COMPOSITIO MATHEMATICA

B. BRINDZA

Zeros of polynomials and exponential diophantine equations

Compositio Mathematica, tome 61, n° 2 (1987), p. 137-157

<http://www.numdam.org/item?id=CM_1987__61_2_137_0>

Compositio Mathematica 61: 137–157 (1987)

Zeros of polynomials and exponential diophantine equations

B. BRINDZA
Mathematical Institute, Kossuth Lajos University 4010 Debrecen Pf.12, Hungary

Received 17 February 1986; accepted 27 March 1986

§1. Introduction

Several diophantine problems can be reduced to the equation

$$f(x) = y^m \tag{1}$$

where f is a given polynomial with rational (or algebraic) integer coefficients. For the study of this equation the distribution of the multiplicities of the zeros of the polynomial f is important. From some inequalities of Mason and Brownawell-Masser concerning $\mathcal{S}$-unit equations over function fields we derive some new bounds for the numbers of distinct and simple zeros of polynomials, respectively, in §2. In §3 we combine these results with known results on equation (1), proved by the Gelfond-Baker method. Doing so we obtain some new results on equations of type (1) where m, x and y are variables, and on some related equations.

§2. Lower bounds for the number of distinct zeros

Let k be an algebraically closed field of characteristic zero and $k(z)$ the rational function field over k. Further, let $\mathbf{K}$ be a finite extension of $k(z)$ of genus g and the height of a non-zero element α of $\mathbf{K}$ defined by

$$H_{\mathbf{K}}(\alpha) = -\sum_{v} \min\{0, v(\alpha)\}$$

where v runs through the (additive) valuations of $\mathbf{K}/k$ with value group $\mathbf{Z}$. The following lemma is the powerful inequality of [Mason, 1984].

LEMMA 1 *Let $\mathcal{S}$ denote a finite set of valuations on $\mathbf{K}$, and suppose that γ_1, γ_2 and γ_3 are non-zero elements of $\mathbf{K}$ such that $v(\gamma_1) = v(\gamma_2) = v(\gamma_3) = 0$ for all*

This paper has been written during a stay at the University of Leiden which was made possible by a scholarship of the Hungarian Academy of Sciences.

$v \notin \mathcal{S}$ and $\gamma_1 + \gamma_2 + \gamma_3 = 0$. Then either γ_1/γ_2 is element of k and so $H_{\mathbf{K}}(\gamma_1/\gamma_2) = 0$, or

$$H_{\mathbf{K}}(\gamma_1/\gamma_2) \leqslant |\mathcal{S}| + 2g - 2$$

where $|\mathcal{S}|$ denotes the number of elements of $\mathcal{S}$.

The following result is a special case of Lemma 1.

COROLLARY A (R.C. Mason) *Let P_1, P_2 and P_3 be coprime polynomials in $k[z]$ with $P_1 + P_2 + P_3 = 0$, and such that their product $P_1P_2P_3$ has N distinct zeros in k. Then either P_1, P_2 and P_3 are all constants, or*

$$\max_{1 \leqslant i \leqslant 3} \{\deg P_i\} < N.$$

For elements $u_1, \ldots, u_n$ of $\mathbf{K}$, not all zero, we define the (projective) height as

$$H_{\mathbf{K}}^P(u_1, \ldots, u_n) = -\sum_v \min\{v(u_1), \ldots, v(u_n)\},$$

(cf. [Brownawell and Masser, to appear]).

It is clear that

$$\max_{i \neq j} H_{\mathbf{K}}(u_i/u_j) \leqslant H_{\mathbf{K}}^P(u_1, \ldots, u_n), \quad (u_j \neq 0).$$

We adopt the terminology of Brownawell and Masser, and say that elements $u_1, \ldots, u_n$ of $\mathbf{K}$ satisfying

$$u_1 + \cdots + u_n = 0 \tag{2}$$

give rise to a non-degenerate solution if there is no non-empty proper subset of $\{u_1, \ldots, u_n\}$ whose elements are linearly dependent over k. Let $\mathcal{S}$ be a finite set of additive valuations of $\mathbf{K}/k$. We recall that an element α of $\mathbf{K}$ is said to be $\mathcal{S}$-unit if $v(\alpha) = 0$ for all valuations v of $\mathbf{K}/k$ not contained in $\mathcal{S}$.

LEMMA 2 [Brownawell and Masser, to appear]. *Suppose $u_1, \ldots, u_n$ $(n > 2)$ give rise to a non-degenerate solution of* (2) *and that $u_1, \ldots, u_n$ are $\mathcal{S}$-units for some finite set $\mathcal{S}$. Then*

$$H_{\mathbf{K}}^P(u_1, \ldots, u_n) \leqslant \tfrac{1}{2}(n-1)(n-2)\{|\mathcal{S}| + 2g - 2\}.$$

LEMMA 3 [Brownawell and Masser, to appear]. *Suppose $u_1, \ldots, u_n$ $(n > 2)$ give*

rise to a non-degenerate solution of (2) *and that for* $1 \leqslant j \leqslant n$ *the element* u_j *is an* $\mathscr{S}_j$*-unit for some finite set* $\mathscr{S}_j$. *Then*

$$H_{\mathbf{K}}^{P}(u_1,\ldots,u_n) \leqslant (n-2)(|\mathscr{S}_1|+\cdots+|\mathscr{S}_n|) + \tfrac{1}{2}(n-1)(n-2)(2g-2).$$

By using Lemma 2 we shall prove the following result.

THEOREM 1 *Let*

$$F(X_1,\ldots,X_n) = \sum_{i=1}^{N} a_i X_1^{k_{1i}}\ldots X_n^{k_{ni}}, \quad (N>1)$$

be a polynomial in $X_1,\ldots,X_n$ *with non-zero complex coefficients and suppose that*

$$m(F) = \min_{1\leqslant P<r\leqslant N} \sum_{j=1}^{n} |k_{jp}-k_{jr}| \deg f_j \Big/ \sum_{j=1}^{n} \deg f_j > N(N-1)$$

where $f_1,\ldots,f_n \in \mathbf{C}[X]$ *are pairwise relatively prime non-constant polynomials. Then the polynomial* $P(X) = F(f_1(X),\ldots,f_n(X))$ *has at least* $\left(\dfrac{m(F)}{N(N-1)}-1\right)\sum_{j=1}^{n}\deg f_j$ *distinct zeros.*

Proof. At first we show that the elements $a_i f_1^{k_{1i}}\ldots f_n^{k_{ni}}$ $(1\leqslant i\leqslant n)$ are linearly independent over $\mathbf{C}$. Supposing the contrary we have

$$\sum_{i=1}^{N} \lambda_i a_i f_1^{k_{1i}}\ldots f_n^{k_{ni}} = 0$$

for some $\lambda_i \in \mathbf{C}$ $(1\leqslant i\leqslant N)$ with at least one λ_j different from zero. Moreover, we may assume that the cardinality of $\mathscr{J}=\{i \mid \lambda_i \neq 0\}$ is minimal. Since $m(F)>0$, there are $p<q<r$ such that $\lambda_p\lambda_q\lambda_r \neq 0$. From Lemma 2 we obtain

$$H_{\mathbf{C}(z)}\left(a_p a_r^{-1}\lambda_p\lambda_r^{-1} f_1^{k_{1p}-k_{1r}}\ldots f_n^{k_{np}-k_{nr}}\right) \leqslant \tfrac{1}{2}(N-1)(N-2)(|\mathscr{S}|-2) \quad (3)$$

where the valuation set $\mathscr{S}$ consists of the infinite valuations of $\mathbf{C}(z)/\mathbf{C}$ and all the finite valuation v such that $v(f_i)\neq 0$ for some $i\in\{1,\ldots,n\}$ whence

$$|\mathscr{S}| \leqslant \deg f_1 + \cdots + \deg f_n + 1.$$

From the definition of the height function we obtain

$$H_{\mathbf{C}(z)}\left(a_p a_r^{-1}\lambda_p\lambda_r^{-1}f_1^{k_{1p}-k_{1r}}\dots f_n^{k_{np}-k_{nr}}\right)$$

$$\geqslant \tfrac{1}{2}\sum_{j=1}^{n}|k_{jp}-k_{jr}|\deg f_j \geqslant \tfrac{1}{2}m(F)\sum_{j=1}^{n}\deg f_j.$$

Comparing these inequalities with (3), we infer

$$m(F)\sum_{j=1}^{n}\deg f_j \leqslant (N-1)(N-2)\left(\sum_{j=1}^{n}\deg f_j - 1\right)$$

which contradicts $m(F) > N(N-1)$.

Now, we can apply Lemma 2 to the equation

$$\sum_{i=1}^{N} a_i f_1^{k_{1i}}\dots f_n^{k_{ni}} - P = 0$$

and we have

$$\tfrac{1}{2}m(F)\sum_{j=1}^{n}\deg f_j \leqslant \tfrac{1}{2}N(N-1)\left(\sum_{j=1}^{n}\deg f_j + s\right) \tag{4}$$

where s denotes the number of distinct zeros of the polynomial $P(X)$. Finally, (4) implies the theorem.

In the special case when $F(X_1,\dots,X_n) = a_1X_1^{k_1} + \cdots + a_NX_N^{k_N}$ we shall prove the following stronger result.

THEOREM 2 *Let $f_1(X),\dots,f_N(X)$ be non-constant pairwise relatively prime polynomials with complex coefficients and $a_1,\dots,a_N$ non-zero complex numbers. Suppose that*

$$\mu = \min_{1\leqslant i\leqslant N} k_i > N(N-1).$$

Then the polynomial $Q(X) = a_1f_1^{k_1}(X) + \cdots + a_Nf_N^{k_N}(X)$ has at least $\dfrac{\mu}{N-1}$ distinct zeros.

Proof. Following the argument of the proof of Theorem 1 we can see that there is no proper subset $\mathscr{J}$ of $\{1,\dots,N\}$ such that $\sum_{i\in\mathscr{J}}\lambda_i a_i f_i^{k_i}$ is identically zero and $\prod_{i\in\mathscr{J}}\lambda_i \neq 0$. Let t denote the number of distinct zeros of the polynomial Q.

Then from Lemma 3 we have

$$k_i \deg f_i \leqslant (N-1)(\deg f_1 + \cdots + \deg f_N + t) - N(N-1), \quad i = 1, \ldots, N.$$

Taking the sum of these inequalities we obtain

$$\mu \sum_{i=1}^{N} \deg f_i \leqslant N(N-1)\left(\sum_{i=1}^{N} \deg f_i + t \right) - N^2(N-1)$$

which implies our theorem.

Now we shall give a lower bound for the number of simple zeros of the polynomial $P(X) = aF^n(X) + bG^m(X)$ where a, b are non-zero complex numbers; n, m are positive integers and F, $G \in \mathbf{C}[X]$ are non-constant relatively prime polynomials.

THEOREM 3 *Suppose that* $\frac{1}{n} + \frac{1}{m} \leqslant \frac{1}{2}$ *and* $n \deg F \geqslant m \deg G$. *Then the polynomial* $P(X)$ *has at least* $2 + n\left(1 - \frac{2}{n} - \frac{2}{m}\right) \deg F$ *simple zeros.*

Since $n \deg F$ is a trivial upper bound for the number of simple zeros of P, our theorem shows that 'almost' all zeros of P are simple and $\deg P \geqslant 2 + n\left(1 - \frac{2}{n} - \frac{2}{m}\right) \deg F$.

Proof. We assume that $P(X)$ has the representation

$$P(X) = A \prod_{i=1}^{k} (X - \alpha_i)^{r_i}$$

with $A \neq 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. By applying Corollary A

$$n \deg F \leqslant \deg F + \deg G + k - 1, \tag{5}$$

$$m \deg G \leqslant \deg F + \deg G + k - 1. \tag{6}$$

From (5) and (6) we deduce

$$\left(n - 1 - \frac{1}{m-1}\right) \deg F \leqslant \frac{m}{m-1}(k-1)$$

and

$$\left(m - 1 - \frac{1}{n-1}\right) \deg G \leqslant \frac{n}{n-1}(k-1).$$

Let l denote the number of simple zeros of $P(X)$. Then

$$2(k-l) + l \leqslant \deg P \leqslant \deg F + \deg G + k - 1, \tag{7}$$

hence

$$k-1 \leqslant \deg F + \deg G + l - 2 \leqslant \frac{n}{n-1} \frac{1}{m-1-\frac{1}{n-1}}(k-1)$$

$$+ \frac{m}{m-1} \frac{1}{n-1-\frac{1}{m-1}}(k-1) + l - 2.$$

This implies that

$$l \geqslant 2 + (k-1)\frac{mn-2m-2n}{mn-m-n}$$

$$\geqslant 2 + \frac{mn-2m-2n}{mn-m-n} \cdot \frac{(n-1)(m-1)-1}{m} \deg F$$

$$= 2 + n \deg F\left(1 - \frac{2}{n} - \frac{2}{m}\right).$$

which proves the theorem.

COROLLARY 1 *Suppose that* $4 \leqslant (m-2)(n-2)$. *Then the polynomial* $P(X)$ *has at least two simple zeros. If* $4 > (m-2)(n-2)$ *then* $P(X)$ *may have less than two simple zeros.*

Proof. In the case when $\min\{m, n\} > 3$ our corollary immediately follows from Theorem 3. So, we may assume that $n = 3$ and $m \geqslant 6$. By Corollary A we have

$$3 \deg F \leqslant \deg F + \deg G + k - 1,$$

$$6 \deg G \leqslant \deg F + \deg G + k - 1.$$

It is easy to see that $\deg F \leqslant \frac{2}{3}(k-1)$ and $\deg G \leqslant \frac{1}{3}(k-1)$. From (7) we obtain

$$l + 2(k-l) \leqslant \deg P \leqslant 2k - 2$$

which proves the first part of the corollary.

The following examples show the necessity of the condition $4 \leqslant (m-2)(n-2)$.

Case 1. $(m-2)(n-2) = 3$. We may assume $m = 5$, $n = 3$, Take

$$1728\left[X(X^{10} + 11X^5 - 1)\right]^5 + \left(X^{20} + 1 - 228(X^{15} - X^5) + 494X^{10}\right)^3$$

$$= \left(X^{30} + 1 - 522(X^{25} - X^5) - 10\,005(X^{20} + X^{10})\right)^2.$$

Case 2. $(m-2)(n-2)=2$. We may assume $m=4$, $n=3$. Take

$$108\left[X(X^4-1)\right]^4-(X^8+14X^4+1)^3=-(X^{12}-33X^8-33X^4+1)^2.$$

Case 3. $(m-2)(n-2)=1$. It follows that $m=n=3$. Take

$$(3X^4+6X^2-1)^3+(-3X^4+6X^2+1)^3=(18X^5+6X)^2.$$

Case 4. $(m-2)(n-2)=0$. We may assume $m=2$. Take

$$\left(\frac{1-X^n}{2}\right)^2+(X^2)^n=\left(\frac{1+X^n}{2}\right)^2.$$

(cf. F. Klein, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, Leipzig 1884.)

§3. Power values of polynomials

Let f be a polynomial with rational integer coefficients and consider the equation

$$f(x)=bz^m$$

where $b\neq 0$, $m>0$, x and z with $|z|>1$ are rational integers. [Tijdeman, 1976] proved m is bounded by a computable constant depending only on f and b. He assumed that f has at least two simple rational zeros. Schinzel and Tijdeman, 1976] showed that the supposition of two distinct roots is sufficient. Later, [Sprindžuk, 1982] and [Turk, to appear] gave explicit upper bounds for the exponent m. These results were extended by [Shorey and Tijdeman, in press, Th. 10.3.) to the more general case, when the ground ring is a ring of integers in an arbitrary algebraic number field and, moreover,

$$b\in\{\pi_1^{\alpha_1}\dots\pi_s^{\alpha_s}\,|\,0\leqslant\alpha_i\in\mathbf{Z},\ i=1,\dots,s\}$$

where $\pi_1,\dots,\pi_s$ $(s\geqslant 0)$ are given non-zero algebraic integers. György, Tijdeman and the author [Brindza *et al.*, 1985] proved the following result which made it possible to give effective bounds for the solutions of some diophantine equations connected with the Fermat equation

THEOREM A *Let* $f(X)\in\mathbf{Z}[X]$ *be a polynomial with at least two distinct zeros. Let* $\{P_1,\dots,P_s\}$ *be a finite set of primes. Put*

$$\mathscr{S}=\{P_1^{\alpha_1}\dots P_s^{\alpha_s}\,|\,0\leqslant\alpha_t\in\mathbf{Z},\ 1\leqslant i\leqslant s\}\quad\text{and}\quad P=\prod_{j=1}^{s}\log P_j.$$

(if $s = 0$ put $\mathcal{S} = \{1\}$ and $P = 1$.) If a, b, m, w, x and y are rational integers with $ab \neq 0$, $w \in \mathcal{S}$ and $|y| > 1$ such that

$$af(x) = bwz^m$$

then

$$m \leqslant \left[(C_1(s+1))^{s+1}P\right]^{C_2}(\log A)(\log\log A)^2$$

where $A = \max\{|a|, |b|, 4\}$ and C_1, C_2 are computable constants depending only on the degree and height of f.

Here, we extend Theorem A to the case of any algebraic number field. Let $\mathbf{L}$ be an algebraic number field with ring of integers $\mathbf{Z}_\mathbf{L}$ and let $f(X) \in \mathbf{Z}_\mathbf{L}[X]$. Further, let a, b, $u_1, \ldots, u_s$ $(s \geqslant 0)$ be given non-zero algebraic integers in $\mathbf{L}$ and consider the equation

$$af(x) = bu_1^{\alpha_1} \ldots u_s^{\alpha_s} z^m \tag{10}$$

where x, $z \in \mathbf{Z}_\mathbf{L}$ and $\alpha_1, \ldots, \alpha_s$, m are non-negative integers.

THEOREM 4 *Suppose $f(X)$ has at least two distinct roots and $0 \neq z$ is not a unit in $\mathbf{L}$. Then* (10) *implies that*

$$m < \left[(C_3(s+1))\log U\right]^{C_4(s+1)}(\log M)(\log\log M)^2$$

where $U = \max_i |N_{\mathbf{L}/\mathbf{Q}}(u_i)|$, $M = \max\{|N_{\mathbf{L}/\mathbf{Q}}(a)|, |N_{\mathbf{L}/\mathbf{Q}}(b)|, 4\}$ and C_3, C_4 are effectively computable constants depending only on f and $\mathbf{L}$. Further, if z is a unit but not a root of unity and $s = 0$ then

$$m < C_5 \log M_1$$

where $M_1 = \max\{H(a), H(b), 2\}$ and C_5 is a computable constant depending only on $\mathbf{L}$ and f. *

The following result is a simple consequence of Theorem 4.

THEOREM 5 *Let A, $B \in \mathbf{Z}_\mathbf{L}[X]$ and c, $d \in \mathbf{Z}_\mathbf{L}$. Then all solutions of the equation*

$$A(t)x^{2t} + B(t)x^t y + cy^2 = d \tag{11}$$

* We denote by $H(\alpha)$ the height of α. By the height of an algebraic number α we mean the height of the minimal defining polynomial of α with rational integer coefficients.

in x, $y \in \mathbf{Z}_{\mathbf{L}}$ *and rational integer* t *with* $t > 1$ *such that* $0 \neq cd(B^2(t) - 4cA(t))$ *and* $0 \neq x$ *is not a unit in* $\mathbf{Z}_{\mathbf{L}}$ *satisfy*

$$\max\{H(x), H(y), t\} < C_6$$

where C_6 *is an effectively computable constant depending only on* $\mathbf{L}$, A, B, c *and* d.

In the special case when $\mathbf{L} = \mathbf{Q}$ and $A(X)$, $B(X)$ are constant polynomials this yields a result of [Shorey and Stewart, 1983].

On the equation $F^n(x) + G^m(x) = y^z$

Let p be a fixed odd prime. [Inkeri, 1946] proved that there exist at most a finite number of relatively prime positive integers x, y, z which satisfy the conditions

$$x^p + y^p = z^p$$

and for which at least one of the differences $|x - y|$, $z - x$, $z - y$ is less than a given positive number $\mathcal{M}$. Later, [Everett, 1973] gave a new proof for this result by using Roth's famous theorem on approximation of algebraic numbers. [Stewart, 1977] and [Inkeri and van der Poorten, 1976] independently, proved that, for any positive number $\mathcal{M}$, all positive integer solutions x, y, z and $n > 2$ of the equation

$$x^n + y^n = z^n \quad \text{with } |x - y| < \mathcal{M}$$

satisfy $\max\{n, x, y, z\} < C_7$ where C_7 is an effectively computable constant depending only on $\mathcal{M}$. [Inkeri, 1976] studied the more general equation

$$h^p(x) + g^p(x) = z^p$$

where h and g are given non-constant polynomials with rational integer coefficients. He gave an upper bound for the solutions which depends only on p, h and g under some conditions. In [Brindza, 1984a] the author proved that these conditions are unnecessary. For some similar and more general results we refer to [Brindza *et al.*, 1985].

Let $\mathbf{K}$ be an algebraic number field and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ $(s \geqslant 0)$ be distinct prime ideals in $\mathbf{K}$. Further, let $\mathcal{S}$ denote the set of all valuations of $\mathbf{K}$ corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and $\mathcal{O}_{\mathbf{K},\mathcal{S}}$ the ring of $\mathcal{S}$-integers of $\mathbf{K}$. We recall that an element α of $\mathbf{K}$ is said to be $\mathcal{S}$-integral if $v(\alpha) \geqslant 0$ for all $v \notin \mathcal{S}$. Using Corollary 1 we shall prove the following result.

THEOREM 6 *Let* $F, G \in \mathbf{K}[X]$ *non-constant polynomials which are relatively prime and* m, n *given positive integers such that* $(m-2)(n-2) \geqslant 4$. *Then all solutions of the equation*

$$F^m(x) + G^n(x) = y^z \quad \text{in } x,\ y \in \mathcal{O}_{\mathbf{K},\mathcal{S}};\ z \in \mathbf{Z} \tag{12}$$

with $z > 2$ *and* $0 \neq y$ *is not a root of unity satisfy*

$$\max\{H(x), H(y), z\} < C_8$$

where C_8 *is an effectively computable number depending only on* m, n, F, G, $\mathbf{K}$, s *and* $\max_{\iota} N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_\iota)$.

In the special case when $\mathbf{K} = \mathbf{Q}$, $s = 0$, $m = n$ and z $(= n)$ is fixed we have the above mentioned result of the author.

On the equation

$$f_1^{k_1}(x) + \cdots + f_N^{k_N}(x) = u_1^{\alpha_1} \ldots u_s^{\alpha_s} y^z$$

Let $\mathbf{K}$ be an algebraic number field with ring of integers $\mathbf{Z}_\mathbf{K}$ and let $f_1(X), \ldots, f_N(X) \in \mathbf{Z}_\mathbf{K}[X]$, $(N > 1)$ be non-constant polynomials which are pairwise relatively prime. Further, let $u_1, \ldots, u_s$ $(s \geqslant 0)$ be given non-zero algebraic integers in $\mathbf{K}$. Write $U = \max_{\iota} |N_{\mathbf{K}/\mathbf{Q}}(u_\iota)|$ again. Combining Theorems 2 and 4 we immediately have the following result.

THEOREM 7 *Suppose* $\min_{\iota} k_\iota > N(N-1)$. *Then all solutions of the equation*

$$f_1^{k_1}(x) + \cdots + f_N^{k_N}(x) = u_1^{\alpha_1} \ldots u_s^{\alpha_s} y^z \tag{13}$$

in x, $y \in \mathbf{Z}_\mathbf{K}$ *and non-negative integers* $\alpha_1, \ldots, \alpha_s$, z *with* $0 \neq y$ *is not a unit (in* $\mathbf{Z}_\mathbf{K}$*) satisfy*

$$z < \left[(C_9(s+1)) \log U\right]^{C_{10}(s+1)}$$

where C_9 *and* C_{10} *are computable numbers depending only on the polynomial* $f_1^{k_1}(X) + \cdots + f_N^{k_N}(X)$ *and* $\mathbf{K}$.

Auxiliary results

Let $\mathbf{K}$ be an algebraic number field of degree d and let $R_\mathbf{K}$, $h_\mathbf{K}$ and $r_\mathbf{K}$ denote the regulator, class number and unit rank of $\mathbf{K}$, respectively. By $|\alpha|$ we shall denote the maximum of the absolute values of the conjugates of an algebraic

number α and we denote by $N(\alpha)$ the norm of α. It is known that if $\alpha \in \mathbf{Z}_{\mathbf{K}}$ with $0 \neq \alpha$ then

$$\lceil \alpha \rceil \leqslant 2H(\alpha) \quad \text{and} \quad H(\alpha) \leqslant (2\lceil \alpha \rceil)^d.$$

LEMMA 4 *Suppose* $r_{\mathbf{K}} \geqslant 1$. *There exist multiplicatively independent units* $\eta_1, \ldots, \eta_{r_{\mathbf{K}}}$ *in* $\mathbf{K}$ *such that*

$$\max_i |\log |\varphi(\eta_i)|| < C_{11} R_{\mathbf{K}}$$

for every $\mathbf{Q}$*-isomorphism* φ *of* $\mathbf{K}$ *where* C_{11} *is a computable constant depending only on* d; *further, there are* $\mathbf{Q}$*-isomorphisms* $\varphi_1, \ldots, \varphi_{r_{\mathbf{K}}}$ *such that*

$$|\mathrm{Det}(\log |\varphi_j(\eta_i)|)| > C_{12} > 0$$

where C_{12} *is an absolute constant.*

Proof. See [Stark, 1973] and [Zimmert, 1981].

LEMMA 5 *Let* α *be a non-zero element of* $\mathbf{K}$. *There exists a unit* ϵ *in the multiplicative group generated by* $\eta_1, \ldots, \eta_{r_{\mathbf{K}}}$ *such that*

$$\exp(-C_{13} R_{\mathbf{K}}) |N(\alpha)|^{1/d} \leqslant |\varphi(\epsilon\alpha)| \leqslant \exp(C_{13} R_{\mathbf{K}}) |N(\alpha)|^{1/d}$$

for every $\mathbf{Q}$*-isomorphism* φ *of* $\mathbf{K}$ *where* C_{13} *is a computable number depending only on* d.

Proof. See e.g. [György, 1980].

The following lemma is a consequence of Baker's estimate concerning linear forms.

LEMMA 6 *Let* $\alpha_1, \ldots, \alpha_N$ $(N \geqslant 2)$ *be non-zero elements of* $\mathbf{K}$ *and let* $A_1, \ldots, A_N$ *(each* $\geqslant 3$*) be upper bounds for the heights of* $\alpha_1, \ldots, \alpha_N$, *respectively. Put*

$$\Omega' = (\log A_1) \ldots (\log A_{N-1}) \quad \text{and} \quad \Omega = \Omega' \log A_N.$$

There exist computable positive absolute constants C_{14} *and* C_{15} *such that for every* $B \geqslant 2$ *the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_N^{b_N} - 1| < \exp\{-(C_{14} N d)^{C_{15} N} \Omega \log \Omega' \log B\}$$

have no solution in rational integers $b_1, \ldots, b_N$ *with absolute values at most* B.

LEMMA 7 *If x_1, x_2 and x_3 are non-zero algebraic integers in* $\mathbf{K}$ *satisfying*

$$x_1 + x_2 + x_3 = 0 \quad \text{and } x_1,\ x_2,\ x_3 \text{ are } \mathcal{S}\text{-units}$$

then for some $\sigma \in \mathbf{Z}_{\mathbf{K}}$ *and* $\rho_i \in \mathbf{Z}_{\mathbf{K}}$ *we have*

$$x_i = o\rho_i;\ i = 1, 2, 3 \quad \text{and } \max_i \lceil \rho_i \rceil < C_{16}$$

where C_{16} is a computable number depending on $\mathbf{K}$ *and* $\mathcal{S}$. *Further, if* $\max_i |N(x_i)| \leqslant M$ *for some positive M, then*

$$\max_i \lceil \rho_i \rceil < C_{17} M$$

where C_{17} is a computable number depending on $\mathbf{K}$ *and* $\mathcal{S}$.

LEMMA 7 is a special case of Lemma 6 of [György, 1979].

LEMMA 8 *Let α be a non-zero algebraic integer of degree n which is not a root of unity. There exists an effectively computable positive number C_{18} depending only on n such that*

$$\lceil \alpha \rceil > 1 + C_{18}$$

Proof. See e.g. [Schinzel and Zassenhaus, 1965].

The following lemma is an effective version of a well-known theorem of LeVeque. Let $f(X) \in \mathbf{K}[X]$ and assume that f has the representation

$$f(X) = a_N X^N + \cdots + a_0 = a_N \prod_{i=1}^{n} (X - \alpha_i)^{t_i}$$

with $a_N \neq 0$, $n > 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Further, let $m > 1$ be a natural number and put

$$t_i = m/(m, r_i), \quad (i = 1, \ldots, n).$$

LEMMA 9 [Brindza, 1984b] *Suppose that $\{t_1, \ldots, t_n\}$ is not a permutation of the n-tuples*

a) $\{t, 1, \ldots, 1\}$ *and* b) $\{2, 2, 1, \ldots, 1\}$.

Then all $\mathcal{S}$-integral solutions of the equation

$$f(x) = y^m$$

satisfy

$$\max\{H(x), H(y)\} < \exp\exp\left\{C_{19}P^2(s+1)^3\right\}$$

where $P = \max_i N\mathfrak{p}_i$ *(if* $s = 0$, *put* $P = 1$*) and* C_{20} *is a computable constant depending on* $\mathbf{K}$, f *and* m.

Proof. See [Brindza, 1984b] (cf. [Shorey and Tijdeman, in press]).

Let $\gamma_1, \ldots, \gamma_n, \pi_1, \ldots, \pi_s$ $(n > 1,\ s \geqslant 0)$ be algebraic integers in $\mathbf{K}$ with $\gamma_i \neq \gamma_j$ for $i \neq j$ and suppose that $0 \neq \pi_i$ is not a unit in $\mathbf{K}$ $(1 \leqslant i \leqslant s)$. Put

$$\mathscr{S}^* = \left\{\pi_1^{k_1} \ldots \pi_s^{k_s} \mid 0 \leqslant k_i \in \mathbf{Z},\ i = 1, \ldots, s\right\}.$$

Consider the equation

$$(x - \gamma_1 z)^{r_1} \ldots (x - \gamma_n z)^{r_n} = \epsilon\gamma y^m \tag{14}$$

where $z, \gamma \in \mathscr{S}^*$, $r_1, \ldots, r_n, m \in \mathbf{N}$, ϵ is a unit and $0 \neq y \in \mathbf{Z}_{\mathbf{K}}$ is not a unit. Let τ be a positive number.

LEMMA 10 ([Shorey and Tijdeman, in press] Th. 10.3) If

$$\min\left\{\operatorname{ord}_{\mathfrak{p}} x, \operatorname{ord}_{\mathfrak{p}} z\right\} \leqslant \tau$$

for all prime ideals $\mathfrak{p}$ *then all solutions* x, z, ϵ, γ, y, m *of equation* (14) *under the above mentioned conditions satisfy* $m < C_{21}$ *where* C_{21} *is an effectively computable constant depending only on* $\mathbf{K}$, $\mathscr{S}^*$, τ *and the binary form*

$$\prod_{i=1}^{n} (X - \gamma_i Z)^{r_i}.$$

Proof of Theorem 4

We shall follow the proof of Theorem A (see [Brindza, 1985]). We assume throughout that $f(X)$ has the representation

$$f(X) = a_k \prod_{i=1}^{n} (X - \gamma_i)^{r_i}$$

with $a_k \neq 0$ and $\gamma_i \neq \gamma_j$ for $i \neq j$. Let $\mathbf{K} = \mathbf{L}(\gamma_1, \ldots, \gamma_n)$ and $d = [\mathbf{K} : \mathbf{Q}]$. Then we have

$$a(x - \beta_1)^{r_1} \ldots (x - \beta_n)^{r_n} = a_k^{k-1} b u_1^{\alpha_1} \ldots u_s^{\alpha_s} z^m \tag{15}$$

where $\beta_i = a_k\gamma_i \in \mathbf{Z}_\mathbf{K}$ and the number x corresponds to $a_k x$ in (10). In this proof $c_1, c_2, \ldots$ will denote positive computable numbers which depend only on f and $\mathbf{L}$.

Let $m, \alpha_1, \ldots, \alpha_s, x, z$ be an arbitrary but fixed solution of (15). Using square brackets to indicate principal ideals in $\mathbf{K}$, we get

$$[a][x-\beta_1]^{r_1}\ldots[x-\beta_n]^{r_n} = [a_k]^{k-1}[b][u_1]^{\alpha_1}\ldots[u_s]^{\alpha_s}[z]^m. \tag{16}$$

At first we suppose that z is a unit but not a root of unity and $s = 0$. Then

$$|N(x-\beta_i)| \leqslant |N(a_k^{k-1}b)|, \quad (i = 1, \ldots, n).$$

Now we can apply Lemma 7 to the equation

$$(x-\beta_1) + (\beta_1 - \beta_2) + (\beta_2 - x) = 0.$$

We obtain $x - \beta_1 = o\rho_1$ and $\beta_1 - \beta_2 = o\rho_2$ where

$$\max\{\lceil\rho_1\rceil, \lceil\rho_2\rceil\} < c_1 M_1,$$

hence $\lceil x\rceil < c_2 M_1$. Since $|\varphi(b^{-1})| > (2M_1)^{-1}$ for every $\mathbf{Q}$-isomorphism φ of $\mathbf{K}$ and $|\varphi(z)| > 1 + c_3$ for some φ, we obtain from (15) that

$$c_4 M_1^{c_5} > (1 + c_3)^m,$$

hence $m < c_6 \log M_1$. This proves the second part of Theorem 4.

In the sequel we assume z is not a unit in $\mathbf{Z}_\mathbf{K}$. Then $|N(z)| \geqslant 2$ and from (15) we deduce

$$2^m \leqslant |N(a)|\Big(\max_i |N(x-\beta_i)|\Big)^k$$

$$< |N(a)|\Big(\max_i H(x-\beta_i)\Big)^{kd} < c_7(M \cdot H(x))^{c_8}, \quad (d = [\mathbf{K}:\mathbf{Q}]).$$

If $H(x) < c_9 M$, then our theorem immediately follows. We may therefore assume without loss of generality that $H(x) > c_{10} M$ for a sufficiently large c_{10} which will be determined later. Then

$$m < c_{11} \log H(x). \tag{17}$$

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the distinct prime ideals of $\mathbf{K}$ which divide $[a_k][u_1]\ldots[u_s][\Delta]$ where

$$\Delta = \prod_{1 \leqslant i < j \leqslant n} (\beta_i - \beta_j)^2.$$

Then we have $N\mathfrak{p}_t < c_{12}U^{c_{13}}$ and $t < c_{14}(s+1)$. We write the ideals $[a]$ and $[b]$ in the following form

$$[a] = \mathfrak{p}_1^{v_1} \dots \mathfrak{p}_t^{v_t}\mathscr{A}, \quad [b] = \mathfrak{p}_1^{w_1} \dots \mathfrak{p}^{w_t}\mathscr{B}$$

where v_t, w_t are non-negative integers and $\mathscr{A}$, $\mathscr{B}$ are integral ideals which are relatively prime to $\mathfrak{p}_1, \dots, \mathfrak{p}_t$. From (16) we obtain

$$\mathscr{A}^* \prod_{i=1}^{n} [x - \beta_i]^{r_i} = \left(\prod_{j=1}^{t} \mathfrak{p}_j^{d_j} \right) \mathscr{B}^* \mathfrak{M}^m \tag{18}$$

where $\mathscr{A}^* \mid \mathscr{A}$, $\mathscr{B}^* \mid \mathscr{B}$ are relatively prime ideals, $\mathfrak{M}$ is an integral ideal which is relatively prime to $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ and $d_1, \dots, d_t$ are non-negative integers. The g.c.d. of any two factors $[x - \beta_i]$, $[x - \beta_j]$ on the left hand side of (18) is a divisor of the ideal $[\Delta]$ and each prime ideal divisor of $\mathscr{A}^*\mathscr{B}^*$ divides at most one of the ideals $[x - \beta_i]$. Hence we can write

$$\mathscr{A}_i [x - \beta_i]^{r_i} = \left(\prod_{j=1}^{t} \mathfrak{p}_j^{d_{ji}} \right) \mathscr{B}_i \mathfrak{M}_i^m \tag{19}$$

where $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathfrak{M}_i$ are integral ideals with $\mathscr{A}_i \mid \mathscr{A}$, $\mathscr{B}_i \mid \mathscr{B}$, $\mathfrak{M}_i \mid \mathfrak{M}$ for $i = 1, \dots, n$ and the d_{ji} are non-negative integers. We take the norm and logarithm of both sides of equation (19); if c_{10} is large enough then we have

$$\sum_{j=1}^{t} d_{ji} \log N(\mathfrak{p}_j) + m \log N(\mathfrak{M}_i)$$

$$\leqslant \log N(\mathscr{A}_i) + r_i |\log |N(x - \beta_i)|| < c_{15} \log H(x). \tag{20}$$

For symmetry let us set $r_1' = r_2$ and $r_2' = r_1$. From (19) we obtain

$$\mathscr{A}_i^{h_{\mathbf{K}} r_i'} [x - \beta_i]^{h_{\mathbf{K}} r_1 r_2} = \left(\left(\prod_{j=1}^{t} \mathfrak{p}_j^{d_{ji}} \right) \mathscr{B}_i \mathfrak{M}_i^m \right)^{h_{\mathbf{K}} r_i'}, \quad i = 1, 2. \tag{21}$$

Putting $\mathscr{A}_i^{h_{\mathbf{K}} r_i'} = [\vartheta_i]$, $\mathscr{B}_i^{h_{\mathbf{K}} r_i'} = [\delta_i]$, $\mathfrak{p}_j^{h_{\mathbf{K}}} = [\pi_j]$ and $\mathfrak{M}_i^{h_{\mathbf{K}} r_i'} = [\tau_i]$ for $i = 1, 2$, we have ϑ_i, δ_i, π_j, $\tau_i \in \mathbf{Z}_{\mathbf{K}}$ and

$$\max\{ |N(\vartheta_i)|, |N(\delta_i)| \} < M^{c_{16}}, \quad i = 1, 2$$

$$|N(\pi_j)| < c_{17}U^{c_{18}}, \quad j = 1, \dots, t.$$

By applying Lemma 5 we may assume that

$$C_{19} |N(\alpha)|^{1/d} \leqslant |\varphi(\alpha)| \leqslant c_{20} |N(\alpha)|^{1/d} \tag{22}$$

for every **Q**-isomorphism φ of **K** and $\alpha \in \{\vartheta_1, \vartheta_2, \delta_1, \delta_2, \tau_1, \tau_2, \pi_1, \ldots, \pi_t\}$. Then

$$\max\{|\vartheta_i|, |\delta_i|\} < M^{c_{21}}, \quad i = 1, 2$$

and

$$H(\pi_j) < c_{22} U^{c_{23}}, \quad j = 1, \ldots, t.$$

By (21) we have

$$\vartheta_i (x - \beta_i)^h = \epsilon_i \left(\prod_{j=1}^{t} \pi_j^{d_{ji}} \right)^{r_i'} \delta_i \tau_i^m, \quad i = 1, 2. \tag{23}$$

where $h = h_{\mathbf{K}} r_1 r_2$ and ϵ_1, ϵ_2 are units in **K**. By Lemmas 4 and 5 ϵ_i can be written in the form

$$\epsilon_i = \mu_i \eta_1^{w_{i1}} \ldots \eta_r^{w_{ir}}, \quad i = 1, 2 \tag{24}$$

where $\eta_1, \ldots, \eta_r$ are units in **K** with $\max_j H(\eta_j) < c_{24}$, $w_{i1}, \ldots, w_{ir}$ are rational integers and $\max_i H(\mu_i) < c_{25}$. From (23) and (24) we deduce

$$\sum_{j=1}^{r} w_{ij} \log |\varphi(\eta_j)| = h \log |\varphi(x - \beta_i)| + \log |\varphi(\vartheta_i)| - \log |\varphi(\mu_i \delta_i)| -$$

$$- r_i' \sum_{j=1}^{t} d_{ij} \log |\varphi(\pi_j)| - m \log |\varphi(\tau_i)|,$$

$$i = 1, 2 \tag{25}$$

for every **Q**-isomorphism φ of **K**. Let us consider the absolute values of the expressions on the right-hand side of (25). It is clear that

$$\max\{ |\log |\varphi(\vartheta_i)||, |\log |\varphi(\mu_i \delta_i)|| \} < c_{26} \log M < c_{27} \log H(x),$$

$$i = 1, 2 \tag{26}$$

and

$$|\varphi(x - \beta_i)| \geqslant \left(\prod_{\sigma \neq \varphi} |\sigma(x - \beta_i)| \right)^{-1} > c_{28} (H(x))^{c_{29}} \tag{27}$$

(where the product is taken over all **Q**-isomorphisms $\sigma \neq \varphi$ of **K**); on the other

hand

$$|\varphi(x-\beta_i)| < c_{30}(H(x))^{c_{31}}. \tag{28}$$

Comparing (27) with (28) we have

$$|h \log|\varphi(x-\beta_i)||| < c_{32} \log H(x). \tag{29}$$

Hence, by (22) and (23), we obtain

$$\max\left\{\left|r_i' \sum_{j=1}^{t} d_{ij} \log|\varphi(\pi_j)|\right|, m|\log|\varphi(\tau_i)|||\right\} < c_{33} \log H(x),$$

$$i = 1, 2. \tag{30}$$

Now, we consider the equation (25) for $r = r_{\mathbf{K}}$ appropriate $\mathbf{Q}$-isomorphisms of $\mathbf{K}$. Using Cramer's rule, Lemma 4, (26), (29) and (30) we have

$$|w_{ij}| < c_{34} \log H(x) \quad \text{for } i = 1, 2 \quad \text{and } j = 1, \ldots, t. \tag{31}$$

Let us write w_{ij} in the form $w_{ij} = mw_{ij}^{(1)} + w_{ij}^{(2)}$ with $w_{ij}^{(1)}, w_{ij}^{(2)} \in \mathbf{Z}$ and $0 \leqslant w_{ij}^{(2)} < m$. Similarly, for $i = 1, 2$ and $j = 1, \ldots, t$ put $d_{ij} = md_{ij}^{(1)} + d_{ij}^{(2)}$ with non-negative integers $d_{ij}^{(1)}$, $d_{ij}^{(2)}$ such that $d_{ij}^{(2)} < m$. For brevity let us write

$$\omega_i = \left(\prod_{j=1}^{r} \eta_j^{w_{ij}^{(1)}}\right)\left(\prod_{j=1}^{t} \pi_j^{d_{ij}^{(1)}}\right)^{r_i'} \tau_i, \quad \xi_i = \mu_i \delta_i / \vartheta_i, \quad i = 1, 2.$$

By applying Lemma 4 and inequalities (20), (22), (30) and (31) we have

$$\log |\overline{\omega_i}| < c_{35} \frac{1}{m} \log H(x).$$

This implies that

$$\log H(\omega_1/\omega_2) < c_{36} \frac{1}{m} \log H(x).$$

Further, by the construction of the elements δ_i, ϑ_i and by (22),

$$\log H(\xi_1/\xi_2) < c_{37} \log M.$$

Setting $w_j = w_{1j}^{(2)} - w_{2j}^{(2)}$ and $D_j = d_{1j}^{(2)} - d_{2j}^{(2)}$ we rewrite (23) in the form

$$\left(\frac{x-\beta_1}{x-\beta_2}\right)^h - 1 = \left(\prod_{j=1}^{r} \eta_j^{w_j}\right)\left(\prod_{j=1}^{t} \pi_j^{D_j}\right)^{r_i'} \left(\frac{\omega_1}{\omega_2}\right)^m \frac{\xi_1}{\xi_2} - 1. \tag{32}$$

If $(x-\beta_1)^h=(x-\beta_2)^h$ then $x=(\epsilon\beta_2-\beta_1)(\epsilon-1)^{-1}$ where $1\neq\epsilon$ is a root of unity in $\mathbf{K}$ and so $H(x)<c_{38}$. We have seen that c_{10} may be chosen arbitrarily large. So, we may assume that $(x-\beta_1)^h\neq(x-\beta_2)^h$. For a $\mathbf{Q}$-isomorphism φ such that $\lceil x\rceil=|\varphi(x)|$ we obtain by taking c_{10} large enough

$$\left|\left(\frac{\varphi(x)-\varphi(\beta_1)}{\varphi(x)-\varphi(\beta_2)}\right)^h-1\right|=\left|\left(1+\frac{\varphi(\beta_2-\beta_1)}{\varphi(x)-\varphi(\beta_2)}\right)^h-1\right|$$

$$<\exp(-c_{39}\log\lceil x\rceil)<\exp(-c_{40}\log H(x)). \tag{33}$$

Now, we apply Lemma 6 to give a lower bound for $\left|\left(\frac{\varphi(x)-\varphi(\beta_1)}{\varphi(x)-\varphi(\beta_2)}\right)^h-1\right|$. Put $N=r+t+2$, $\alpha_i=\varphi(\eta_i)$, $i=1,\ldots,r$, $\{\alpha_{r+1},\ldots,\alpha_{N-2}\}=\{\varphi(\pi_1),\ldots,\varphi(\pi_t)\}$, $\alpha_{N-1}=\varphi(\xi_1\xi_2^{-1})$, $\alpha_N=\varphi(\omega_1\omega_2^{-1})$ and $B=c_{41}m$. Then, we have, by (32),

$$\left|\left(\frac{\varphi(x)-\varphi(\beta_1)}{\varphi(x)-\varphi(\beta_2)}\right)^h-1\right|>\exp\Big\{-(c_{42}(s+1)\log U)^{c_{43}(s+1)}(\log M)$$

$$\times(\log\log M)\left(\frac{1}{m}\log H(x)\right)\log m\Big\}. \tag{34}$$

Finally, comparing (33) with (34) we have Theorem 4.

Proof of Theorem 5

From (11) we have

$$\big(B^2(t)-4cA(t)\big)x^{2t}=\xi^2+4cd$$

for some $\xi\in\mathbf{Z}_{\mathbf{L}}$. It is clear that

$$|N\big(B^2(t)-4cA(t)\big)|\leqslant\big[H\big(B^2(t)-4cA(t)\big)\big]^{[\mathbf{L}:\mathbf{Q}]}<c_{44}(t+1)^{c_{45}}$$

where c_{44} and c_{45} are computable constants depending only on $\mathbf{L}$, B, A and c. From Theorem 4 with $s=0$ we infer that t is bounded. We may therefore assume that t is fixed. Hence, Lemma 9 implies Theorem 5.

Proof of Theorem 6

We assume the polynomial $P(X) = F^m(X) + G^n(x)$ has the representation

$$P(X) = a\prod_{i=1}^{k}(X-\alpha_i)^{r_i}$$

with $a \neq 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. From Corollary 1 we have $P(X)$ has at least two simple zeros. Let $\mathbf{K}$ be the splitting field of $P(X)$ with ring of integers $\mathbf{Z}_{\mathbf{K}}$, and write $x_1 = ax$ and $\beta_i = a\alpha_i$, $i = 1, \ldots, k$. Then the elements $\beta_1, \ldots, \beta_k$ are algebraic integers in $\mathbf{K}$ and equation (12) can be written in the form

$$\prod_{i=1}^{k}(x_1-\beta_i)^{r_i} = a^{d-1}y^z$$

where $d = \deg P$. Using square brackets to indicate principal ideals in $\mathbf{K}$ we obtain

$$[x_1] = \frac{\mathfrak{X}}{\mathfrak{p}_1^{a_1}\ldots\mathfrak{p}_s^{a_s}}, \quad [y] = \frac{\mathfrak{M}}{\mathfrak{p}_1^{b_1}\ldots\mathfrak{p}_s^{b_s}}$$

where $a_1, b_1, \ldots, a_s, b_s$ are non-negative integers and $\mathfrak{X}$, $\mathfrak{M}$ are integral ideals such that

$$(\mathfrak{X}, \mathfrak{p}_1^{a_1}\ldots\mathfrak{p}_s^{a_s}) = (\mathfrak{M}, \mathfrak{p}_1^{b_1}\ldots\mathfrak{p}_s^{b_s}) = [1].$$

It is known that there is an integral ideal $\mathfrak{q}$ with bounded norm such that $\mathfrak{p}_1^{a_1}\ldots\mathfrak{p}_s^{a_s}\mathfrak{q} = [\omega]$ for some algebraic integer ω. Then

$$\prod_{i=1}^{k}(x_2-\beta_i\omega)^{r_i} = a^{d-1}\omega^d y^z \tag{35}$$

and $x_2 := x_1\omega \in \mathbf{Z}_{\mathbf{K}}$. Further, it is clear that the greatest prime factor of the norm of the g.c.d. of $[x_2]$ and $[\omega]$ is bounded. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the prime ideal divisors of $[a^{d-1}\omega^d]$ and let h denote the class number of $\mathbf{K}$. Write $\mathfrak{q}_i^h = [\pi_i]$, $i = 1, \ldots, t$. By Lemma 5 we may assume that $\max_i \overline{[\pi_i]}$ is bounded. From (35) we have

$$\prod_{i=1}^{k}(x_2-\beta_i\omega)^{r_i h} = \epsilon\pi_1^{\delta_1}\ldots\pi_t^{\delta_t}y^{zh}$$

where ϵ is a unit and $\delta_1, \ldots, \delta_t$ are non-negative integers. If y is not a unit then from Lemma 10 we have z is bounded. If y is a unit then $x_2 - \beta_1\omega$,

$x_2 - \beta_2\omega$ and $(\beta_2 - \beta_1)$ are $\mathcal{S}_1$-units where $\mathcal{S}_1$ consists of all the valuations v of $\mathbf{K}$ such that $v(\omega\pi_1 \dots \pi_t(\beta_2 - \beta_1)) \neq 0$. By applying Lemma 7 to

$$(x_2 + \beta_2\omega) + (\beta_1\omega - x_2) + (\beta_2 - \beta_1)\omega = 0$$

we infer that

$$H\left(\frac{x_2 - \beta_1\omega}{(\beta_1 - \beta_2)\omega}\right) = H\left(\frac{x_1 - \beta_1}{\beta_1 - \beta_2}\right)$$

is bounded. It follows that $H(x_1)$, $H(x)$ and $H(y^z)$ are also bounded. Then there is a bounded non-zero rational integer A such that $Ay^z \in \mathbf{Z}_\mathbf{K}$ therefore $\mathfrak{p}_i^{b_i z} \mid [A]$, $i = 1, \dots, s$. If at least one b_i is positive then we have z is bounded; if $b_1 = \cdots = b_s = 0$ then $y \in \mathbf{Z}_\mathbf{K}$ and y is not a root of unity, so, by Lemma 8 we obtain

$$(1 + C_{16})^z < |y|^z \leqslant 2H(y^z)$$

which also implies z is bounded. Finally, from Lemma 9 we have the theorem.

Acknowledgements

I would like to thank Prof. Dr. R. Tijdeman and Dr. F. Beukers for their valuable suggestions.

References

Baker, A.: The theory of linear forms in logarithms. In: *Transcendence Theory: Advances and Applications*. Academic Press, London (1977) 1–27.

Brindza, B.: On a diophantine equation connected with the Fermat equation. *Acta Arith.* 44 (1984a) 357–363.

Brindza, B.: On $\mathcal{S}$-integral solutions of the equation $f(x) = y^m$. *Acta Math. Hung.* 44 (1984b) 133–139.

Brindza, B., Győry, K. and Tijdeman, R.: The Fermat equation with polynomial values as base variables. *Invent. Math.* 80 (1985) 139–151.

Brownawell, W.D. and Masser, D.W.: Vanishing sums in function fields. to appear in *Math. Proc. Camb. Phil. Soc.*

Everett, C.J.: Fermat's conjecture, Roth's theorem, Pythagorean triangles and Pell's equation. *Duke Math. J.* 40 (1973) 801–804.

Győry, K.: Résultats effectifs sur la représentation des entiers par des formes décomposables. *Queen's Papers in Pure and Applied Mathematics* No. 56. Kingston, Canada (1980).

Győry, K.: On the number of solutions of linear equations in units of an algebraic number field. *Comment. Math. Helv.* 54 (1979) 583–600.

Inkeri, K.: Untersuchungen über die Fermatsche Vermutung. *Ann. Acad. Sc. Fenn.* Ser. AI No. 33 (1946) 1–60.

Inkeri, K.: A note on Fermat's conjecture. *Acta Arith.* 39 (1976) 251–256.

Inkeri, K. and van der Poorten, A.J.: Some remarks on Fermat's conjecture. *Acta Arith.* 39 (1976) 251–256.

Mason, R.C.: Diophantine equations over function fields. *LMS Lecture Notes*, No. 96. Cambridge University Press (1984).

Schinzel, A. and Tijdeman, R.: On the equation $P(x) = y^z$. *Acta Arith.* 31 (1976) 199–204.

Schinzel, A. and Zassenhaus, H.: A refinement of two theorems of Kronecker. *Michigan Math. J.* 12 (1965) 81–85.

Shorey, T.N. and Stewart, C.L.: On the diophantine equation $ax^{2t} + bx^t y + y^2 = d$ and pure powers in recurrence sequences. *Math. Scand.* 52 (1983) 24–26.

Shorey, T.N. and Tijdeman, R.: *Exponential diophantine equations*. Cambridge University Press, New York, in press.

Sprindžuk, V.G.: Classical diophantine equations in two unknowns (Russian). Nauka, Moskva (1982).

Stark, H.M.: Effective estimates of solutions of some diophantine equations. *Acta Arith.* 24 (1973) 251–259.

Stewart, C.L.: A note on the Fermat equation. *Mathematika* 24 (1977) 130–132.

Tijdeman, R.: Applications of the Gel'fond-Baker method to rational number theory. *Topics in Number Theory*, Proc. Conf. Debrecen 1974, *Colloq. Math. Soc. János Bolyai* 13. North Holland, Amsterdam (1976) 399–416.

Turk, J.: On the difference between perfect powers. *Acta Arith.*, to appear.

Zimmert, R.: Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung. *Invent. Math.* 62 (1981) 367–380.