COMPOSITIO MATHEMATICA

GIDEON SCHECHTMAN

More on embedding subspaces of L_p in l_r^n

Compositio Mathematica, tome 61, nº 2 (1987), p. 159-169

<http://www.numdam.org/item?id=CM_1987__61_2_159_0>

Compositio Mathematica 61: 159–169 (1987)

More on embedding subspaces of L_p in l_r^n

GIDEON SCHECHTMAN
Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

Received 11 May 1986; accepted 1 April 1986

Given two normed spaces X, Y and a real number $1 \leqslant K < \infty$, we say that X K-embeds into Y (denoted $X \overset{K}{\hookrightarrow} Y$) if there is a one to one linear operator,

$$T: X \to T(X) \subseteq Y \quad \text{with } \|T\|\,\|T^{-1}\| \leqslant K.$$

We are concerned here mostly with the situation where Y is one of the sequence spaces $l_r^n \left(= \left\{ x \in \mathbb{R}^n;\ \|x\|_r = \left(\sum_{i=1}^n |x_i|^r \right)^{1/r} < \infty \right\} \right)$ and X is a general m-dimensional subspace of one of the function spaces $L_p(0, 1)(= \{f;\ \|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p} < \infty\})$.

The expressions $\|x\|_r$, $\|f\|_p$ are norms only for r, $p \geqslant 1$. We shall need, however, to use these expressions also for r or p smaller than 1. We shall continue to refer to them as norms also in this situation. The notion 'X K-embeds into Y' has meaning, with the same definition, also in this case $\left(\text{e.g. } \|T\| = \sup\left\{ \frac{\|Tx\|_r}{\|x\|_p};\ x \in X,\ x \neq 0 \right\} \right)$.

We continue here the investigation of the following question: fixing K, p, r and m, how small can we take n to be? Following is a sample of some of the results of this paper:

i) For $0 < r \leqslant p < 2$, $X \overset{1+\epsilon}{\hookrightarrow} l_r^n$, where $n \leqslant C(p, r, \epsilon) m^{1+r/p}$.

ii) For $2 < r = p < \infty$, $X \overset{1+\epsilon}{\hookrightarrow} l_p^n$, where $n \leqslant C(p, \epsilon) m^{1+r/2}$.

Changing the small constant, $1 + \epsilon$, in (i) with a large one, we get a much better estimate on the relation between n and m for $r < p$.

iii) For $0 < r < p < 2$, there exists a $K = K(p, r)$ such that $X \overset{K}{\hookrightarrow} l_r^n$ for $n \leqslant C\, m(\log m)^4$, C absolute.

The proofs here are much simpler than in the related papers [Johnson and Schechtman, 1982; Pisier, 1983; Schechtman, 1984/85, 1985].

For $1 = r < p$, i) is an important special case of Theorem 1 of [Schechtman, 1985] (except for a missing log factor. What is special here is that the range space is l_1^n rather than more general spaces).

The case $r=1=p$ in i) is an improvement of Theorem 2 in [Schechtman, 1985].

We refer the reader to [Milman and Schechtman, 1986] for the background to the subjects discussed here.

The main results are contained in Theorems 5 and 6. Proposition 4 is the main tool in proving these theorems. We begin with three lemmas, versions of which were used also in [Schechtman, 1985]. The first two are versions of Lemma 1 in [Schechtman, 1985]. Note again that $\|x\|_r$ denotes the homogeneous 'norm' in $L_r(0,1)$ or l_r^n also for $0<r<1$ $\left(\left(\int_0^1 |x|^r\right)^{1/r} \text{ or } \left(\sum_{i=1}^n |x_i|^r\right)^{1/r}\right)$. The Banach-Mazur distance, $d(X, Y)$, between a subspace X of L_r and a subspace Y of L_s is defined, as for normed spaces,

$$d(X,Y)=inf\left\{ab;\ a^{-1}\|x\|_r\leqslant \|Tx\|_s\leqslant b\|x\|_r,\ T: X\overset{1-1}{\underset{\text{onto}}{\rightarrow}} Y,\quad T \text{ linear}\right\}.$$

Lemma 1. *Let $0<r\leqslant 2$ and let Z be an m-dimensional subspace of $L_r(0,1)$, then*

a) there exist a probability measure μ on $[0,1]$ and a subspace W of $L_r(\mu)$ isometric to Z and satisfying

$$\sup\{\|w\|_\infty;\ \|w\|_r\leqslant 1,\ w\in W\}\leqslant em^{1/2}\,\mathrm{d}(Z, l_2^m)$$

b) $\mathrm{d}(Z, l_2^m)\leqslant \mathrm{e}^{(2/r)-1}m^{(1/r)-(1/2)}$.

Consequently,

$$\sup\{\|w\|_\infty;\ \|w\|_r\leqslant 1,\ w\in W\}\leqslant \mathrm{e}^{2/r}m^{1/r}.$$

Proof. As in [Schechtman, 1985], let $x_1,\dots,x_m$ be a basis for Z satisfying

$$a^{-1}\left(\sum_{i=1}^m a_i^2\right)^{1/2}\leqslant\left\|\sum_{i=1}^m a_i x_i\right\|_r\leqslant b\left(\sum_{i=1}^m a_i^2\right)^{1/2}$$

with $ab=d(Z, l_2^m)$.

Define

$$\mathrm{d}\mu=\left[\left(\sum_{i=1}^m x_i^2\right)^{r/2}\Big/\int_0^1\left(\sum_{i=1}^m x_i^2\right)^{r/2}\right]\mathrm{d}t$$

and $T: Z \to L_r(\mu)$ by

$$Tx = \frac{x}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \left(\int \left(\sum_{i=1}^m x_i^2\right)^{r/2}\right)^{1/r}.$$

T is clearly an isometry. Let $W = TZ$. For $w = T(\sum a_i x_i)$ of norm one, we have

$$|w| = \frac{\left|\sum_{i=1}^m a_i x_i\right|}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \left(\int \left(\sum_{i=1}^m x_i^2\right)^{r/2}\right)^{1/r}. \tag{1}$$

Now,

$$\frac{\left|\sum_{i=1}^m a_i x_i\right|}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \leqslant \left(\sum_{i=1}^m a_i^2\right)^{1/2} \leqslant a \left\|\sum_{i=1}^m a_i x_i\right\|_r = a, \tag{2}$$

and, with g_i being independent standard gaussian variables,

$$\begin{aligned} \left(\int \left(\sum_{i=1}^m x_i^2\right)^{r/2}\right)^{1/r} &= \left(E \int \left|\sum_{i=1}^m x_i g_i\right|^r\right)^{1/r} \Big/ (E\,|g_1|^r)^{1/r} \\ &\leqslant b \left(E\left(\sum_{i=1}^m g_i^2\right)^{r/2}\right)^{1/r} \Big/ (E\,|g_1|^r)^{1/r} \\ &\leqslant b\sqrt{m} \,/ (E\,|g_1|^r)^{1/r}. \end{aligned} \tag{3}$$

To evaluate $(E\,|g_1|^r)^{1/r}$ from below, use integration by parts to get

$$E\,|g_1|^r = \frac{1}{r+1} E\,|g_1|^{r+2} \geqslant \frac{1}{r+1}.$$

Thus,

$$(E\,|g_1|^r)^{1/r} \geqslant \left(\frac{1}{r+1}\right)^{1/r} \geqslant \frac{1}{e}.$$

Combining this with (1), (2) and (3), we get a).

To prove b), notice that, for $w \in W$,

$$\|w\|_r \leqslant \|w\|_2 \leqslant \|w\|_\infty^{1-(r/2)} \|w\|_r^{r/2} \leqslant \left(e\, m^{1/2}\, d(Z, l_2^m)\right)^{1-(r/2)} \|w\|_r.$$

Consequently,

$$d(Z, l_2^m) = d(W, l_2^m) \leqslant \left(e\, m^{1/2}\, d(Z, l_2^m)\right)^{1-(r/2)}.$$

Rearranging we get b). □

LEMMA 2. *Let $2 < r < \infty$ and let Z be a m-dimensional subspace of $L_r(0, 1)$. Then there exist a probability measure μ on $[0, 1]$ and a subspace W of $L_r(\mu)$ isometric to Z such that*

$$\{ \|w\|_\infty ;\ \|w\|_r = 1,\ w \in W \} \leqslant m^{1/2}.$$

Proof. By Theorem 1 in [Lewis, 1978] there is a probability measure μ $(= f^r dt$ in the notation of [Lewis, 1978]) and a basis $(x_i)_{i=1}^m (x_i = f_i/f)$ of a space $W \subseteq L_r(\mu)$, isometric to Z $(W = f^{-1}Z)$, such that

a) $$\left\| \sum_{i=1}^m a_i x_i \right\|_2 = \frac{\left(\sum_{i=1}^m a_i^2 \right)^{1/2}}{\sqrt{m}} \quad \text{for all } a_1, \ldots, a_m \in \mathbb{R}$$

b) $$\left(\sum_{i=1}^m x_i^2 \right)^{1/2} \equiv 1.$$

Now, for all $a_1, \ldots, a_m \in \mathbb{R}$,

$$\left\| \sum_{i=1}^m a_i x_i \right\|_\infty \leqslant \left(\sum_{i=1}^m a_i^2 \right)^{1/2} \left\| \left(\sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_\infty$$

$$= \sqrt{m} \left\| \sum_{i=1}^m a_i x_i \right\|_2 \leqslant \sqrt{m} \left\| \sum_{i=1}^m a_i x_i \right\|_r . \qquad \square$$

The next lemma is a standard large deviation inequality for sums of indendent random variables. We give a proof for completeness.

LEMMA 3. *Let $(d_i)_{i=1}^n$ be independent random variables with*

$$E|d_i| \leqslant A, \quad Ed_i = 0, \quad \|d_i\|_\infty \leqslant B, \quad i = 1, \ldots, n,$$

then

$$P\left(\left|\sum_{i=1}^{n} d_i\right| > C\right) \leqslant 2\exp\left(\frac{-C^2}{4eABn}\right)$$

for all $C \leqslant 2eAn$.

Proof. First notice that for all $p \geqslant 2$

$$E|d_i|^p \leqslant E|d_i| \, \|d_i\|_\infty^{p-1} \leqslant AB^{p-1}, \quad i = 1, \ldots, n.$$

For all $\lambda \geqslant 0$ and all $i = 1, \ldots, n$

$$E\,e^{\lambda d_i} \leqslant 1 + \sum_{k=2}^{\infty} \frac{\lambda^k E|d_i|^k}{k!} \leqslant 1 + \sum_{k=2}^{\infty} \frac{\lambda^k AB^{k-1}}{k!} \leqslant 1 + \lambda^2 AB \sum_{k=0}^{\infty} \frac{\lambda^k B^k}{k!}$$

$$\leqslant \exp(\lambda^2 AB\, e^{\lambda B}).$$

Independence implies

$$E\,e^{\lambda \Sigma_{i=1}^n d_i} \leqslant \exp(\lambda^2 ABn\, e^{\lambda B}).$$

Consequently, for $0 < \lambda \leqslant B^{-1}$,

$$P\left(\sum_{i=1}^{n} d_i > C\right) \leqslant e^{\lambda \Sigma_{i=1}^n d_i - \lambda C} \leqslant \exp(\lambda^2 ABne - \lambda C).$$

Choosing $\lambda = \frac{1}{2}\frac{C}{ABne}\left(\leqslant \frac{1}{B} \textit{ for } C \leqslant 2eAn\right)$, we get

$$P\left(\sum_{i=1}^{n} d_i > C\right) \leqslant \exp\left(-\frac{C^2}{4eABn}\right).$$

The same inequality holds for $-\sum_{i=1}^{n} d_i$ and we get the desired result. □

PROPOSITION 4. *Let X be an m-dimensional subspace of $L_r(\Omega, \mathscr{F}, \mu)$ for some probability space $(\Omega, \mathscr{F}, \mu)$ sand some $0 < r < \infty$. Assume*

$$M = \sup\{\|x\|_\infty;\ x \in X,\ \|x\|_r = 1\} < \infty.$$

Then, for all $\epsilon > 0$, $X \overset{1+\epsilon}{\hookrightarrow} l_r^n$ for some

$$n \leqslant C(\epsilon, r) m M^r.$$

Moreover, for some absolute constant C,

$$C(\epsilon, r) \leqslant \frac{C \log \frac{1}{r\epsilon}}{r^3 \epsilon^2} \quad \text{for } 0 < r < 1$$

and

$$C(\epsilon, r) \leqslant \frac{C \log \frac{1}{\epsilon}}{\epsilon^2} \quad \text{for } r > 1.$$

Proof. For $t \in [0, 1]^n$ and $x \in X$ define $x_i(t) = x(t_i)$. Then x_i are independent random variables and for all t the map

$$x \to (x_1(t), \ldots, x_n(t))$$

is linear. Define, for $t \in [0, 1]^n$, an operator

$$T_t: X \to l_r^n$$

by

$$T_t x = \frac{1}{n^{1/r}} \sum_{i=1}^{n} x_i(t)\, \mathrm{e}_i$$

($(\mathrm{e}_i)_{i=1}^n$ is the canonical basis of l_r^n). Then

$$E \| Tx \|_r^r = \| x \|_r^r$$

(E denotes expectation with respect to P – the product measure on $[0, 1]^n$).
For $x \in X$ with $\| x \|_r = 1$,

$$\| T_t x \|_r^r - 1 = \frac{1}{n} \sum_{i=1}^{n} (\, | x_i(t) |^r - 1).$$

Each of the summands $y_i = | x_i(t) |^r - 1$ is bounded by M^r and satisfies $E | y_i | \leqslant 2$. Plugging these estimates in Lemma 3, with $\mathrm{d}_i = \frac{y_i}{n}$, $A = \frac{2}{n}$ and $B = \frac{M^r}{n}$, we get, for $0 < \eta < \frac{1}{3}$ and an absolute constant $\delta > 0$,

$$P(\, | \, \| T_t x \|_r^r - 1 | > \eta) \leqslant 2\, \mathrm{e}^{-\delta \eta^2 n / M^r}. \tag{4}$$

We now distinguish between the two cases $0 < r < 1$ and $1 \leqslant r < \infty$. If

$0 < r < 1$ choose an η-net, N, in the sphere of X in the metric $d(x, y) = \|x - y\|_r^r$. One can do that with

$$|N| \leqslant \left(1 + \frac{2}{\eta}\right)^{m/r} \leqslant e^{(m/r)\log(2/\eta)}$$

(the proof is standard, see e.g. [Johnson and Schechtman, 1982] Lemma 2). Using (4) we get in this case that if

$$m \leqslant c(\eta) rn/M^r \tag{5}$$

$(c(\eta) \approx \eta^2/\log\frac{1}{\eta})$, then, for some t

$$1 - \eta \leqslant \|T_t x\|_r^r \leqslant 1 + \eta$$

for all $x \in N$. Using a standard successive approximation argument (see e.g. [Johnson and Schechtman, 1982] Lemma 3) we get that

$$\frac{1 - 3\eta}{1 - \eta} \leqslant \|T_t x\|_r^r \leqslant \frac{(1 + \eta)^2}{(1 - \eta)} \tag{6}$$

for all $x \in X$, $\|x\|_r = 1$. This concludes the proof in the case $0 < r < 1$ except for the evaluation of the constant $C(\epsilon, r)$. Given $0 < r < 1$ and $0 < \epsilon < 1$ choose a δ such that $(1 + \delta)^{1/r} = 1 + \epsilon$ ($\delta \approx \epsilon r$ with absolute constants) then choose an $0 < \eta < \frac{1}{3}$ such that $\frac{(1+\eta)^2}{(1 - 3\eta)} = 1 + \delta$ ($\eta \approx \delta$ with absolute constants). Then, in the construction above, $\|T_t\| \, \|T_t^{-1}\| \leqslant (1 + \delta)^{1/r} = 1 + \epsilon$ and by (5) we may choose

$$n \approx C(\epsilon, r) m M^r$$

where

$$C(\epsilon, r) = \frac{1}{c(\eta) r} \approx \frac{\log\frac{1}{\eta}}{\eta^2 r} \approx \frac{\log\frac{1}{\epsilon r}}{\epsilon^2 r^3}.$$

The proof for $r > 1$ is very similar. Here we work with an η-net N in the metric given by the norm. Its size is

$$|N| \leqslant e^{m \log(2/\eta)}$$

(see e.g. [Figiel, Lindenstrauss and Milman, 1977]) and we get that if

$$m \leqslant c(\eta) n/M^r$$

$(c(\eta) \approx \eta^2/\log\frac{1}{\eta})$ then there exists a t such that

$$\|T_t\| \, \|T_t^{-1}\| \leqslant \left[\frac{(1+\eta)^2}{(1-3\eta)}\right]^{1/r} \leqslant \frac{(1+\eta)^2}{(1-3\eta)}.$$

Taking η of order ϵ we get the desired result. □

THEOREM 5.

a) *For* $0 < r \leqslant p < 2$ *any m-dimensional subspace X of* $L_p(0,1)$ $(1+\epsilon)$*-embeds into* l_r^n *for some*

$$n \leqslant K(\epsilon, r) m^{1+(r/p)}.$$

b) *For* $0 < r \leqslant 1$ *any m-dimensional normed subspace X of* $L_r(0,1)$ $(1+\epsilon)$*-embeds into* l_r^n *for some*

$$n \leqslant K(\epsilon, r) m^{1+r}.$$

c) *For* $2 < r < \infty$ *any m-dimensional subspace X of* $L_r(0,1)$ $(1+\epsilon)$*-embeds into* l_r^n *for some*

$$n \leqslant K(\epsilon) m^{1+(r/2)}.$$

The constants $K(\epsilon, r)$ *in a) and b) are dominated by* $10C(\epsilon, r)$ *of Proposition* 4. *The constant in c) depends only on* ϵ.

Proof. Since $L_p(0,1) \overset{1}{\hookrightarrow} L_r(0,1)$, $0 < r < p < 2$, we may assume in all three cases that $X \subseteq L_r(0,1)$. By Lemmas 1 and 2, we may assume in addition that, putting

$$M = \sup\{\|x\|_\infty;\ x \in X,\ \|x\|_r = 1\},$$

$$M \leqslant e^{2/p} m^{1/p} \quad \text{in } a)$$
$$M \leqslant e\, m \quad \text{in } b)$$
$$M \leqslant m^{1/2} \quad \text{in } c).$$

Now apply Proposition 4. □

Remarks

i) In a) and b) nothing is known about lower bounds (i.e. the case may be that n can be chosen proportional to m). In c) there is a lower bound: $n \geqslant k(\epsilon, r) m^{r/2}$ (see [Bennett *et al.*, 1977] or [Milman and Schechtman, 1986]).

ii) Following the proofs of Proposition 4 and Theorem 5, one can easily prove:
Let F be a finite set in L_r, $1 \leqslant r < 2$, $|F| = m$. Then for each $\epsilon > 0$ and $n \geqslant c(\epsilon) m \log m$ there exists a function $f: F \to f(F) \subset l_r^n$ with

$$\|f\|_{Lip} \|f^{-1}\|_{Lip} \leqslant 1 + \epsilon$$

$$\left(\|f\|_{Lip} = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}, \ c(\epsilon) \text{ depends only on } \epsilon. \right)$$

This should be compared with a result of [Ball, 1984]: *For $\epsilon = 0$, n must be of order at least m^2 (and $n \approx m^2$ is always enough).*

We suspect that the right order of n (for $(1 + \epsilon)$-Lipschitz embeddings) is some power of $\log m$.

THEOREM 6

a) Given $0 < q < p < 2$ there exists a $K = K(q, p)$ such that any m-dimensional subspace X of $L_p(0, 1)$ K-embeds into l_q^n for some $n \leqslant C\, m(\log m)^3 \log(\log m)$, C absolute.

b) For any $0 < r < q < 1$, any m-dimensional normed subspace X of $L_r(0, 1)$ $K(q)$-embeds into l_q^n for some $n \leqslant C\, m(\log m)^3 \log(\log m)$, C absolute and $K(q)$ depends only on q (and not on r).

Proof. In both cases X can be considered as a subspace of $L_s(0, 1)$ for any $0 < s \leqslant r$. Thus, by Theorem 5, X 2-embeds into l_s^n for

$$n \leqslant \frac{C \log \frac{1}{s}}{s^3} m^{1 + (s/p)} \text{ in } a)$$

and

$$n \leqslant \frac{C \log \frac{1}{s}}{s^3} m^{1 + s} \text{ in } b).$$

The choice $s = \dfrac{p}{\log m}$ in a) and $s = \dfrac{1}{\log m}$ in b) gives that X 2-embeds into l_s^n for

$$n \leqslant C(p)(\log m)^3 (\log(\log m)) m$$

for some constant $C(p)$, depending only on p, in a), and for

$$n \leqslant C(\log m)^3 (\log(\log m)) m$$

for some absolute constant C in b).

Now apply one of Maurey's factorization theorems, Theorem 2 of [Maurey, 1974], to get an embedding of X into l_q^n via a change of measure. One should notice that s does not affect the constants. □

There are several problems which suggest themselves naturally. We shall mention explicitly only one with a possible way of attack (toward a negative solution).

PROBLEM 7. *Is there a function* $C(\epsilon)$, $\epsilon > 0$ *such that any* m*-dimensional subspace* X *of* $L_1(0, 1)$ $(1+\epsilon)$*-embeds into* l_1^n *for some* $n \leqslant C(\epsilon)m$?

Denote by $R(Y)$ the K-convexity constant of Y [Maurey and Pisier, 1976], that is, the norm of the projection $R \otimes I$ in $L_2(Y)$, where R is the orthogonal projection onto the span of the Rademacher functions. As is well known $R(X) \leqslant C\sqrt{\log m}$ for any m-dimensional subspace X of $L_1(0, 1)$. Inspecting the proof of this fact one easily gets an estimate on C

$$R(X) \leqslant \left(\sqrt{2} + \mathrm{o}(1)\right)\sqrt{\log m}, \quad \dim X = m \to \infty. \tag{7}$$

In particular,

$$R(l_1^m) \leqslant \left(\sqrt{2} + \mathrm{o}(1)\right)\sqrt{\log m}, \quad m \to \infty. \tag{8}$$

We are mainly interested in whether the numerical constants in (7) and (8) are the same or different. Indeed if for some $\alpha > 1$ and $\epsilon > 0$

$$\limsup_{m \to \infty} \left[\sup_{\dim X = m} R(x)/R\left(l_1^{m^\alpha}\right) \right] \geqslant 1 + \epsilon$$

then for some m there exists an m-dimensional subspace X of L_1 which does not $1+\epsilon$ embed into $l_1^{m^\alpha}$.

PROBLEM 8. *What are the best numerical constants in* (7) *and* (8)?

Acknowledgement

Supported in part by the U.S.-Israel Binational Science Foundation.

References

Ball, K.: Isometric embedding in l_p spaces. Preprint (1984).

Bennett, G., Dor, L.E., Goodman, V., Johnson, W.B. and Newman, C.M.: On uncomplemented subspaces of L_p, $1 < p < 2$. *Israel J. Math.* 26 (1977) 178–187.

Figiel, T., Lindenstrauss, J. and Milman, V.D.: The dimension of almost spherical sections of convex bodies. *Acta Math.* 139 (1977) 53–94.

Johnson, W.B. and Schechtman, G.: Embedding l_p^m into l_1^n. *Acta Math.* 149 (1982) 71–85.

Lewis, D.R.: Finite dimensional subspaces of L_p. *Studia Math.* 63 (1978) 207–212.

Maurey, B.: *Théorèmes de Factorisation pour les Opérateurs à Valeurs dans un Espace L^p. Astérisque Soc. Math. France* #11 (1974).

Maurey, B. and Pisier, G.: Séries de variables aléatoires vectorielles indépendentes et propriétés géométriques des espaces de Banach. *Studia Math.* 58 (1976) 45–90.

Milman, V.D. and Schechtman, G.: *Asymptotic Theory of Finite Dimensional Normed Spaces. Lecture Notes in Math.* 1200 Springer, (1986).

Pisier, G.: On the dimension of the l_p^n-subspaces of Banach spaces, for $1 \leqslant p < 2$. *Trans. A.M.S.* 276 (1983) 201–211.

Schechtman, G.: Fone embeddings of finite dimensional subspaces of L_p, $1 \leqslant p < 2$, into finite dimensional normed spaces II. *Longhorn Notes.* Univ. of Texas (1984/85).

Schechtman, G.: Fine embeddings of finite dimensional subspaces of L_p, $1 \leqslant p < 2$, into l_1^m. *Proc. A.M.S.* 94 (1985) 617–623.

Added in proof: J. Bourgain, J. Lindenstrauss and V.D. Milman (private communication) improved recently the results of this paper. They proved that for an m-dimensional subspace, X, of L_p,

$$X \overset{1+\epsilon}{\hookrightarrow} l_1^n \quad \text{for} \quad n \leqslant C(p, \epsilon)m \qquad \text{if } 1 < p \leqslant 2$$

$$X \overset{1+\epsilon}{\hookrightarrow} l_1^n \quad \text{for} \quad n \leqslant C(\epsilon)m(\log m)^5 \quad \text{if } p = 1$$

$$X \overset{1+\epsilon}{\hookrightarrow} l_p^m \quad \text{for} \quad n \leqslant C(p, \epsilon)m^{p/2+\epsilon} \quad \text{if } 2 < p < \infty.$$

Their proof is based on the results and method developed here.