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DIANE MEUSER

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## On the degree of a local zeta function

DIANE MEUSER

*Department of Mathematics, Harvard University, Cambridge, MA 02138, USA; present address:  
Dept. of Mathematics, Boston University, 111 Cunningham Street, Boston, MA 02215, USA*

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Let  $K$  be a finite algebraic extension of  $\mathbb{Q}$ ,  $R$  the ring of integers of  $K$  and  $\{v\}$  the set of finite places of  $K$ . For  $v \in \{v\}$  let  $|\cdot|_v$  be the non-archimedean absolute value on  $K$  and  $K_v$  the completion of  $K$  with respect to this absolute value. Let  $R_v$  be the ring of integers of  $K_v$ ,  $P_v$  the unique maximal ideal of  $R_v$  and  $k_v = R_v/P_v$ . Then  $k_v$  is a finite field and we let  $q_v = \text{card } k_v$ . Let  $f(x) = f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$  be a homogeneous polynomial of degree  $m$ . Then for any  $v$  we can consider

$$Z(t) = \int_{R_v^{(n)}} |f(x)|_v^s |dx|_v$$

where  $s \in \mathbb{C}$ ,  $\text{Re}(s) > 0$  and  $t = q_v^{-s}$ . This has been shown to be a rational function of  $t$  by Igusa in [Igusa, 1977]. Writing  $Z(t) = P(t)/Q(t)$  we define  $\deg Z(t) = \deg P(t) - \deg Q(t)$ . Igusa has conjectured in [Igusa, 1984], p. 1027, and [Igusa, 1986], that for almost all  $v$ , i.e. except for a finite number of  $v$ , one has  $\deg Z(t) = -m$ . In this paper Igusa gives many examples where  $f$  satisfies the additional property that it is the single invariant polynomial for a connected irreducible simple linear algebraic group.

In this paper we show this conjecture is true if  $f$  is non-degenerate with respect to its Newton Polyhedron. This establishes the conjecture for “generic” homogeneous polynomials in a sense to be described below.

### §1. The Newton polyhedron of $f$ and its associated toroidal modification

We first recall some of the terminology and basic properties of the Newton polyhedron of an arbitrary polynomial. Other references for this include [Danilov, 1978; Kouchnirenko, 1976; Lichtin, 1981; Varchenko, 1977].

Let  $f \in K[x_1, \dots, x_n]$ . We write  $f = \sum_{I \in \mathbb{N}^n} a_I x^I$ , where  $I = (i_1, \dots, i_n)$  and  $x^I = x_1^{i_1} \cdots x_n^{i_n}$ . Let  $\text{Supp}(f) = \{I \in \mathbb{N}^n \mid a_I \neq 0\}$ . Let  $S(f)$  denote the convex hull of  $\cup_{I \in \text{Supp}(f)} (I + \mathbb{R}_+^n)$ . Let  $\Gamma_+(f)$  be the union of all faces of  $S(f)$ . Let  $\Gamma(f)$  be the union of compact faces only.  $\Gamma_+(f)$  is called the Newton polyhedron of  $f$  and  $\Gamma(f)$  is called the Newton diagram. We will denote a

fixed Newton polyhedron and diagram by  $\Gamma_+$  and  $\Gamma$  respectively. Given a Newton polyhedron  $\Gamma_+$  and its associated Newton diagram  $\Gamma$  we define  $\Omega_{\Gamma_+} = \{g \in K[x_1, \dots, x_n] \mid \Gamma_+(g) = \Gamma_+\}$ . If  $g \in \Omega_{\Gamma_+}$  and  $\gamma$  is a face of  $\Gamma$ , we define  $g_\gamma$  to be  $\sum_{I \in \gamma} b_I x^I$  if  $g = \sum_{I \in \gamma} b_I x^I + \sum_{I \in \gamma} b_I x^I$ . Then we define non-degeneracy as in [Kouchnirenko, 1976].

*Definition:*  $f$  is non-degenerate with respect to its Newton polyhedron if for any face  $\gamma$  of  $\Gamma_+(f)$  the functions  $(x_i \cdot \partial f / \partial x_i)$ , have no common zero in  $(\bar{K} - \{0\})^n$ , where  $\bar{K}$  denotes the algebraic closure of  $K$ .

Fix  $m$  and  $n$ . Identify homogeneous polynomials of degree  $m$  in  $n$  variables with  $\mathbf{P}_K^N$ , where  $N = \binom{m+n-1}{m} - 1$ . For  $\Gamma_+$  a fixed Newton polyhedron  $X_{\Gamma_+} = \{f \mid \Gamma_+(f) = \Gamma_+\}$  is a Zariski subset of  $\mathbf{P}_K^N$ . Let

$$Y_{\Gamma_+} = \{f \mid f \text{ is non-degenerate with respect to } \Gamma_+\}.$$

Then in a completely analogous manner to the proof of Theorem 6.1 in [Kouchnirenko, 1976] we have the following result which shows the non-degeneracy condition is generic.

**PROPOSITION 1:**  $Y_{\Gamma_+}$  is a Zariski open, dense subset of  $X_{\Gamma_+}$ .

Let  $K$  be a finite algebraic extension of  $\mathcal{Q}$ ,  $\{v\}$  the finite places of  $K$ , and  $K_v, R_v, P_v$  and  $k_v$  as defined in the introduction. Let  $U_v = R_v - P_v$  be the units of  $R_v$ . We first recall some definitions concerning the reduction of varieties modulo  $P_v$ .

For  $g \in R[x_1, \dots, x_n]$ ,  $v$  a finite place of  $K$ , let  $\bar{g}_v$  denote the polynomial in  $k_v[x_1, \dots, x_n]$  obtained by reducing the coefficients of  $g$  modulo  $P_v$ . We shall abbreviate this to  $\bar{g}$  when  $v$  is understood and use the same notation when  $g$  is a constant in  $R$ . Let  $V$  be an algebraic set defined over  $K$ , i.e.,  $V = \{x \in \bar{K}^n \mid f_i(x) = 0, 1 \leq i \leq r\}$ , where  $f_i(x) \in K[x_1, \dots, x_n]$ .

Let  $I(V)$  be the ideal of  $V$ , i.e.,  $I(V) = \{f \in \bar{K}[x_1, \dots, x_n] \mid f(x) = 0 \ \forall x \in V\}$ . Then we define the reduction of  $V$  modulo  $P_v$ , denoted  $\bar{V}_v$  by

$$\bar{V}_v = \{x \in \bar{k}_v^n \mid \bar{f}_v(x) = 0 \ \forall f \in I(V) \cap R_v[x_1, \dots, x_n]\}.$$

If  $f \in R[x_1, \dots, x_n]$  then for any finite place  $v$  of  $K$  we can consider the non-degeneracy of  $\bar{f}_v$ . We have:

**PROPOSITION 2:** Let  $f \in R[x_1, \dots, x_n]$  be non-degenerate with respect to its Newton polyhedron. Then for almost all  $v$

- a)  $\Gamma_+(\bar{f}_v) = \Gamma_+(f)$
- b)  $\bar{f}_v$  is non-degenerate with respect to its Newton polyhedron.

*Proof.* Let  $S = \{v \mid \text{all coefficients of } f \text{ are in } U_v\}$ . Then for  $v \in S$ ,  $\Gamma_+(\bar{f}_v) = \Gamma_+(f)$  and a) follows since almost all  $v$  are in  $S$ .

Let  $\tau$  be a face of  $\Gamma(f)$ , and write  $f_\tau = f_{\tau,1}, \dots, f_{\tau,t}$  where each  $f_{\tau,i}$  is absolutely irreducible. Let  $V_{\tau,i}$  be the variety defined by  $f_{\tau,i} = 0$ ,  $Y_i$  the hyperplane defined by  $x_i = 0$ , and  $Y = \bigcup_{i=1}^n Y_i$ . The condition that  $f$  is non-degenerate is equivalent to the condition that for any face  $\tau$  of  $\Gamma(f)$ , and  $V_{\tau,i}$  as above, the singular points of each  $V_{\tau,i}$  are contained in  $Y$  and for any  $i, j, i \neq j$  we have  $V_{\tau,i} \cap V_{\tau,j} \subset Y$ .

Let  $L$  be a finite extension of  $K$  such that the coefficients of  $f_{\tau,i}$  for any  $\tau, i$  are in  $L$ . To each place of  $v$  of  $K$  let  $v'$  be any place of  $L$  dividing  $v$ . As a straightforward consequence of Hilbert's Nullstellensatz, for any  $\tau, i, j$  we have  $(\bar{V}_{\tau,i})_{v'} \cap (\bar{V}_{\tau,j})_{v'} \subset \bar{Y}_{v'}$  for all  $v, v'$ . As a consequence of Proposition 30 in [Shimura, 1955],  $(\bar{V}_{\tau,i})_{v'}$  is absolutely irreducible and its singularities are contained in  $\bar{Y}_{v'}$  for almost all places  $v'$  of  $L$ . Let  $\tilde{S}$  be the set of  $v \in S$  satisfying the above property for all  $\tau, i$  and all  $v' | v$ . Then almost every place of  $K$  is in  $\tilde{S}$  and  $\tilde{f}_v$  is non-degenerate for all  $v \in \tilde{S}$ . Q.E.D.

We next describe a toroidal modification of  $K_v^n$  that we shall use to prove the conjecture for homogeneous  $f$  that are non-degenerate with respect to their Newton polyhedron. The modification we use is not the one utilized in [Lichtin, 1981] or [Lichtin and Meuser, 1985], which gives a nonsingular variety  $Y_v$  and a morphism  $h: Y_v \rightarrow K_v^n$  such that  $f \circ h = 0$  is a divisor with normal crossings, but a weaker modification that has also been used by Denef in [Denef, not yet published].

Let  $(\mathbf{R}_+^n)^* = \mathbf{R}_+^n - \mathbf{0}$ . Let  $a^1, \dots, a^l$  be vectors in  $\mathbf{R}_+^n$  and  $\sigma = \{\alpha_1 a^1 + \dots + \alpha_l a^l \mid \alpha_i \in \mathbf{R}_+, 1 \leq i \leq l\}$ .  $\sigma$  is called a closed cone which we denote by  $\langle a^1, \dots, a^l \rangle$ .  $\check{\sigma} = \{\alpha_1 a^1 + \dots + \alpha_l a^l \mid \alpha_i > 0, 1 \leq i \leq l\}$  is called an open cone. The dimension of any cone is the dimension of the smallest vector subspace of  $\mathbf{R}^n$  containing it.  $\sigma$ , or  $\check{\sigma}$ , is called a simplicial cone if  $a^1, \dots, a^l$  are linearly independent over  $\mathbf{R}$ . If  $\sigma$  is a closed cone spanned by integral vectors, then we have the following well known result on  $\sigma \cap \mathbf{Z}_+^n$  which we shall later use.

LEMMA 1. Let  $\sigma = \langle a^1, \dots, a^l \rangle$  be a closed cone in  $\mathbf{R}_+^n$ , where each  $a^i, 1 \leq i \leq l$ , is an integral vector. There are a finite number of integral vectors  $w^1, \dots, w^r$  such that

$$\sigma \cap \mathbf{Z}_+^n = \bigsqcup_{i=1}^r \left\{ w^i + \sum_{j=1}^l \alpha_j a^j \mid \alpha_j \in \mathbf{Z}_+ \right\}.$$

*Proof:* It is well known that  $\sigma$  has a partition into closed simplicial cones where each such cone is spanned by a subset of  $\{a^1, \dots, a^l\}$ . Thus we can assume  $\sigma$  is simplicial. We form the parallelotope  $P_\sigma = \left\{ \sum_{j=1}^l \alpha_j a^j \mid 0 \leq \alpha_j < 1 \right\}$ .

Let  $w^1, \dots, w^r$  be the points in  $P_\sigma \cap \mathbf{Z}_+^n$ . Then these  $w^i$  satisfy the statement of the lemma. Q.E.D.

Associated to any Newton polyhedron  $\Gamma_+$  we have a partition of  $(\mathbf{R}_+^n)^*$  into open cones. For  $a \in (\mathbf{R}_+^n)^*$  we let  $m(a) = \inf_{y \in \Gamma_+} \{a \cdot y\}$  and  $\tau_a = \{y \in \Gamma_+ \mid y \cdot a = m(a)\}$ .  $\tau_a$  is called the meet locus of  $a$ . We define an equivalence relation  $\sim$  by  $a^1 \sim a^2$  if  $\tau_{a^1} = \tau_{a^2}$ . This equivalence relation satisfies the following properties:

- i) If  $a \in (\mathbf{R}_+^n)^*$ ,  $\tau_a$  is a face of  $\Gamma_+$ .
- ii) Let  $\tau$  be a face of  $\Gamma_+$ . Let  $F_1, \dots, F_r$  be the facets of  $\Gamma_+$  containing  $\tau$ . Let  $a^i$  denote a vector dual to  $F_i$ ,  $1 \leq i \leq r$ . Then

$$\{a \in (\mathbf{R}_+^n)^* \mid \tau_a = \tau\} = \{\alpha_1 a^1 + \dots + \alpha_r a^r \mid \alpha_i > 0\}.$$

We denote the cone in the above formula by  $\check{\sigma}_\tau$ . Then its closure  $\sigma_\tau$  satisfies  $\sigma_\tau = \{a \in (\mathbf{R}_+^n)^* \mid \tau_a \supseteq \tau\}$ . A vector  $a = (a_1, \dots, a_n)$  in  $\mathbf{Z}_+^n - \mathbf{0}$  is called primitive if the greatest common divisor of the  $a_j$ ,  $1 \leq j \leq n$ , is one. For each facet of  $\Gamma_+$  there is a unique primitive integral vector dual to that facet. The above properties imply each equivalence class under  $\sim$  is an open cone spanned by a subset of primitive integral vectors dual to facets.

If  $f$  is a homogeneous polynomial of degree  $m$  in  $n$  variables note that all  $I \in \text{Supp}(f)$  lie on the hyperplane  $I \cdot x = m$ , where  $I = (1, \dots, 1)$ . Let  $F$  be a face of  $\Gamma(f)$ . It is straightforward to see that if  $P$  is an exposed point of  $F$  then  $P = I$  for some  $I \in \text{Supp}(f)$ . Hence  $\Gamma(f)$  is a single face with supporting hyperplane  $\bar{1} \cdot x = m$ . Let  $E(\Gamma_+)$  be the exposed points of  $\Gamma_+$ . Every  $P \in E(\Gamma_+)$  lies in  $\Gamma$  hence  $I \in \sigma P$ . We can partition  $\sigma_P$  into simplicial cones of the form  $\{\alpha_1 a^1 + \dots + \alpha_n a^n \mid \alpha_i \in \mathbf{R}, \alpha_i > 0\}$  where we may assume  $a^1 = I$ , and  $a^2, \dots, a^n$  are primitive integral vectors dual to noncompact facets of  $\Gamma_+$  containing  $P$ .

Let  $\sigma = \langle a^1, \dots, a^n \rangle$  be the closure of one of the maximum dimension cones corresponding to  $P \in E(\Gamma_+)$ . Write  $a^i = (a_{i1}, \dots, a_{in})$  and let  $M = [a_{ij}]$ . Then  $M$  determines a morphism  $\theta: K_v^n \rightarrow K_v^n$  defined by  $\theta(y_1, \dots, y_n) = (x_1, \dots, x_n)$  where

$$x_j = y_1^{a_{1j}} \cdots y_n^{a_{nj}}. \quad (1)$$

Let  $dx$  be the differential  $dx_1 \dots dx_n$  and  $\theta^*(dx)$  its pullback under  $\theta$ . Then for  $f \in R[x_1, \dots, x_n]$ ,  $\Gamma_+$ , and  $\theta$  as above we have the following result.

PROPOSITION 3:

- a)  $(f \circ \theta)(y) = y_1^m \prod_{i=2}^n y_i^{m(a^i)} f_\theta(y)$  where  $f_\theta(y) \in R[y_2, \dots, y_n]$ ,  $f_\theta(\mathbf{0}) \neq 0$ .
- b)  $\theta^*(dx) = (\det M) y_1^{n-1} \prod_{i=2}^n y_i^{|a^i|-1} dy$  where  $|a^i| = \sum_{j=1}^n a_{ij}$ .
- c) Let  $S = \{v \mid \Gamma_+(\bar{f}_v) = \Gamma_+(f), \bar{f}_v \text{ non-degenerate with respect of } \Gamma_+, \text{ and}$

$(\overline{\det M})_v \neq 0\}$ . Then for  $v \in S$ ,  $(\tilde{f}_\theta)_v(\mathbf{0}) \neq 0$ , and if  $b \in k_v^n$  satisfies  $(\tilde{f}_\theta)_v(b) = 0$  then

$$y_j \frac{\partial (\tilde{f}_\theta)_v}{\partial y_j}(b) \neq 0$$

for some  $2 \leq j \leq n$ .

*Proof:* a) and b) are just specializations of Varchenko's result, Lemma 10.2 in [Varchenko, 1977]. We write  $f = a_p x^P + \sum_I$  with  $P$  as in the discussion above and  $x^I = x_1^{i_1} \cdots x_n^{i_n}$ . Then under the map  $\theta$  the monomial  $x^I$  is transformed to  $\sum_{i=1} y_i^{I \cdot a^i}$ . For a) we denote that  $I \in \Gamma(f)$  implies  $I \cdot a^1 = m$  and  $I \cdot a^i \geq m(a^i)$  for  $2 \leq i \leq n$ . Furthermore  $P \cdot a^i = m(a^i)$  for all  $i$ , and  $P$  is the only point of  $\Gamma(f)$  having this property, so this gives the above factorization of  $(f \circ \theta)(y)$ . The formula  $\theta^*(dx)$  is a straightforward consequence of (1).

For c), we first observe that for  $v \in S$  we have  $(\bar{a}_p)_v \neq 0$ , hence  $(\tilde{f}_\theta)_v(\mathbf{0}) \neq 0$ . The proof of the rest of c) is identical to Lichtin's proof of Proposition 2.3 in [Lichtin, 1981]. Q.E.D.

Let  $K_v$  be the completion of  $K$  corresponding to any finite place  $v$  of  $K$ . Using the same notation as in the introduction, for every such place we fix  $\pi_v \in P_v - P_v^2$ . Let  $U_v = R_v - P_v$ . For  $x \in K_v^*$  we can write  $x = \pi_v^{\text{ord } x} u$  where  $u \in U_v$ . Let  $R_v^{(n)} = R_v \times \cdots \times R_v$  ( $n$  times) with a similar meaning for  $U_v^{(n)}$ ,  $P_v^{(n)}$ .

Let  $\sigma = \langle a^1, \dots, a^l \rangle$  be the closure of a cone in the partition corresponding to  $\Gamma_+$ . To each such cone we associate a maximal dimension closed cone  $\tilde{\sigma}$  containing  $\sigma$ , and note that it is not unique. For any place  $v$ , associated to  $\sigma$  we consider the subset of  $R_v^{(n)}$  defined by

$$X_\sigma = \{x \in R_v^{(n)} \mid (\text{ord } x_1, \dots, \text{ord } x_n) \in \sigma\}.$$

Let  $Y_\sigma = R_v^{(l)} \times U_v^{(n-l)}$  and consider the morphism  $\theta|_{Y_\sigma} : Y_\sigma \rightarrow R_v^{(n)}$  where  $\theta$  is the morphism associated to  $\tilde{\sigma}$  defined by (1). We observe that  $(\text{ord } x_1, \dots, \text{ord } x_n) = \sum_{i=1} (\text{ord } y_i) a^i$ , hence  $\theta(Y_\sigma) \subseteq X_\sigma$ . The next Lemma gives the properties of  $\theta|_{Y_\sigma}$  and the decomposition of  $X_\sigma$  that were established by Denef, Lemma 3 in [Denef, not yet published]. For  $\gamma = (\gamma_1, \dots, \gamma_n) \in K_v^n$ , and  $T$  any subset of  $K_v^n$ , denote by  $\gamma T$  the set  $\{(\gamma_1 x_1, \dots, \gamma_n x_n) \mid (x_1, \dots, x_n) \in T\}$ .

**LEMMA 2.** a) The map  $\theta|_{Y_\sigma} : Y_\sigma \rightarrow \theta(Y_\sigma)$  is locally bianalytic and each fiber has cardinality  $\kappa_\theta(v) = \text{card ker } \theta|_{U_v^{(n)}}$ . b) If  $w^i = (w_{i1}, \dots, w_{in})$ ,  $1 \leq i \leq r$ , are the vectors in  $\sigma \cap \mathcal{Z}_+^n$  given by Lemma 1, let  $\pi^{w^i}$  denote  $(\pi^{w_{i1}}, \dots, \pi^{w_{in}})$ . Let  $u_1, \dots, u_{s(v)}$  be the coset representatives for  $U_v^{(n)}/\theta(U_v^{(n)})$ . Then

$$X_\sigma = \prod_{\substack{1 \leq i \leq s(v) \\ 1 \leq j \leq r}} u_i \pi^{w^j}(\theta(Y_\sigma)).$$

**§2. The degree of  $Z(t)$**

Let  $v$  be a finite place of  $K$ . Using the same notation as in the preceding sections, we define an absolute value on  $K_v^*$  by  $|x|_v = q_v^{-\text{ord } x}$ . We let  $|dx|_v$  be the Haar measure on  $K_v$  normalized so that the measure of  $R_v$  is one. Then the measure of  $a + P_v$  for any  $a \in K_v$  is  $q_v^{-1}$ . If  $a \in R_v^{(n)}$ ,  $a + P_v^{(n)}$  will denote a coset modulo  $P_v^{(n)}$ , i.e.  $(a_1 + P_v) \times \cdots \times (a_n + P_v)$  where  $a = (a_1, \dots, a_n)$ .

We shall also use  $|dx|_v$ , defined above for  $n = 1$ , to be the measure  $\prod_{i=1}^n |dx_i|_v$  on  $R_v^{(n)}$ . When  $v$  is fixed we denote  $\pi_v$ ,  $|dx|_v$  and  $q_v$  by  $\pi$ ,  $|dx|$  and  $q$  respectively. Letting  $t = q^{-s}$  we have the following basic formulas for  $N$ ,  $n \in \mathbf{Z}$ ;  $N, n \geq 0$ .

$$\int_R |x|^{Ns+n-1} |dx| = \frac{q^n(1 - q^{-1})}{q^n - t^N} \tag{2}$$

$$\int_P |x|^{Ns+n-1} |dx| = \frac{(1 - q^{-1})t^N}{q^n - t^N}.$$

For  $f \in K[x_1, \dots, x_n]$ , and any finite place  $v$ , we can consider the zeta function  $Z(t)$  associated to  $f$  as defined in the introduction. We then have the following result.

**THEOREM.** *Let  $f(x) = f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$  be a homogeneous polynomial that is non-degenerate with respect to its Newton polyhedron. Then for almost every place  $v$  of  $K$ ,  $\text{deg } Z(t) = -\text{deg } f(x)$ .*

*Proof:* Let  $\text{deg } f(x) = m$ , and  $\Gamma_+$  be the Newton polyhedron of  $f$ . As explained in the previous section, associated to this Newton polyhedron we have a partition of  $\mathbf{R}_+^n$  into open cones. For  $P$  an exposed point of  $\Gamma_+$ , let  $\check{\sigma}_P$  be the associated maximal dimension open cone. As previously observed we can partition  $\check{\sigma}_P$  into simplicial cones of the form  $\{\alpha_1 a^1 + \cdots + \alpha_n a^n \mid \alpha_i > 0\}$  where  $a^1 = \mathbf{I}$ , if  $\check{\sigma}_P$  is not already in this form. The  $a^i$ ,  $2 \leq i \leq n$ , are dual to noncompact facets of  $\Gamma_+$ . Repeating this process for all points of  $E(\Gamma_+)$  let  $\check{\sigma}_1, \dots, \check{\sigma}_K$  denote the resulting simplicial cones, and let  $\sigma_1, \dots, \sigma_K$  denote the corresponding closed cones.  $\mathbf{R}_+^n \subseteq \bigcup_{i=1}^K \sigma_i$  and if  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, K\}$  then

$\bigcap_{j=1}^k \sigma_{i_j}$  is a closed cone, which is a face of each  $\sigma_{i_j}$ , hence is a simplicial cone. Furthermore the closed cone  $\{\alpha \mathbf{I} \mid \alpha \geq 0\}$  is contained in every such cone.

Consider

$$\begin{aligned} & \bigcup_{i=1}^K \sigma_i - \bigcup_{1 \leq i_1 < i_2 \leq K} (\sigma_{i_1} \cap \sigma_{i_2}) + \cdots + (-1)^{j-1} \\ & \times \bigcup_{1 \leq i_1 < \cdots < i_j \leq K} (\sigma_{i_1} \cap \cdots \cap \sigma_{i_j}) + \cdots + (-1)^{K-1} (\sigma_1 \cap \cdots \cap \sigma_K). \end{aligned} \tag{3}$$

Since every  $(k_1, \dots, k_n) \in \mathbf{Z}_+^{(n)}$  occurs exactly once in (3) we can write  $Z(t)$  as the sum and difference of integrals of the form

$$\int_{X_\sigma} |f(x)|_v^s |dx|_v \tag{4}$$

where  $\sigma = \langle \mathbf{I}, a^2, \dots, a^l \rangle$  for some  $l, 1 \leq l \leq n$ , where the  $l = 1$  case is  $\sigma = \langle \mathbf{I} \rangle$ .

For each maximal dimension cone  $\sigma_k = \langle \mathbf{I}, a^2, \dots, a^n \rangle$  write  $a^l = (a_{i1}, \dots, a_{in})$ , let  $M_k = [a_{ij}]$ , and let  $\theta_k$  be the morphism defined by (1) in §1. Let  $S$  be the set of places satisfying the conditions in Proposition 3 c) for  $M_k, 1 \leq k \leq K$ .

We now fix  $v \in S$ , and  $\sigma = \langle \mathbf{I}, a^2, \dots, a^l \rangle$ . Choose a maximal dimension cone  $\sigma_k, 1 \leq k \leq K$ , such that  $\sigma_k$  contains  $\sigma$ . We denote this choice by  $\tilde{\sigma} = \langle \mathbf{I}, a^l, a^{l+1}, \dots, a^n \rangle$  and let  $M, \theta$  be the matrix and morphism associated to  $\tilde{\sigma}$ . Referring to the decomposition of  $X_\sigma$  in Lemma 2 b) we can write (4) as a sum of integrals of the form

$$\int_{u\pi^w\theta(Y_\sigma)} |f(x)|^s |dx| \tag{5}$$

for some  $u = u_i, 1 \leq i \leq s(v)$ , and  $w = w^j, 1 \leq j \leq r$ , where  $Y_\sigma = R_v^{(l)} \times U_v^{(n-l)}$ .

Write  $f = \sum_I a_I x^I$ , then  $f(u\pi^w x) = \sum_I a_I u^I \pi^{w \cdot I} x^I$ . We have  $w \cdot I \geq m(w)$  for all  $I \in \Gamma_+$ , so we let

$$f_{u,w}(x) = \sum_I a_I u^I \pi^{w \cdot I - m(w)} x^I. \tag{6}$$

Then the integral in (5) equals

$$q^{-|w|} |t|^{m(w)} \int_{\theta(Y_\sigma)} |f_{u,w}(x)|^s |dx|.$$

By applying a) and b) in Proposition 3, in addition to the above observations,



we have that the integral in the above is

$$\frac{1}{\kappa_\theta(v)} \int_{R_v} |y_1|^{ms+n-1} |dy_1| \cdot \int_{Y'_\sigma} \prod_{i=2}^n |y_i|^{m(a^i)s+|a^i|-1} |g_{u,w}(y)|^s |dy|$$

where  $Y'_\sigma = R_v^{(l-1)} \times U_v^{(n-l)}$  and  $g_{u,w}(y) \in R_v[y_2, \dots, y_n]$ . Applying (2) to the first integral we have that the contribution to  $Z(t)$  from (5) is

$$q^n(1 - q^{-1})(q^n - t^m)^{-1} \tag{7}$$

times

$$\frac{q^{-|w|}}{\kappa_\theta(v)} t^{m(w)} \int_{Y'_\sigma} \prod_{i=2}^n |y_i|^{m(a^i)s+|a^i|-1} |g_{u,w}(y)|^s |dy_2 \cdots dy_n|. \tag{8}$$

By our observations above the factor (7) occurs for any integral of the form (5), so we can write

$$Z(t) = \frac{q^n(1 - q^{-1})}{q^n - t^m} \tilde{Z}(t)$$

where  $\tilde{Z}(t)$  is the sum and difference of expressions in the form of (8) for all possible  $\sigma, u, w$ . We shall show that (8) can be written in the form  $P_{\sigma,u,w}(t)/Q(t)$  where  $Q(t) = (q - t)\prod(q^{|a^i|} - t^{m(a^i)})$  and the product is over all  $a^i$  dual to a noncompact facet of  $\Gamma_+$ . We then write  $\tilde{Z}(t) = P(t)/Q(t)$  and

$$P(t) = \sum_{\sigma} (\text{sign } \sigma) \sum_{u,w} P_{\sigma,u,w}(t) \tag{9}$$

where  $\text{sign } \sigma = \pm 1$  is the coefficient of  $\sigma$  in the decomposition (3). Let  $D = 1 + \sum m(a^i) = \deg Q(t)$ . We shall show that after possibly excluding an additional finite set of places in  $S$ , that  $\deg P(t) = D$ , in which case the theorem follows.

Now consider  $g_{u,w}(y)$ . Referring back to  $f_{u,w}(x)$  as given in (6) we see that  $\bar{f}_{u,w}(x) = \sum_{I \in \tau_w} \bar{a}_i \bar{u}^I x^I$ , where  $\tau_w$  is a face of  $\Gamma_+$ . We have that  $\bar{f}_{u,w}$  is nondegenerate with respect to its Newton polyhedron since if  $\tau'$  is a face of  $\tau_w$  and  $b \in (\bar{k}_v - \{0\})^n$  is a solution to

$$\left( x_j \frac{\partial \bar{f}_{u,w}}{\partial x_j} \right)_{\tau'} = 0 \quad 1 \leq j \leq n$$

then  $\bar{u}b$  would be a solution to

$$\left( x_j \frac{\partial \bar{f}}{\partial x_j} \right)_{\tau'} = 0 \quad 1 \leq j \leq n.$$

But  $\tau'$  is a face of  $\Gamma_+$ , hence this contradicts the non-degeneracy of  $\bar{f}$ . Thus by applying Proposition 3 we have  $\bar{g}_{u,w}(b) = 0$  implies

$$\left( y_j \frac{\partial \bar{g}_{u,w}}{\partial y_j} \right) (b) \neq 0 \tag{10}$$

for some  $j$ ,  $2 \leq j \leq n$ .

First consider the case where  $\sigma = \langle I, a^2, \dots, a^l \rangle$  with  $l \geq 2$ . We shall show that  $\deg P_{\sigma,u,w}(t) \leq D$ . Then writing the coefficient  $c_{\sigma,u,w}$  of  $t^D$  in  $P_{\sigma,u,w}(t)$  as  $(\kappa_\theta(v))^{-1} \tilde{c}_{\sigma,u,w}$  we show  $q^{n-1} \tilde{c}_{\sigma,u,w} \equiv 0 \pmod{q}$ .

If  $w \neq \mathbf{0}$ , since  $w \in \sigma \cap \mathbf{Z}_+^{(n)}$  by permuting the vectors  $\{a^2, \dots, a^l\}$  we may suppose  $w = \alpha_1 \mathbf{1} + \alpha_2 a^2 + \dots + \alpha_k a^k$  where  $0 < \alpha_i < 1$ ,  $2 \leq i \leq k$ ,  $0 \leq \alpha_1 < 1$  and  $k \leq l$ . When  $w = \mathbf{0}$  set  $k = 1$ . Then we write the integral in (8) as

$$\int_{R_v^{(l-k)} \times U_v^{(n-l)}} \int_{R_v^{(k-1)}} \prod_{i=2}^l |y_i|^{m(a^i)s + |a^i| - 1} |g_{u,w}(y)|^s |dy|. \tag{11}$$

We have

$$g_{u,w}(y) = \sum_I a_I u^I \pi^{w \cdot I - m(w)} y_2^{I \cdot a^2 - m(a^2)} \dots y_n^{I \cdot a^n - m(a^n)}.$$

Observing that  $I \in \tau_w$  implies  $I \cdot a^i = m(a^i)$ ,  $2 \leq i \leq k$ , we have  $\bar{g}_{u,w} \in k_v[y_{k+1}, \dots, y_n]$ . Thus in this case (10) specializes to  $\bar{g}_{u,w}(b) = 0$ ,  $b \in \bar{k}_v^n$  implies  $(y_j \partial \bar{g}_{u,w} / \partial y_j)(b) \neq 0$  for some  $j$ ,  $k < j \leq n$ ; which implies the system of congruences

$$\begin{aligned} g_{u,w}(y) &\equiv 0 \pmod{P_v} \\ \left( y_j \frac{\partial g_{u,w}}{\partial y_j} \right) (y) &\equiv 0 \pmod{P_v}, \quad k < j \leq n \end{aligned} \tag{12}$$

has no solution in  $R_v^{(n)}$ .

For any subset  $J \subseteq \{k+1, \dots, l\}$  consider cosets  $(c_{k+1}, \dots, c_n) + P_v^{(n-k)_v}$  of  $R_v^{(l-k)} \times U_v^{(n-l)}$  satisfying

$$\begin{aligned} c_i &\equiv 0 \pmod{P_v} & i \in J \\ c_i &\not\equiv 0 \pmod{P_v} & i \notin J \end{aligned} \tag{13}$$

and call these cosets of type  $J$ . We distinguish the cosets of type  $J$  further by saying a coset is of type  $J_1$  if it satisfies  $g_{u,w} \not\equiv 0 \pmod{P_v}$  in addition to the above conditions and say it is of type  $J_2$  if it satisfies  $g_{u,w} \equiv 0 \pmod{P_v}$  in addition to the above conditions. We then write (11) as a sum over varying  $J$

of integrals of type

$$\int_{C_J} \int_{R_v^{(k-1)}} \prod_{i=2}^l |y_i|^{m(a^i)s + |a^i| - 1} |g_{u,w}(y)|^s |dy| \tag{14}$$

where  $C_J$  is a coset of type  $J$ .

If  $C_J$  is a coset of type  $J_1$ , by applying the formulas (2), we have that the integral in (14) is of the form  $P_1(t)/Q_1(t)$ , where  $\deg P_1(t) = \sum_{i \in J} m(a^i)$  and

$\deg Q_1 = \sum_{i=2}^k m(a^i) + \sum_{i \in J} m(a^i)$ . If  $C_J$  is of type  $J_2$  by (12) we can choose

$j, k < j \leq n$ , such that  $y_j \partial g_{u,w} / \partial y_j \not\equiv 0 \pmod{P_v}$ . We then make the change of variables  $\tilde{y}_j = g_{u,w}$ ,  $\tilde{y}_i = y_i$ ,  $i \neq j$ . Then the integral (14) is of the form

$P_2(t)/Q_2(t)$ , where  $\deg P_2(t) = 1 + \sum_{i \in J} m(a^i)$  and  $\deg Q_2(t) = 1 + \sum_{i=2}^k m(a^i)$

+  $\sum_{i \in J} m(a^i)$ . In either case we have  $P_i(t)/Q_i(t) = R_i(t)/Q(t)$  where

$\deg R_i(t) = D - \sum_{i=2}^k m(a^i)$ . Thus (11) is the sum of rational functions with

this property, hence referring to (8) we see that for  $w \neq \theta$

$$\deg P_{\sigma,u,w}(t) \leq D + m(w) - \sum_{i=2}^k m(a^i).$$

Moreover the coefficient of the highest degree term in  $P_{\sigma,u,w}(t)$  is

$$\begin{aligned} & \pm \kappa_\theta(v)^{-1} q^{-|w| + \sum_{i=2}^k |a^i|} \sum_J (-1)^{|J|} (1 - q^{-1})^{|J| + k - 1} q^{-(n - k - |J| - 1)} \\ & \times [N_{J_1} q^{-1} + N_{J_2} (1 - q^{-1})], \end{aligned}$$

where  $N_{J_i}$  is the number of cosets of type  $J_i$ .

If  $w = \theta$ , we have  $\deg P_{\sigma,u,w}(t) \leq D$ . If  $w \neq \theta$  in order to show this we must show  $m(w) \leq \sum_{i=2}^k m(a^i)$ . We have  $w_j = \alpha_1 + \sum_{i=2}^k \alpha_i a_{ij}$  where  $\alpha_i < 1$ ,  $1 \leq i \leq k$ ,

hence  $w_j < 1 + \sum_{i=2}^k a_{ij}$ , and  $w_j \in \mathbb{Z}$  implies  $w_j \leq \sum_{i=2}^k a_{ij}$ . Now let  $P \in E(\Gamma_+)$

be such that  $\tilde{\sigma}$  is obtained from the partition of  $\check{\sigma}_p$  into simplicial cones. Write  $P = (P_1, \dots, P_n)$ . We have  $m(w) = P \cdot w$  and  $m(a^i) = P \cdot a^i$ ,  $2 \leq i \leq k$ . Hence

$$m(w) \leq \sum_{j=1}^n P_j \left( \sum_{i=2}^k a_{ij} \right) = \sum_{i=2}^k P \cdot a^i.$$

Thus  $m(w) \leq \sum_{i=2}^k m(a^i)$ , which implies  $\deg P_{\sigma,u,w}(t) \leq D$ .

Now consider  $c_{\sigma,u,w}$ . If  $\deg P_{\sigma,u,w}(t) < D$  then  $c_{\sigma,u,w} = 0$ . If  $\deg P_{\sigma,u,w}(t) = D$  then by observing that  $N_{J_1} + N_{J_2} = (q-1)^{n-k-|J|}$  we have

$$q^{n-1}\tilde{c}_{\sigma,u,w} = \pm q^{-|w|+\sum_{i=2}^k|a^i|} \sum_J (-1)^{|J|} (q-1)^{|J|+k-1} \\ \times \left[ (q-1)^{n-k-|J|} - N_J - N_J(q-1) \right],$$

where we let  $N_J = N_{J_2}$ . If  $|w| < \sum_{i=2}^k |a^i|$  then  $q^{n-1}\tilde{c}_{\sigma,u,w}$  is clearly congruent to zero mod  $q$ , but if  $|w| = \sum_{i=2}^k |a^i|$  then

$$q^{n-1}\tilde{c}_{\sigma,u,w} \equiv \pm \sum_J (-1)^{|J|} \pmod{q} \equiv 0 \pmod{q}. \tag{15}$$

This proves our assertion about the case  $\sigma = \langle \mathbf{1}, a^2, \dots, a^l \rangle$  with  $l \geq 2$ .

The only remaining cases to consider are those where  $u$  varies and  $\sigma = \langle \mathbf{1} \rangle$ : In this case we show that (8) can be written as  $P_{1,u}(t)/Q(t)$  where  $\deg P_{1,u}(t) \leq D$ . Denoting the coefficient of  $t^D$  by  $c_{1,u}$  and defining  $\tilde{c}_{1,u}$  as in the previous case we show  $q^{n-1}\tilde{c}_{1,u}$  is an integer and  $q^{n-1}\tilde{c}_{1,u} \not\equiv 0 \pmod{q}$ .

In this case the integral in (8) is

$$\int_{U_v^{(n-1)}} |g_u(y)|^s |dy|.$$

Consider the cosets mod  $P_v^{(n-1)}$  of  $U_v^{(n-1)}$ . Letting  $N$  be the number of cosets satisfying  $g_u \equiv 0 \pmod{P_v}$  and applying entirely similar reasoning as before we have that the above integral equals

$$(q^{-1})^{(n-2)} \left[ ((q-1)^{n-1} - N)q^{-1} + \frac{N(1-q^{-1})t}{(q-t)} \right].$$

Then examination of the above shows  $\deg P_{1,u}(t) \leq D$  and

$$q^{n-1}\tilde{c}_{1,u} = (-1)^D \left[ (q-1)^{n-1} - N - N(q-1) \right].$$

Hence  $q^{n-1}\tilde{c}_{1,u}$  is an integer and

$$q^{n-1}\tilde{c}_{1,u} \equiv \pm 1 \pmod{q}. \tag{16}$$

Furthermore we note that the value on the right of the congruence is independent of  $u$ .

Let  $c_v$  denote the coefficient of  $t^D$  in  $P(t)$ , which we wish to show is nonzero for almost all  $v$ . We have  $c_v = \sum_{\sigma} (\text{sign } \sigma) \sum_{u,w} c_{\sigma,u,w}$ . Recalling that the

morphisms associated to the maximal dimension cones were denoted  $\theta_1, \dots, \theta_K$  we define  $\kappa(v) = \prod_{i=1}^K \kappa_{\theta_i}(v)$ . If  $\sigma$  is a cone, and the morphism associated to the maximal dimension cone  $\tilde{\sigma}$  is  $\theta_j$ , define  $\kappa_\sigma = \prod_{\substack{i=1 \\ i \neq j}}^K \kappa_{\theta_i}(v)$ . We assume  $\theta_1$  is the morphism associated to  $\langle I \rangle$ , and let  $\kappa_1(v) = \prod_{i=2}^K \kappa_{\theta_i}(v)$ . Then

$$q_v^{n-1} \kappa(v) c_v = \sum_{\sigma \neq \langle I \rangle} (\text{sign } \sigma) \sum_{u,w} q_v^{n-1} \kappa_\sigma(v) \tilde{c}_{\sigma,u,w} \pm \sum_u q_v^{n-1} \kappa_1(v) \tilde{c}_{1,u}.$$

Let  $s_1(v)$  denote the number of coset representatives in  $U_v^n / \theta_1(U_v^n)$ . Then the congruences in (15) and (16) give

$$q_v^{n-1} \kappa(v) c_v \equiv \pm \kappa_1(v) s_1(v) \pmod{q_v}.$$

Now

$$\kappa_1(v) \leq \prod_{i=2}^K \text{card } W_{v, |M_i|}^{(n)} \leq n^{K-1} \prod_{i=2}^K |M_i|$$

where  $|M_i|$  is the determinant of the matrix  $M_i$  associated to  $\theta_i$  and  $W_{v, |M_i|}$  is the  $|M_i|$ -th roots of unity in  $U_v$ . We also have

$$s_1(v) \leq n \cdot [U_v : U_v^{|M_1|}]$$

where  $[U_v : U_v^{|M_1|}] = \text{card } W_{v, |M_1|}$  for almost all  $v$ . Hence for almost all  $v$

$$\kappa_1(v) s_1(v) \leq n^K \prod_{i=1}^K |M_i|$$

which implies  $\kappa_1(v) s_1(v) \not\equiv 0 \pmod{q_v}$  for almost all  $v$ . Therefore  $c_v \neq 0$  for almost all  $v$ , which concludes the proof. Q.E.D.

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**Addendum.** J. Denef has recently given a proof of Igusa’s conjecture in the general case.