(j^i_0 g(x_1), \ldots, j^i_0 g(x_m)) \in (\pi_i)^{-1}(W) - \Sigma_w$, there exists a neighborhood $U_{g,x}$ of $x$ in $(R^n)^m$ such that

the mapping $g$ has the property $T(X, \mu)$ on $U_{g,x} \cap (R^n)^m_x$. \hfill (4.1)

We want to show that

the mapping $g$ has the property $T(X, \mu)$ on $(R^n)_0^m$. \hfill (4.2)

And we also want to show that

the intersection of $A_f$ and the set of such mappings is a dense subset of $A_f$. \hfill (4.3)

Since (4.2) is much stronger than (4.1), we cannot conclude (4.2) and (4.3) from (4.1) by the direct application of Mather’s Multitransversality Theorem.

By this reason, we continue our proof as follows, which is slightly different from the one of Wilson’s Multijet Transversal Extension Theorem ([8], page 677), to obtain our conclusion.

$(R^n)_0^m$ can be covered by a countable collection of compact sets $U_1^i \times \cdots \times U_m^i$, where for each $i$ $U_1^i, \ldots, U_m^i$ are compact, mutually disjoint semi-algebraic coordinate patches of $R^n - \{0\}$. Set

$\Sigma_w^i = \pi_1^{-1}(U_1^i \times \cdots \times U_m^i) \cap (\Sigma_w \times (R^n)^m)$

where $\pi_1: (J^i(R^n, R^n))^m \rightarrow (R^n)^m$ denotes the $s$-jet bundle projection $\pi_1(j^i g_1(x_1), \ldots, j^i g_m(x_m)) = (x_1, \ldots, x_m)$. If the set $\Sigma_w^i$ is not empty, $\Sigma_w^i$ is a semi-algebraic subset of $(J^i(R^n, R^n))^m$. $\Sigma_w^i$ can be covered by a countable collection of compact submanifolds $M_j^i$ (with boundary), furthermore the $M_j^i$ may be chosen so that, for any $i, j$, $M_j^i$ is a submanifold of a stratum of $\mathcal{S}(\Sigma_w^i)$, where $\mathcal{S}(\Sigma_w^i)$ is a semi-algebraic stratification of $\Sigma_w^i$. Set

$B_j^i = \{ g \in C^\infty(R^n, R^n), j^i g \mid \Sigma_w^i \text{ on } M_j^i \}$ \text{ and } $A_f^i = A_f \cap B_j^i$.

By the proof of Wilson’s Multijet Transversal Extension Theorem ([8], p. 677), $A_f^i$ is an open dense subset of $A_f$ for any $i, j$. Set $A^i = \bigcap_j A_f^i$. Since the
The codimension of $M_j$ in $(J^i(R^n, R^p))^m$ is greater than $nm$ for any $i, j$, we see
\[ A^i = \{ g \in A^i \mid \exists g(U^i_1 \times \cdots \times U^i_m) \cap (\Sigma_w \times (R^p)^m) = \emptyset \}. \]

Since $U^i_1 \times \cdots \times U^i_m$ is compact, for any mapping $g: R^n \to R^p$ with $g \in A^i$ we can choose finite multipoints $x^1, \ldots, x^q$ of $U^i_1 \times \cdots \times U^i_m$ such that for any multipoint $x$ of $U^i_1 \times \cdots \times U^i_m$ ($\subset (R^n)^{(m)}$) there exists $x^k \in \{x^1, \ldots, x^q\}$ with $x \in U_{g,x_k} \cap (R^n)^{(m)}$. Hence by (4.1) we have:

for any mapping $g: R^n \to R^p$ with $g \in A^i$,

\[ \text{the mapping } g \text{ has the property } T(X, \mu) \text{ on } U^i_1 \times \cdots \times U^i_m. \quad (4.4) \]

Therefore for any mapping $g: R^n \to R^p$ with $g \in \bigcap_i A^i$, we have (4.2).

Since $A^i = \{ g \in C^\infty(R^n, R^p) \mid j^\infty f(0) = j^\infty g(0) \}$ is a Baire space (see p. 677 of [8]) and $\bigcap_i A^i = \bigcap_{i,j} A^i_j$ is a residual subset of $A^i$, $\bigcap_i A^i$ is dense in $A^i$. Hence $A^i_j(X, \mu) (\supset \bigcap_i A^i)$ is dense in $A^i$. Q.E.D.

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References