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Closed transverse (p, p)-forms on compact complex manifolds

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Abstract. We define p-Kähler manifolds requiring the existence of closed (p, p)-forms transverse to the complex structure and then characterize them by a condition on the space of positive currents of the manifolds. The behaviour of the p-Kähler condition with respect to holomorphic submersions and immersions is also studied.

Introduction

The classical examples of compact complex non-Kähler manifolds are the parallelisable compact manifolds which are not tori, and the Calabi-Eckmann spheres. In this paper (Chapter 3) we study this type of manifolds as non trivial examples of p-Kähler manifolds. These are defined in Chapter 1 by requiring the existence of a closed (p, p)-form 'transverse' to the complex structure, and are precisely the Kähler manifolds for $p = 1$ and the balanced (cosymplectic) manifolds for $p = \dim_{\mathbb{C}} M - 1$.

They can be also characterized by a condition on the space of positive currents of the manifold; this condition turns out to be simpler for p-symplectic manifolds (see Def. 1.11).

The behaviour of the p-Kähler condition with respect to holomorphic submersions and immersions is studied in Chapter 2, and this is perhaps the simplest way for testing the 'Kähler degree' of a compact complex manifold.

Preliminaries and notation

A manifold M is always supposed to be complex, compact and connected. Let $\mathscr{E}^{p,q}(M)$ ($\mathscr{E}^{p}(M)$) denote the Fréchet space of complex valued (p, q)-differential forms (p-differential forms), while $\mathscr{E}'_{p,q}(M)$ ($\mathscr{E}'_{p}(M)$) denotes it's dual space of complex currents of bidimension (p, q) (dimension p). The complex structure of M induces an $\mathbb{R}$-linear conjugation on $\mathscr{E}^{p}(M)$ sending dz_j to $d\bar{z}_j$. A p-form ω is *real* if $\bar{\omega} = \omega$, and a p-current T is *real* if $\bar{T} = T$, in the sense that $\overline{T(\varphi)} = T(\bar{\varphi})$ for all $\varphi \in \mathscr{E}^{p}(M)$.

$\mathscr{E}^p(M)_{\mathbb{R}}$ and $\mathscr{E}'_p(M)_{\mathbb{R}}$ denote respectively the space of real p-forms and real p-currents, and analogously for $\mathscr{E}^{p,p}(M)_{\mathbb{R}}$ and $\mathscr{E}'_{p,p}(M)_{\mathbb{R}}$.

We recall also the following definitions:

Definition

A symplectic manifold (M, σ) is a pair consisting of a $2n$-dimensional real manifold M together with a closed real 2-form σ which is non-degenerate (i.e. σ^n never vanishes).

Definition

A balanced (cosymplectic) manifold M is a complex compact manifold admitting an hermitian metric h with Kähler form ω such that $\mathrm{d}\omega^{n-1} = 0$ *($n = \dim_{\mathbb{C}} M$).*

1.

In order to expose the main ideas of the paper, we need a few concepts concerning a real differentiable manifold M introduced by [Sullivan, 1976]. For the comfort of the reader, we recall them here.

1.1. Definition

A compact convex cone *C in a (locally convex topological) vector space over $\mathbb{R}$ is a convex cone such that, for some (continuous) linear functional L, $L(x) > 0$ for $x \neq 0$ in C and $L^{-1}(1) \cap C$ is compact. The latter set is called a base for the cone. We will sometime identify a base with the set of rays in the cone, denoted by ς.*

1.2. Definition

A cone structure *on a manifold M is a continuous field of compact convex cones $\{C_x\}_{x \in M}$ in the vector spaces $\Lambda_p(x)$ of real tangent p-vectors on M. Continuity of cones is defined by the Hausdorff metric on the compact subsets of the rays in Λ_p. Namely the bases of the cones move continuously relatively to the metric* $h(\varsigma,\varsigma') = \max(\sup_{c \in \varsigma}\rho(c, \varsigma'), \sup_{c' \in \varsigma'}\rho(c', \varsigma))$ *where ρ is a convenient metric on rays defined in some local trivialisation of Λ_p.*

1.3. Definition

A differential p-form ω *(of class* C^∞*) on M is* transversal to the cone structure *C if* $\omega_x(v) > 0$ *for each* $v \neq 0$ *in* $C_x \subset \Lambda_p(x)$, $x \in M$.

1.4. Proposition

([Sullivan, 1976], Prop. 1.4). *A cone structure C admits p-forms transversal to C.*
□

1.5 Definition

A Dirac current *is a current determined by the evaluation of p-forms on a single p-vector at a point. The* cone of structure currents *associated to the cone structure C is the closed convex cone of currents generated by the Dirac currents associated to elements of* C_x, $x \in M$.

Now, let M be again a compact complex manifold of complex dimension n. M has natural cone structures $C_1, \dots, C_n$ defined by the almost complex structure J as follows: at a point x, $C_p(x)$ is the compact convex cone in $\Lambda_{2p}(x)$ generated by the positive linear combinations of complex subspaces of $\mathbb{C}$-dimension p (i.e. finite sums of the type $\Sigma\lambda_i V_i$, $\lambda_i \geqslant 0$); (see also [Sullivan, 1976], p. 251).

1.6. Definition

The complex currents on M obtained by extending $\mathbb{C}$*-linearly the structure currents of the cone structure* C_p *are called* positive currents of bidimension (p, p). *We denote the cone of these currents by* $P_{p,p}(M)$.

1.7. Proposition

The cone of positive currents of bidimension (p, p) *on a compact complex manifold M is a compact convex cone.*

1.8. Proposition

For any positive current T of bidimension (p, p) *there is a non negative measure* $\|T\|$ *on M and a* $\|T\|$*-integrable function* $\boldsymbol{T}$ *into* Λ^c_{2p} *satisfying* $\boldsymbol{T}_x \in C^c_p(x)$, *such that* $T = \int_M \boldsymbol{T}\, \|T\|$ *(the upperscript c denotes the complexification of the real vector space).*

The proofs of Propositions 1.7. and 1.8. are the same as those of Proposition 1.5. and Proposition 1.8. in [Sullivan, 1976], but for the complex case. To prove Proposition 1.7. and to have uniqueness of the representation in Proposition 1.8., we need an auxiliar hermitian metric on M.

1.9. Definition

The complex $2p$*-forms on* M *obtained by extending* $\mathbb{C}$*-linearly the* $2p$*-forms transversal to the cone structure* C_p *are called* complex transverse 2p-forms.

1.10. Remarks

a) In [Harvey, 1977], the elements in $C_p(x)$ are called strongly positive (p, p)-vectors (p. 312); our complex transverse (p, p)-forms belong to the interior of the cone of strongly positive (p, p)-forms (p. 323) and our definition of positive currents agrees with that of strongly positive currents (p. 326).
b) Positive currents and complex transverse forms are *real* in the sense that $\bar{T} = T$ or $\bar{\omega} = \omega$. Moreover, any complex current (or form) which is real (in this sense) is in fact the $\mathbb{C}$-linear extension of a real current (or form).

We define now two classes of complex manifolds generalizing symplectic and Kähler manifolds.

1.11. Definition

A complex manifold M *is called* p-Kähler *if it admits a closed complex transverse* (p, p)*-form, called the* p-Kähler form. *The integer p will be called* Kähler degree of M. M *is called* p-symplectic *if it admits a closed complex transverse* $2p$*-form, called* p-symplectic form.

(We could give the definitions more generally for an almost complex manifold, but this is far from the aim of this paper).

Note that every M of dimension n is simultaneously n-Kähler and n-symplectic. Moreover, for $p < n - 1$, if ω is the Kähler form of an hermitian metric on M and ω^p turns M into a p-Kähler manifold, then M was already 1-Kähler; in fact to prove that $d\omega^p = 0$ implies $d\omega = 0$ for $p < n - 1$ is merely a linear algebra computation (not a short one!).

The following propositions give a first motivation to the definitions.

1.12. Proposition

If M is 1-symplectic, then M is symplectic.

Proof

If $\psi^{\#}$ is a 1-symplectic form of M, consider the real 2-form ψ of which $\psi^{\#}$ is the $\mathbb{C}$-extension. ψ is always closed, and of maximal rank if $\ker_x \psi := \{X \in T_x M / \psi_x(X, Y) = 0 \ \forall Y \in T_x M\} = 0 \ \forall x \in M$. Suppose $X \in \ker_x \psi$. Then $\psi_x(X, JX) = 0$ but $(X, JX) \in C_1(x)$ so $X = 0$ (J is the complex structure of M). □

To have the converse of this result, standing the definition of a symplectic manifold usually in the realm of real geometry, we need the following remark: given a symplectic structure ψ on M, since $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ and $Sp(2n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ have the same maximal compact $U(n) \subset GL(2n, \mathbb{R})$ there is a well defined contractible set of almost complex structures J determined by ψ. They are in fact characterized by the transversality condition $\psi(X, JX) > 0$.

1.13. Definition

On a complex manifold (M, J) a symplectic structure ψ is said to be compatible with the complex structure *if J belongs to the set of almost complex structures determined by ψ.*

We have immediately

1.14. Proposition

If M has a symplectic structure compatible with the complex structure, then M is 1-symplectic. □

The definition of p-Kähler manifold is more natural in the context of complex geometry and in fact we have:

1.15 Proposition

a) *M is 1-Kähler iff M is Kähler.*
b) *M is $(n-1)$-Kähler if and only if it is balanced.*

Proof

a) Suppose M Kähler with Kähler form ω. It is known that ω is a complex (1, 1)-form which is real and since the metric is positive $\omega_x(X, JX) > 0$ $\forall X \in T_x M$. Being ω closed by definition, M is 1-Kähler. Conversely, suppose M 1-Kähler and ω a closed complex transverse (1, 1)-form. We define, for all $x \in M$ and for all $X, Y \in T_x M$, $h(X, Y) = \omega(JX, Y) + i\omega(X, Y)$; h becomes an hermitian metric on M with Kähler form ω. The positivity of h descends from the transversality of ω.
b) If M is balanced with Kähler form ω, then ω^{n-1} is, analogously to the case a), a closed complex transverse $(n-1, n-1)$-form. Let now Ω be a closed complex transverse $(n-1, n-1)$-form. We claim that $\Omega = \omega^{n-1}$ where ω is a complex transverse (1, 1)-form. This is simply a matter of multilinear algebra which can be found in ([Michelson, 1983], p. 279). As in the case a) we can find an hermitian metric on M with Kähler form ω. By the hypothesis, $d(\omega^{n-1}) = d\Omega = 0$. □

1.16. Remarks

a) If M is p-Kähler, then M is p-symplectic.
b) If M is 1-Kähler (1-symplectic), then M is p-Kähler (p-symplectic) for $1 \leqslant p \leqslant n$. More generally if M is p-Kähler (p-symplectic) then M is rp-Kähler (rp-symplectic) for $1 \leqslant rp \leqslant n$.

Now we give the main result which follows readily from an application of the Hahn-Banach theorem [Schäfer, 1971] and a finite dimensionality theorem descending from a particular resolution of the sheaf $\mathscr{H}$ of pluriharmonic functions.

1.17. Theorem

a) M is p-Kähler if and only if there are no non trivial positive currents of bidimension (p, p) which are (p, p)-components of boundaries.
b) M is p-symplectic if and only if there are no non trivial positive currents of bidimension (p, p) which are boundaries.

To prove the theorem we need a few more definitions and lemmas.

1.18. Definition

Let $[\mathscr{E}'_{p,p+1}(M) \oplus \mathscr{E}'_{p+1,p}(M)]_{\mathbb{R}}$ denote the space of real currents dual to $[\mathscr{E}^{p,p+1}(M) \oplus \mathscr{E}^{p+1,p}(M)]_{\mathbb{R}}$ and let

$\mathrm{d}_{p,p}: [\mathscr{E}'_{p,p+1}(M) \oplus \mathscr{E}'_{p+1,p}(M)]_{\mathbb{R}} \to [\mathscr{E}'_{p,p}(M)]_{\mathbb{R}}$ *denote the differential operator dual to*

$$d\,|_{[\mathscr{E}^{p,p}(M)]_{\mathbb{R}}}: [\mathscr{E}^{p,p}(M)]_{\mathbb{R}} \to [\mathscr{E}^{p,p+1}(M) \oplus \mathscr{E}^{p+1,p}(M)]_{\mathbb{R}}$$

defined by

$$d_{p,p} = \pi_{p,p} o d\,|_{[\mathscr{E}'_{p,p+1}(M) \oplus \mathscr{E}'_{p+1,p}(M)]_{\mathbb{R}}}$$

where $\pi_{p,p}$ *denotes the natural projection*

$$\pi_{p,p}: \mathscr{E}'_{2p}(M)_{\mathbb{R}} \to \mathscr{E}'_{p,p}(M)_{\mathbb{R}}$$

and d is the usual differential operator for currents.

1.19. Theorem

$$\dim_{\mathbb{C}} \frac{\{\alpha \in [\mathscr{E}^{p,p+1}(M) \oplus \mathscr{E}^{p+1,p}(M)]_{\mathbb{R}} / \mathrm{d}\alpha = 0\}}{\{\mathrm{d}\beta \text{ for } \beta \in [\mathscr{E}^{p,p}(M)]_{\mathbb{R}}\}} < \infty.$$

Proof

It follows from the resolution of the sheaf $\mathscr{H}$ of pluriharmonic functions studied in [Alessandrini and Andreatta, 1985]. □

1.20. Corollary

The operator

$$\mathrm{d}_{p,p}: [\mathscr{E}'_{p,p+1}(M) \oplus \mathscr{E}'_{p+1,p}(M)]_{\mathbb{R}} \to [\mathscr{E}'_{p,p}(M)]_{\mathbb{R}}$$

has closed range.

Proof

As noted above $\mathrm{d}_{p,p}$ is the adjoint operator of

$$d\,|_{[\mathscr{E}^{p,p}(M)]_{\mathbb{R}}}: [\mathscr{E}^{p,p}(M)]_{\mathbb{R}} \to [\mathscr{E}^{p,p+1}(M) \oplus \mathscr{E}^{p+1,p}(M)]_{\mathbb{R}}.$$

From the closed range theorem [Schäfer, 1971] it is sufficient to prove that d has closed range. This follows from the open mapping theorem and Theorem 1.19. □

1.21. Lemma

The operator d_{2p}: $\mathscr{E}'_{2p+1}(M)_{\mathbb{R}} \to \mathscr{E}'_{2p}(M)_{\mathbb{R}}$ *has closed range.*

Proof

Analogous to but easier than that of Corollary 1.20. □

We will denote by $B_{p,p}(M)$ the range of $d_{p,p}$ and by $B_{2p}(M)$ the range of d_{2p}.

1.22. Lemma

Let ω *be a p-Kähler (p-symplectic) form on a p-Kähler (p-symplectic) manifold* M. *For every* $T \in P_{p,p}$, $T \neq 0$, *we have* $T(\omega) > 0$. *For every* $T \in B_{p,p}(B_{2p})$, *we have* $T(\omega) = 0$.

Proof

If $T \in P_{p,p}$, it follows from Proposition 1.8. that $T(\omega) = \int_M \omega(\boldsymbol{T}) \| T \|$. By definition, if $T \neq 0$, $\omega_x(\boldsymbol{T}_x) > 0$ in both cases and consequently $T(\omega) > 0$.

If $T \in B_{p,p}(B_{2p})$ then $T = d_{p,p}S$ $(T = d_{2p}S)$. From the definition of dual operators we have the equalities: $0 = (d\omega, S) = (\omega, d_{p,p}S) = (\omega, T) = T(\omega)$ $(0 = (d\omega, S) = (\omega, d_{2p}S) = (\omega, T) = T(\omega))$. □

Proof of theorem 1.17

The 'only if' part follows from Lemma 1.22. in both cases. On the contrary, from Proposition 1.7. we have that $P_{p,p}(M)$ is a compact convex cone in $\mathscr{E}'_{p,p}(M)_{\mathbb{R}}$ $(\mathscr{E}'_{2p}(M)_{\mathbb{R}})$. Now as for part a), Corollary 1.20. says that $B_{p,p}(M)$ is a closed subspace of $\mathscr{E}'_{p,p}(M)_{\mathbb{R}}$, and as for part b) by Lemma 1.21. $B_{2p}(M)$ is closed in $\mathscr{E}'_{2p}(M)_{\mathbb{R}}$. So the Hahn-Banach separation theorem applies to tell us that there exists a form $\omega \in [\mathscr{E}^{p,p}(M)]_{\mathbb{R}}([\mathscr{E}^{2p}(M)]_{\mathbb{R}})$ which is zero on $B_{p,p}(M)(B_{2p}(M))$ and strictly positive on $P_{p,p}(M)$.

Now, $T(\omega) = 0$ for all $T \in B_{p,p}(M)(B_{2p}(M))$ implies $d\omega = 0$. Choose $\boldsymbol{T}_x \in C_p^c(x)$, $x \in M$. Then $T = \boldsymbol{T}_x \delta_x \in P_{p,p}$ (δ_x is the Dirac measure in x) and so

$T(\omega)>0$, but $\omega_x(T_x)=\int_M \omega(T_x)\delta_x = T(\omega)>0$. This can be done for every $x \in M$ and every $T_x \in C_p^c(x)$ completing the proof. □

1.23. Remarks

In the case $p=1$ Theorem 1.17. a) provides the same characterization of Kähler manifolds already given in [Harvey and Lawson, 1983] and Theorem 1.17. b) is in [Sullivan, 1976]. On the other side, in the case $p=n-1$, Theorem 1.17. a) gives the characterization of balanced manifolds in [Michelson, 1983].

2.

In this chapter we examine firstly the behaviour of p-Kähler manifolds with respect to holomorphic submersions. This will provide useful criterions expecially in the study of examples of p-Kähler manifolds for the various degrees p.

Then we extabilish other results regarding submanifolds of p-Kähler manifolds and the fundamental class of analytic varieties in p-Kähler manifolds. We don't consider here the p-symplectic case, for which anyway analogous results hold.

2.1. Theorem

Suppose f: $M \to N$ is a holomorphic submersion with p-dimensional fibres onto a p-Kähler manifold ($p \leqslant n/2$, $n = \dim M$). Then there exists a p-Kähler form on M if and only if the fibre of f is not the (p, p)-component of a boundary.

2.2. Remarks

a) Any two fibres of f are homologous. Hence, if a fibre is a (p, p)-component of a boundary, then so are the others.
b) The condition on the fibres of the submersion is necessary as we shall show with some examples in Chapter 3.

For the proof of Theorem 2.1., we need a lemma.

2.3. Lemma

Choose an auxiliar hermitian metric on M. Suppose f: $M \to N$ as in Theorem 2.1. *and that T is a positive current of bidimension (p, p) on M. Then the* *push-forward f_*T of T to N is zero if and only if $T = \int_M \vec{F} \, \|T\|$ where $\vec{F}$ is the field of unit $2p$-vectors tangent to the fibre (and $\|T\|$ is a non negative measure on M).*

If in addition $\partial\bar{\partial}T = 0$ then $T = f^*(\mu)$ for some non negative measure μ on N.

Proof

Suppose $T = \int_M \vec{F} \, \|T\|$. For any $2p$-form ω on N we have $f^*\omega(\vec{F}) = 0$. Hence $f_*T(\omega) = T(f^*\omega) = 0$, thus $f_*T = 0$.

On the contrary suppose $f_*T = 0$ and represent T as in Proposition 1.8. $T = \int_M \boldsymbol{T} \, \|T\|$ with $\boldsymbol{T}_x \in C_p^c(x)$ and of unit norm for every $x \in M$. Let ω be any transverse $2p$-form on N. We have

$$0 = (f_*T)(\omega) = T(f^*\omega) = \int_M (f^*\omega)(\boldsymbol{T}) \, \|T\|.$$

Now $df_x(T) \in C_p^c(f(x))$ because f is holomorphic and so from the transversality of ω, $(f^*\omega)_x(\boldsymbol{T}_x) = \omega_{f(x)}(df_x(\boldsymbol{T})) > 0$ unless $df_x(\boldsymbol{T}) = 0$. We conclude that $(df(\boldsymbol{T}))_{f(x)} = 0$ $\|T\|$-a.e., and consequently that $\boldsymbol{T} = \vec{F}$ as claimed.

Now suppose in addition $\partial\bar{\partial}T = 0$. One can think a positive (p, p)-current as a $(n-p, n-p)$-form with measure coefficients, and so our T can be written as

$$T = \|T\| \, f^*(\Lambda),$$

where Λ is a volume form on N. But $\partial\bar{\partial}T = 0$ implies $(\partial\bar{\partial}\|T\|) \wedge f^*(\Lambda) = 0$, so that $\|T\|$ is harmonic in the fibre directions, and then constant on the fibres. Therefore $\|T\|$ is the pull-back of a measure μ' on N, and then

$$T = f^*(\mu'\Lambda) = f^*(\mu). \qquad \square$$

Proof of theorem 2.1

The 'only if' part follows trivially from Theorem 1.17. As for the 'if' part, suppose that M is not p-Kähler. Then by Theorem 1.17. there exists a positive current T of bidimension (p, p) on M which is the (p, p)-component of a boundary, i.e. $T = d_{p,p}S$ for some $(2p+1)$-current S. Since f is holomorphic, f_*T is a positive current of bidimension (p, p) on N and $f_*T = d_{p,p}(f_*S)$. Thus, since N is p-Kähler, we conclude that $f_*T = 0$. From $T = d_{p,p}S$ we have that $\partial\bar{\partial}T = 0$, so Lemma 2.3. implies that $T = f^*(\mu)$ for some non negative measure μ on N.

Put now $c = \int_N \mu$ and recall that any two measures with the same total mass are homologous on N. So, for any $y \in N$, if δ_y is the Dirac measure at y, we have $c\delta_y - \mu = \mathrm{d}R$ for some current R on N. Pulling back by f we have that

$$c[f^{-1}(y)] - T = \mathrm{d}f^*(R).$$

(We denote by $[f^{-1}(y)]$ the current given by integration along the fibre $f^{-1}(y)$). Therefore the fibre $[f^{-1}(y)]$ is the (p, p)-component of a boundary. □

We have the following corollary to Lemma 2.3.:

2.4. Proposition

Suppose that f: $M \to N$ is a holomorphic submersion with p-dimensional fibres of a non p-Kähler manifold M onto a p-Kähler manifold N ($p \leqslant n/2$, $n = \dim M$). Then the cone of all positive currents which are (p, p)-components of boundaries is equal to $\{T/T = f^*(\mu)$ for some non negative measure μ on $N\}$. □

We give now the dual theorem ($p > n/2$) for which we still need the closure property stated in Corollary 1.20.

2.5. Theorem

Let f: $M^n \to N^{n-p}$ be a holomorphic submersion, where the fibre is p-dimensional and $(2p - n)$-Kähler ($p > n/2$). Then M is p-Kähler if and only if the fibre of f is not the (p, p)-component of a boundary.

Proof

The 'only if' part is obvious from Theorem 1.17.; on the contrary, suppose M not p-Kähler, and let T be a positive current on M such that $T = \mathrm{d}_{p,p}S$ for some real $(2p + 1)$-current S.

The proof follows that of Theorem 5.5. of [Michelson, 1983], and we refer to this paper for a technical lemma which we shall use.

Let us construct a tubular neighbourhood of the fibre well behaved with regard to the complex structure, that is fix a point $y \in Y$ and let $z = (z_1, \dots, z_{n-p})$, $|z| < 1$ a chart on N centered at y. Let $\Delta = \{|z| < \epsilon_0\}$ be a sufficiently small disk such that $D := f^{-1}(\Delta)$ is a tubular neighborhood of $F := f^{-1}(y)$ and g: $D \to \Delta \times F$ is a $\mathscr{C}^\infty$ product structure with the property that the complex structure makes 'infinite order contact with the Δ-factors along $(0) \times F$'. This means: let J be the almost complex structure on D and carry J over $\Delta \times F$ via the diffeomorphism g. Let J_0 be the natural product almost complex structure on $\Delta \times F$. Then we want the tensor $J - J_0$ to be zero to infinite order at all points of $\{0\} \times F$. This can be done by exponentiating the normal bundle of F with any hermitian ($\nabla J = 0$) connection on D.

Now consider the family of $(n-p,\ n-p)$-forms on N given by

$$\varphi_\epsilon = \left(i/\epsilon^{2(n-p)}\right)\varphi(|z|/\epsilon)\ \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

where $\varphi \in \mathscr{C}_0^\infty(-1, 1)$ is a bump function and $\int_M \varphi_\epsilon = 1$, and define the currents $T'_\epsilon := f^*\varphi_\epsilon \wedge T$, $S'_\epsilon := f^*\varphi_\epsilon \wedge S$ which are positive currents with compact support in D. They are related by

$$\mathrm{d}_{2p-n,2p-n}(S'_\epsilon) = (f^*\varphi_\epsilon \wedge dS)_{2p-n,2p-n} = f^*\varphi_\epsilon \wedge d_{p,p}(S) = T'_\epsilon$$

because of the maximal dimension of φ_ϵ.

Set $m_\epsilon := \max\{1, \| T_\epsilon \|\}$, and define $T_\epsilon := T'_\epsilon/m_\epsilon$, $S_\epsilon := S'_\epsilon/m_\epsilon$. We still have that T_ϵ is the $(2p-n, 2p-n)$-component of a boundary, namely S_ϵ. By general compactness theorems, (the T'_ϵ*s* have bounded supports and bounded masses), given a sequence $\epsilon_m \to 0$, there is a subsequence $\{\epsilon_{m_j}\}$ such that $T_j := T(\epsilon_{m_j}) \to T_\infty$ (weakly) where T_∞ is a positive $(2p-n, 2p-n)$-current with support on F.

Claim $T_\infty = 0$, so that $\lim_{\epsilon \to 0} T_\epsilon = 0$.

Furthermore, by positivity, $\lim_j \| T_j \| = \| T_\infty \|$, and so we get that $f^*\varphi_\epsilon \wedge T \to 0$ in the mass norm on M. Now, let ω be a volume form on N: then $f^*\omega \wedge T = 0$ on M, so that $\vec{T}_x = \vec{F}_x$ (the field of unit $2p$-vectors tangent to the fibre) for $\| T \|$ – a.a. x in M. This fact, together with the assumption $T = \mathrm{d}_{p,p}S$, allows us to write $T = f^*(\mu)$ for some non negative measure μ on N, as in the proof of Lemma 2.3., and to conclude the proof as in Theorem 2.1.

Proof of the claim

Consider ρ: $D \to F$, given by $\rho :=$ proj.og, and the push-forward currents $\rho_* T_\epsilon$ and $\rho_* S_\epsilon$, for ϵ small. Since supp$T_\infty \subset F$ and T_∞ is tangent to F at $\| T_\infty \|$-a.a. points, $\rho_* T_\infty = T_\infty$. Then
$(\rho_* T_\epsilon)_{2p-n,2p-n} = (\rho_* \mathrm{d}_{2p-n,2p-n} S_\epsilon)_{2p-n,2p-n} = (\rho_*(dS_\epsilon - \oplus_{r \neq s} d_{r,s} S_\epsilon))_{2p-n,2p-n} = (d(\rho_* S_\epsilon))_{2p-n,2p-n} + E_\epsilon$ where E_ϵ is a sum of terms of the type $(\rho_*(d_{r,s} S_\epsilon))_{2p-n,2p-n}$ for which $\lim_{j \to \infty} E_{\epsilon j} = 0$ (it is a consequence of the 'infinite order contact structure' which we choosed above: see ([Michelson, 1983], Lemma 5.8).

Then $T_\infty = \rho_* T_\infty = \lim_j \rho_* T_j = \lim_j d_{2p-n,2p-n}(\rho_* S_{\epsilon j})$ but the subspace of $(2p-n, 2p-n)$-components of boundaries in F is closed (Corollary 1.20) so that $T_\infty = d_{2p-n,2p-n}(S_\infty)$ for some real $(4p-2n+1)$-current S_∞ on F. Since the fibre is $(2p-n)$-Kähler, we conclude that $T_\infty = 0$. □

Notice that a submanifold of a Kähler manifold is Kähler and analogously for the dual statement: if M is balanced and there exists a holomorphic submersion $f: M \to N$, then N is balanced. For p-Kähler manifolds these statements generalize as follows:

2.6. Proposition

Let f: $M^n \to N^{n-p}$ be a holomorphic submersion with p-dimensional fibres. If M is q-Kähler with $n \geqslant q > p$, then N is $(q-p)$-Kähler.

Proof

Suppose $q < n$ otherwise there is nothing to prove. Let ω_M be a q-Kähler form on M; since M and N are compact, we can define $\omega_N := f_*\omega_M$ where $f_*\omega_M$ is the push-forward of ω_M regarded as a $(n-q,\ n-q)$-current. In local coordinates, if $\omega_M = \Sigma_{|J|=q}\varphi_J\, dz_J \wedge d\bar{z}_J$, then $f_*\omega_M = \Sigma_{|K|=q-p}\psi_K\, dz_K \wedge d\bar{z}_K$ where

$$\psi_K = \int_{\text{Fibre}} \varphi_J\, dz_{n-p+1} \wedge \ldots \wedge dz_n \wedge d\bar{z}_{n-p+1} \wedge \ldots \wedge d\bar{z}_n.$$

$d\omega_N = 0$ because ω_M is closed. Now fix a point $y \in N$ and let $F = f^{-1}(y)$; let $\{e_1, Je_1, \ldots, e_{n-p}, Je_{n-p}\}$ be a basis for T_yN and extend it to a basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$ for T_xM, $x \in F$. If

$$v = \sum_{|K|=q-p 1 \leqslant k_m \leqslant n-p} \lambda_K e_{K1} \wedge Je_{K1} \wedge \ldots \wedge e_{Kq-p} \wedge Je_{Kq-p} \in C^c_{q-p}, \quad v \neq 0,$$

$$\omega_N(v) = \int_{\text{Fibre}} \omega_M\big(v \wedge e_{Kn-p+1} \wedge Je_{Kn-p+1} \wedge \ldots \wedge e_n \wedge Je_n\big) > 0$$

so that ω_N is positive. Then N is $(q-p)$-Kähler. □

2.7. Proposition

If M is a p-Kähler manifold of dimension m and N is a submanifold of dimension $n \geqslant p$, then N is p-Kähler.

Proof

Let i: $N \to M$ be the inclusion map, and ω_M be a p-Kähler form on M. $\omega_N := i^*\omega_M$ is a closed (p, p)-form on N, and if $v \in \Lambda_p(N)$ is not zero, $\omega_N(v) = (i^*\omega_M)(v) = \omega_M(\mathrm{d}i(v)) > 0$ because $\mathrm{d}i$ is injective. □

2.8. Proposition (corollary to Theorem 1.17)

In a p-Kähler manifold M, the fundamental class of any analytic subvariety $V \subset M$ of dimension p is non zero. □

3.

Let us now consider complex compact (holomorphically) parallelisable manifolds. By ([Wang, 1954], Theorem 1), they are homogeneous manifolds G/Γ, where G is a complex Lie group and Γ a discrete uniform subgroup of G.

In [Wang, 1954] it is also shown that the only 1-Kähler manifolds among them are the complex tori.

Let us now prove the following.

3.1. Proposition

On a complex compact parallelisable manifold $M = G/\Gamma$ there is a G-invariant hermitian metric such that the corresponding hermitian connection has zero curvature.

Proof

(see also [Goldberg, 1962], Chapter 6)

Let $\{\vartheta_1, \ldots, \vartheta_n\}$ be holomorphic vector fields everywhere linearly independent on M which give a basis for $\mathfrak{g}$, the Lie algebra of G, and let $\{\varphi_1, \ldots, \varphi_n\}$ be the dual basis of $\mathfrak{g}^*$. Define a connection by requiring $\nabla_{\vartheta_i}\vartheta_j = \nabla_{\bar{\vartheta}_i}\vartheta_j = \nabla_{\vartheta_i}\bar{\vartheta}_j = \nabla_{\bar{\vartheta}_i}\bar{\vartheta}_j = 0$ for i, $j = 1, \ldots, n$. To show that this is the hermitian connection of the metric $h = \Sigma\varphi_i\bar{\varphi}_i$ and that the curvature R is zero is a routne computation. □

Consider now the following result due to Gauduchon:

3.2. Proposition

Let (M, h) be an hermitian manifold of dimension n. If the curvature of the associated hermitian connection is zero, then M is $(n-1)$-Kähler.

Proof ([Gauduchon, 1977], p. 140). □

Combining the last two propositions, we conclude that every complex compact parallelisable manifold of dimension n is $(n-1)$-Kähler; it is Kähler if and only if it is a complex torus. On the contrary, note that the Calabi-Eckmann spheres are examples of non balanced manifolds.

We will now say more about a subclass of the class of complex parallelisable manifolds, i.e. the nilmanifolds.

3.3. Definition

M is said to be a nilmanifold (solvmanifold) if M is a homogeneous space G/Γ, where G is a complex, connected, simply connected, nilpotent (solvable) Lie group which is biholomorphically equivalent to the universal covering of M, and Γ is the fundamental group of M, a discrete uniform subgroup of G.

In particular, M is holomorphically parallelisable and has $\mathbb{C}^n$ as universal covering; we denote by $*$ the product on $\mathbb{C}^n$ which makes $(\mathbb{C}^n, *)$ isomorphic to G as a Lie group. More about nilmanifolds can be found in [Alessandrini and Andreatta, 1986]; we recall here only a characterization which we shall use later.

3.4. Definition

A principal torus tower of height one *is a complex torus. A* principal torus tower of height m, $m > 1$, *is a holomorphic principal bundle with a complex torus as fibre and a principal torus tower of height $m-1$ as basis. We shall call* base torus *the last torus which results from the backwards inductive decompositions of a principal torus tower.*

3.5. Theorem [Barth and Otte, 1969]

Let M be a compact homogeneous manifold. Then M is a principal torus tower if and only if M is a nilmanifold. □

In order to compute the Kähler-degree of a nilmanifold, we need to know the De Rham cohomology groups $H^p_{DR}(M)$ which can be calculated using the Leray spectral sequence, as in [Alessandrini and Andreatta, to appear]. We begin with the following:

3.6. Proposition

Let M be a homogeneous principal torus tower with fibre T_j of dimension j; then M is not p-Kähler for $1 \leqslant p \leqslant j$.

Proof

We shall exhibit a p-dimensional submanifold of M which is homologous to zero, getting then the thesis from Theorem 1.17. Let M be $\mathbb{C}^n/\Gamma$, $(\mathbb{C}^n, *) \cong G$, $\{\vartheta_1, \ldots, \vartheta_n\}$ be a basis for $\mathfrak{g}$, the Lie algebra of G, such that $[\vartheta_i, \vartheta_h] = \sum_{k > \max(i,h)} c_{ih}^k \vartheta_k$, $c_{ih}^k \in \mathbb{C}$ (the existence of such a basis is guaranted by a well known theorem of Lie), and let $\{\varphi_1, \ldots, \varphi_n\}$ be the dual basis of $\mathfrak{g}^*$. As shown in [Alessandrini and Andreatta, 1986], we can find coordinates on $\mathbb{C}^n$ such that $T_j = \{z_1 = \text{const.}, \ldots, z_{n-j} = \text{const.}\}$ and such that the 1-forms φ_i and $\bar{\varphi}_i$ are of the form

$$\varphi_1 = \mathrm{d}z_1, \ldots, \varphi_r = \mathrm{d}z_r, \quad \varphi_k = \mathrm{d}z_k + \sum_{h<k} a_{hk}\, \mathrm{d}z_h,$$

$$\bar{\varphi}_1 = \mathrm{d}\bar{z}_1, \ldots, \bar{\varphi}_r = \mathrm{d}\bar{z}_r,$$

$$\bar{\varphi}_k = \mathrm{d}\bar{z}_k + \sum_{h<k} \bar{a}_{hk}\, \mathrm{d}\bar{z}_h, \quad \text{for } k = r+1, \ldots, n-j.$$

For $1 \leqslant p < j$, let $T_p := T_j \cap \{z_{n-j+1} = \text{const.}, \ldots, z_{n-p} = \text{const.}\}$. T_p represents a class in $H_{2p}(M, \mathbb{Z})$, and by De Rham's theorem, T_p is a boundary iff for every $\alpha \in H_{DR}^{2p}(M)$, $\int_{T_p} \alpha = 0$, or considering the Leray spectral sequence on M, iff $\int_{T_p} \varphi = 0$ for every $\varphi \in E_3^{a,b}$ with $a + b = 2p$. But (see [Alessandrini and Andreatta, to appear] from which we also take the notation) $E_3^{0,2p} = 0$, so every non trivial element of $E_3^{a,b}$ contains a least one φ_k or $\bar{\varphi}_k$ for $k = 1, \ldots, n-j$. Restricting the forms of $E_3^{a,b}$ on T_p, $\varphi_k|_{T_p} = \bar{\varphi}_k|_{T_p} = 0$ for $k = 1, \ldots, n-j$. So we get $\int_{T_p} \varphi = 0$ for every $\varphi \in E_3^{a,b}$, $a + b = 2p$. □

Now we give examples of manifolds which are p-Kähler. For $n = \dim M = 3$, the typical example is the Iwasawa manifold, which, as said before in general, is not 1-Kähler but is 2-Kähler (balanced) and 3-Kähler.

For $n \geqslant 4$, the simplest but very interesting example is a 'generalised Iwasawa manifold', I_n which we shall describe now. Let π: $(\mathbb{C}^n, *) \to (\mathbb{C}^{n-p}, +)$ the projection $(z_1, \ldots, z_n) \to (z_1, \ldots, z_{n-p})$ for $1 < p \leqslant n/2$ and $n \geqslant 4$, where $(y_1, \ldots, y_n) * (z_1, \ldots, z_n) = (y_1 + z_1, \ldots, y_{n-1} + z_{n-1}, y_n + z_n + y_{n-2} z_{n-1})$ (see [Alessandrini and Andreatta, 1986]) and $+$ is the usual abelian sum. The map π is a Lie group homomorphism.

Let $\Gamma < \mathbb{C}^n$ be a discrete uniform subgroup (for instance $(\mathbb{Z}[i])^n$), and let $\Gamma' := \pi(\Gamma) \subset \mathbb{C}^{n-p}$; Γ' is still a discrete uniform subgroup of $\mathbb{C}^{n-p}$ and we have π': $I_n := \mathbb{C}^n/\Gamma \to \mathbb{C}^{n-p}/\Gamma' = T_{n-p}$ which is a holomorphic submersion.

$(\mathbb{C}^n, *)$ is a Lie group of dimension n whose Lie algebra $\mathfrak{g}$ has a Lie basis $\{\vartheta_1, \ldots, \vartheta_n\}$ such that $d\varphi_1 = 0, \ldots, d\varphi_{n-1} = 0$, $d\varphi_n = -\varphi_{n-2} \wedge \varphi_{n-1}$ where $\{\varphi_i\}$ is the dual basis. Moreover, in coordinates we have

$$\varphi_1 = dz_1, \ldots, \varphi_{n-1} = dz_{n-1}, \quad \varphi_n = dz_n - z_{n-2}\, dz_{n-1}.$$

Let ω be the following d-closed (p, p)-form on I_n:

$$\omega = \varphi_{n-p+1} \wedge \ldots \wedge \varphi_n \wedge \bar{\varphi}_{n-p+1} \wedge \ldots \wedge \bar{\varphi}_n.$$

For $q \in T_{n-p}$, we get

$$\int_{\pi^{-1}(q)} \omega = \int_{\pi^{-1}(q)} dz_{n-p+1} \wedge \ldots \wedge dz_n \wedge d\bar{z}_{n-p+1} \wedge \ldots \wedge d\bar{z}_n > 0.$$

The fibre $\pi^{-1}(q)$ is not a (p, p) component of a boundary; for if $\pi^{-1}(q) = d_{p,p}(S)$, we get a contradiction by

$$0 < \int_{\pi^{-1}(q)} \omega = \int_{d_{p,p}(S)} \omega = \int_{d(S)} \omega = \int_S d\omega = 0.$$

Then, since T_{n-p} is Kähler and hence p-Kähler, we conclude from Theorem 2.1. that I_n is p-Kähler. We cannot extend this procedure to the case $p = 1$ because ω is not closed.

So we have proved that the generalized Iwasawa manifold I_n is 2-Kähler, 3-Kähler, ..., $[n/2]$-Kähler, and we have noticed that it is not 1-Kähler but is $(n-1)$-Kähler. I_4 is then completely solved from this point of view. If $n \geqslant 5$, what can we say about the degrees between $[n/2] + 1$ and $n - 2$?

Let j be an integer between 3 and $[(n-1)/2]$, and consider

$$\sigma\colon (\mathbb{C}^n, *) \to (\mathbb{C}^j, *_1)$$

$(z_1, \ldots, z_n) \to (z_{n+1-j}, \ldots, z_n)$ where $*$ is as above and $(y_{n+1-j}, \ldots, y_n) *_1 ((z_{n+1-j}, \ldots, z_n) = (y_{n+1-j} + z_{n+1-j}, \ldots, y_{n-1} + z_{n-1}, \quad y_n + z_n + y_{n-2} z_{n-1})$

The map σ is a Lie group homomorphism. As above, we obtain a holomorphic submersion σ': $I_n \to I_j$. The fibre is a torus of dimension $n-j$, which is Kähler and so $2(n-j) - n (= n - 2j)$-Kähler. Then we get from Theorem 2.5. that I_n is $(n-j)$-Kähler if we prove that T_{n-j} is not the $(n-j, n-j)$-component of a boundary. Let us consider

$$\omega = \varphi_1 \wedge \ldots \wedge \varphi_{n-j} \wedge \bar{\varphi}_1 \wedge \ldots \wedge \bar{\varphi}_{n-j}.$$

ω is a closed form, and for $q \in I_j$, $\int_{\sigma^{-1}(q)} \omega =$ volume of $\sigma^{-1}(q) > 0$, so we conclude as above that I_n is $(n-j)$-Kähler for $3 \leqslant j \leqslant [(n-1)/2]$.

The case $(n-2)$ requires a particular examination, because the above proof does not work. To prove that I_n is $(n-2)$-Kähler, let us suppose first $n \geqslant 6$ and consider

$$\tau\colon (\mathbb{C}^n, *) \to (\mathbb{C}^2, +)$$

$$(z_1, \ldots, z_n) \to (z_1, z_2).$$

Then τ induces a holomorphic submersion $\tau'\colon I_n \to T_2$ with fibre I_{n-2}. But the fibre is a submanifold of I_n, which is $(n-4)$-Kähler by the above proof, so that I_{n-2} is $(n-4)$-Kähler (Proposition 2.7). Now from Theorem 2.5. (the fibre is $2(n-2)-n(=n-4)$-Kähler) I_n is $(n-2)$-Kähler if I_{n-2} is not the $(n-2, n-2)$-component of a boundary. But consider the closed form $\omega = \varphi_3 \wedge \ldots \wedge \varphi_n \wedge \bar{\varphi}_3 \wedge \ldots \wedge \bar{\varphi}_n$: the integration of ω on the fibre gives us the volume of the fibre, so we can conclude as above. For $n = 5$, we cannot use this proof, because I_3 is not 1-Kähler. But I_5 is the fibre of $\tau'\colon I_7 \to T_2$ as above, and I_7 is 3-Kähler so that I_5 is 3-Kähler too. We have then proved

3.7. Proposition

The generalized Iwasawa manifold I_n is j-Kähler for $j = 2, \ldots, n$ but is not 1-Kähler. □

3.8.

We make here a brief digression about the generalized Iwasawa manifold. The computation of the Betti numbers of I_n done as indicated in [Alessandrini and Andreatta, to appear] shows that $b_{2p}(I_n) > 0$ for $p = 0, \ldots, n$ and $b_{2p+1}(I_n) = 2k$ for $p = 0, \ldots, n-1$. This is not peculiar to I_n: if M is a nilmanifold, the odd order Betti numbers are even because if ψ is a d-closed form which represent a cohomology class, $\bar{\psi}$ too represents a class which is clearly different from $[\psi]$ if the degree of ψ is odd. Moreover, if a manifold is p-Kähler, then $b_{2rp}(M) > 0$ for $p \leqslant rp \leqslant n$ (using the p-Kähler form). So the Betti numbers are of no use to decide that these manifold don't support a Kähler metric.

The techniques employed for the generalized Iwasawa manifold can be used for many other classes of examples; for instance for $n \geqslant 5$ consider $t_n = G/\Gamma$ where the 1-forms $\{\varphi_j\}$ dual to the Lie basis for $\mathfrak{g}$ satisfy $d\varphi_1 = 0, \ldots, d\varphi_{n-2} = 0$, $d\varphi_{n-1} = -\varphi_1 \wedge \varphi_2$, $d\varphi_n = -\varphi_1 \wedge \varphi_3$. t_n is a principal torus tower of height two and fibre T_2 [Alessandrini and Andreatta, to appear]. From Proposition 3.6., t_n is not 2-Kähler, and obviously is not 1-Kähler either, but it is $(n-1)$-Kähler. We prove that t_n is p-Kähler, for $3 \leqslant p \leqslant n/2$. Indeed consider the map

$$\pi\colon (\mathbb{C}^n, *) \to (\mathbb{C}^{n-p}, +)$$

$(z_1, \ldots, z_n) \to (z_{p-1}, \ldots, z_{n-2})$ where $*$ is the product which makes G isomorphic to $(\mathbb{C}^n, *)$ as a Lie group: explicitely, $(y_1, \ldots, y_n) * (z_1, \ldots, z_n) = (y_1 + z_1, \ldots, y_{n-2} + z_{n-2}, y_{n-1} + z_{n-1} + y_1 z_2, y_n + z_n + y_1 z_3)$. π is a Lie groups homomorphism, so it induces a holomorphic submersion $\pi'\colon t_n := \mathbb{C}^n/\Gamma \to \mathbb{C}^{n-p}/\Gamma' = T_{n-p}$ where Γ is a discrete uniform subgroup of G. Since $p \leqslant n/2$, T_{n-p} is a p-Kähler manifold; we can exhibite a (p, p)-closed form whose integral over a fibre gives the volume of the fibre: this form is

$$\omega = \varphi_1 \wedge \ldots \wedge \varphi_{p-2} \wedge \varphi_{n-1} \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \ldots \wedge \bar{\varphi}_{p-2} \wedge \bar{\varphi}_{n-1} \wedge \bar{\varphi}_n .$$

The conclusion follows from Theorem 2.1.

Now we prove that t_n is p-Kähler for $n/2 < p \leqslant n-2$. First consider p, $n/2 < p \leqslant n-5$, and let

$$\sigma\colon (\mathbb{C}^n, *) \to (\mathbb{C}^{n-p}, *_1)$$

$(z_1, \ldots, z_n) \to (z_1, \ldots, z_{n-p-2}, z_{n-1}, z_n)$ where $*$ is as above and

$$(y_1, \ldots, y_{n-p}) *_1 (z_1, \ldots, z_{n-p}) = (y_1 + z_1, \ldots, y_{n-p-2} + z_{n-p-2}, y_{n-p-1} + z_{n-p-1} + y_1 z_2, y_{n-p} + z_{n-p} + y_1 z_3).$$

σ induces $\sigma'\colon t_n \to t_{n-p}$, whose fibre is a T_p. The closed (p, p)-form ω is now $\omega = \varphi_{n-p-1} \wedge \ldots \wedge \varphi_{n-2} \wedge \bar{\varphi}_{n-p-1} \wedge \ldots \wedge \bar{\varphi}_{n-2}$ and we conclude from Theorem 2.5.

For $n-4 \leqslant p \leqslant n-3$, $p > n/2$, we must consider

$$\nu\colon (\mathbb{C}^n, *) \to (\mathbb{C}^{n-p}, +)$$

$(z_1, \ldots, z_n) \to (z_4, \ldots, z_{n-p+3})$ and then $\nu'\colon t_n \to T_{n-p}$

which is a holomorphic submersion with fibre t_p. t_n is $(2p-n)$-Kähler, because $n-4 \leqslant p \leqslant n-3$, and so t_p, which is a submanifold of t_n, is $(2p-n)$-Kähler. Now use again Theorem 2.5. considering

$$\omega = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_{n-p+4} \wedge \ldots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \bar{\varphi}_2 \wedge \bar{\varphi}_3 \wedge \bar{\varphi}_{n-p+4} \wedge \ldots \wedge \bar{\varphi}_n .$$

For $p = n-2$, we can now repete the argument. So we get

3.9. Proposition

t_n is p-Kähler for $p = 3, \ldots, n$ and is not 1-Kähler and 2-Kähler. □

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