COMPOSITIO MATHEMATICA

PETER HELLEKALEK

Ergodicity of a class of cylinder flows related to irregularities of distribution

Compositio Mathematica, tome 61, nº 2 (1987), p. 129-136

<http://www.numdam.org/item?id=CM_1987__61_2_129_0>

Compositio Mathematica 61: 129–136 (1987)

Ergodicity of a class of cylinder flows related to irregularities of distribution

PETER HELLEKALEK
Institut für Mathematik, Universität Salzburg, Hellbrunnerstrasse 34, A-5020 Salzburg, Austria

Received 18 November 1985; accepted 10 March 1986

Abstract. We study ergodicity of cylinder flows $(x, t) \mapsto (Tx, t+\varphi(x))$, where T is a von Neumann-Kakutani adding machine transformation on $\mathbb{R}/\mathbb{Z}$ and $\varphi(x) = 1_A(x) - \beta$, A an arc in $\mathbb{R}/\mathbb{Z}$ of length β.

Introduction

We shall be interested in cylinder flows of the following type. Let $T: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto Tx$, be measure preserving and ergodic with respect to Lebesgue measure λ on $\mathbb{R}/\mathbb{Z}$, let G be either a closed subgroup of $\mathbb{R}$ or $G = \mathbb{R}/a\mathbb{Z}$ with a in $\mathbb{R}$. Let h denote Haar measure on G and let $\varphi: \mathbb{R}/\mathbb{Z} \to G$ be measurable with $\int \varphi \, d\lambda = 0$.

The cylinder flow $T_\varphi(x, t) = (Tx, t + \varphi(x))$ acts on the measure theoretic product space $X = \mathbb{R}/\mathbb{Z} \otimes G$ and preserves the product measure $\lambda \otimes h$ on X. We shall study ergodicity (with respect to $\lambda \otimes h$) of the following class:

Example 1

Let T_φ be the cylinder flow where $T: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a generalized von Neumann-Kakutani adding machine transformation (definition in Part II of this paper), and let $\varphi(x) = 1_A(x) - \beta$, where A is an arc in $\mathbb{R}/\mathbb{Z}$ of length β, $0 < \beta \leqslant 1$. Let G be the closed subgroup of $\mathbb{R}$ generated by 1 and β. If β is irrational, then $G = \mathbb{R}$ and h will denote Lebesgue measure. If $\beta = r/s$, r and s positive integers, $(r, s) = 1$, then $G = (1/s)\ \mathbb{Z}$ and h will stand for the counting measure.

The ergodicity of this class of cylinder flows is directly related to irregularities in the distribution of generalized van-der-Corput sequences. For this reason, necessary conditions for the ergodicity of T_φ and hence for β follow from results in [Hellekalek, 1984]. For the general background, in particular the important coboundary theorem and its consequences, the reader is referred to [Liardet, 1982, 1985].

Results on ergodicity of cylinder flows date back to [Anzai, 1951] (T an irrational rotation, $G = \mathbb{R}/\mathbb{Z}$). The following class is now well-known.

Example 2

Let T_φ be the cylinder flow where T: $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \alpha \bmod 1$, α irrational, and $\varphi(x) = 1_{[0,\beta[}(x) - \beta$, $0 < \beta \leqslant 1$. Let G be as in Example 1.

Ergodicity of Example 2 was studied by [Oren, 1983], completing an earlier result of [Conze, 1980]. Oren has proved: T_φ is ergodic if and only if β is rational or 1, α and β are linearly independent over $\mathbb{Z}$.

Example 2 is also related to a class of sequences well-known in the theory of uniform distribution modulo 1, the sequences $(n\alpha)_{n \geqslant 0}$. Good references are [Petersen, 1973] and, in particular, [Liardet, 1985].

I. Remarks

From now on it will be assumed that T_φ is the cylinder flow of Example 1, although the following remarks can easily be generalized to cover Example 2 and a large class of other cylinder flows as well.

T_φ is ergodic if and only if, for every T_φ-invariant measurable subset B of $\mathbb{R}/\mathbb{Z} \otimes G$, either B or its complement has measure zero. We study ergodicity of T_φ by reducing the problem from the infinite case (i.e. T_φ on $\mathbb{R}/\mathbb{Z} \otimes G$) to a finite case (i.e. T_φ on $\mathbb{R}/\mathbb{Z} \otimes G/a\mathbb{Z}$; $a \in G$, $a \neq 0$).

Definition: An element c of G is called a *period* of T_φ if, for every T_φ – invariant function 1_B, B a measurable subset of the product space $\mathbb{R}/\mathbb{Z} \otimes G$, the equality $1_B(x, t) = 1_B(x, t + c)$ holds $\lambda \otimes h$ – a.e..

The set P_φ of periods of T_φ is a subgroup of G. [Schmidt, 1976] has extensively studied what he calls 'essential values' of a cylinder flow. It follows from Theorem 5.2. in [Schmidt, 1976] that essential values and periods are the same.

Remarks: it is not difficult to see that

i) if $\varphi = g - g \circ T$ λ – a.e., g: $\mathbb{R}/\mathbb{Z} \to G$ measurable, then $P_\varphi = \{0\}$;

ii) if $\varphi = \psi + g - g \circ T$ λ – a.e., ψ, g: $\mathbb{R}/\mathbb{Z} \to G$ measurable, then $P_\varphi = P_\psi$.

Let a be an element of G and let S_a denote the cylinder flow T_φ on $\mathbb{R}/\mathbb{Z} \otimes G/a\mathbb{Z}$:

$$S_a\colon \mathbb{R}/\mathbb{Z} \otimes G/a\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \otimes G/a\mathbb{Z}$$

$$S_a(x, t) = (Tx, t + \varphi(x) \bmod a).$$

Ergodicity of T_φ and S_a are related as follows. S_a is a factor of T_φ, hence ergodicity of T_φ implies ergodicity of S_a. If a is a period of T_φ and if S_a is ergodic then T_φ is ergodic. We shall use this observation later on.

Ergodicity of T_φ is associated with the following type of functional equation. Define

- $\Gamma := \{\chi \in \hat{G}$: the functional equation $h \circ T = \chi(\varphi)h \ \lambda -$ a.e. has a nontrivial measurable solution $h: \mathbb{R}/\mathbb{Z} \to \mathbb{C}\}$, and, for a in G,
- $\Gamma_a := \{\chi \in (G/a\mathbb{Z})\hat{}: h \circ T = \chi(\varphi)h \ \ \lambda -$ a.e. has a nontrivial measurable solution $h : \mathbb{R}/\mathbb{Z} \to \mathbb{C}\}$. The sets Γ and Γ_a are subgroups.

LEMMA 0: *Let $a \in G$, $a \neq 0$. Then S_a is ergodic if and only if Γ_a is trivial.*

Proof: This result is classical, see [Anzai, 1951]. □

THEOREM 1 *If $c \in P_\varphi$, then $\Gamma = \Gamma_c$.*

Proof: Clearly Γ_c is a subset of Γ. Let χ be an arbitrary element of Γ and let h be a nontrivial measurable solution of the equation $h \circ T = \chi(\varphi)h \ \lambda -$ a.e.. The measurable function $f(x, t) = h(x)\bar{\chi}(t)$ is invariant under T_φ, hence $f(x,t+c) = h(x)\bar{\chi}(t)\bar{\chi}(c) = f(x, t)\lambda \otimes h -$ a.e.. This implies $\chi(c) = 1$, thus χ belongs to Γ_c. □

COROLLARY: *The following are equivalent:*
i) T_φ is ergodic;
ii) $P_\varphi \neq \{0\}$ and Γ is the trivial subgroup of $\hat{G}$.
We shall now study example 1. We ask under which conditions for β and γ will Γ_1 be trivial (hence S_1 ergodic) and 1 be a period of $T_\varphi = T_\varphi(\beta, \gamma)$.

II. A class of cylinder flows

We shall consider the following generalization of the von Neumann-Kakutani adding machine transformation on $\mathbb{R}/\mathbb{Z}$. Let $q = (q_i)_{i \geq 1}$ be a bounded sequence of integers q_i, $2 \leq q_i \leq K$ for all i, with some positive constant K.

If $\mathbb{A}(q)$ denotes the compact Abelian group of q-adic integers, then the transformation $z \mapsto z + 1$ on $\mathbb{A}(q)$ is uniquely ergodic with respect to normalized Haar-measure on $\mathbb{A}(q)$ (see [Hewitt and Ross, 1963] for details on $\mathbb{A}(q)$).

Consider next the one-dimensional torus $\mathbb{R}/\mathbb{Z}$ with Haar measure λ. We shall write

$$p(k) = q_1 \cdot \ldots \cdot q_k, \quad k = 1, 2, \ldots$$

$$p(0) := 1.$$

If

$$z = \sum_{i=0}^{\infty} z_i p(i), \quad z_i \in \{0, 1, \ldots, q_{i+1} - 1\}$$

is an element of $\mathbb{A}(q)$, then

$$\Phi(z) = \sum_{i=0}^{\infty} z_i/p(i+1) \mod 1$$

belongs to $\mathbb{R}/\mathbb{Z}$. The map Φ: $\mathbb{A}(q) \to \mathbb{R}/\mathbb{Z}$ is measure-preserving and injective on $\mathbb{A}(q)$ except on a subset of Haar measure zero.

The q-adic representation of an element x of $\mathbb{R}/\mathbb{Z}$,

$$x = \sum_{i=0}^{\infty} x_i/p(i+1), \quad x_i \in \{0, 1, \ldots, q_{i+1} - 1\},$$

is unique under the condition $x_i \neq q_{i+1} - 1$ for infinitely many i. We shall call x *non-q-adic* if x has infinitely many nonzero digits x_i. The uniqueness condition for the representation ensures that the following transformation T: $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is well-defined:

$$Tx := \Phi(z+1), \quad \text{where } z = z(x) = \sum_{i=0}^{\infty} x_i p(i).$$

T is ergodic with respect to λ and $T \circ \Phi(z) = \Phi(z+1)$ for almost all z. For further properties of T see [Hellekalek, 1984]. T may be called a (generalized) *von Neumann-Kakutani adding machine transformation* (see [Petersen, 1983]).

A rational number β in $]0, 1[$, $\beta = r/s$, r and s positive integers, $(r, s) = 1$, is called *strictly non-q-adic* if k/s is non-q-adic for all k, $1 \leqslant k \leqslant s-1$; equivalently, if no prime divisor of s divides an element of the sequence q.

THEOREM 2: *Let T be the q-adic transformation defined above and let* $\varphi(x) = 1_{[0,\beta[}(x) - \beta$, $0 < \beta \leqslant 1$. *Let T_φ be the cylinder flow defined in Example* 1.

Then the following are equivalent:
i) T_φ is ergodic;
ii) β is irrational or strictly non-q-adic.

We can generalize this result to:

THEOREM 3: *Let T be as in Theorem* 2 *and let* $\varphi(x) = 1_A(x) - \beta$, *where A is an arc in* $\mathbb{R}/\mathbb{Z}$ *of length* β, $0 < \beta \leqslant 1$, $A = \gamma + [0, \beta[$ *mod* 1 *with* $0 \leqslant \gamma < 1$. *Define T_φ as in Example* 1. *Then*
i) T_φ ergodic implies β irrational or strictly non-q-adic;
ii) β irrational implies T_φ ergodic, for all γ;
iii) β strictly non-q-adic and γ q-adic (i.e. $\gamma = a/p(g)$ with nonnegative integers a and g, $a < p(g)$) imply T_φ ergodic;
iv) $q_i = q \geqslant 2$ for all i, and β strictly non-q-adic imply T_φ ergodic for almost all γ.

The proof of these two theorems will be given by Lemmata 1 to 6 and their corollaries. In Lemma 6 we will prove a stronger result than iv) of Theorem 3.

In the sequel we shall write φ_n for the sum $\varphi + \varphi \circ T + \cdots + \varphi \circ T^{n-1}$, $n = 1, 2\ldots$ Lemma 1 below indicates how to obtain periods. The idea is due to [Oren, 1983] (Proposition 1).

LEMMA 1: *Let $(k_n)_{n=1}^{\infty}$ be a subsequence of $(n)_{n=1}^{\infty}$ and let $(A_{k_n})_{n=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}/\mathbb{Z}$ such that*

i) $\varphi_{p(k_n)}$ is constant on A_{k_n}

ii) $\lim_{n\to\infty} \varphi_{p(k_n)}(A_{k_n})$ exists and

iii) $\inf_n \lambda(A_{k_n}) > 0$.

Then $c = \lim_{n\to\infty} \varphi_{p(k_n)}(A_{k_n})$ will be a period of T_φ.

Proof: Let 1_B be an arbitrary T_φ-invariant measurable function on $\mathbb{R}/\mathbb{Z} \otimes G$. The set $M = \{x \in \mathbb{R}/\mathbb{Z}$ such that $1_B(x, t) = 1_B(x, t + c)$ for almost all t in $G\}$ is invariant under T, thus of measure 0 or 1. We shall find a subset of M of positive measure. This will prove the lemma.

Let $a_{k_n} = \varphi_{p(k_n)}(A_{k_n})$ and put $g_{k_n}(x, t) = |1_B(T^{p(k_n)}x, t + a_{k_n}) - 1_B(x, t + c)|$. Let $X_N = \mathbb{R}/\mathbb{Z} \times [-N, N]$, N a positive integer. We note that $|T^{p(k)}x - x| < 1/p(k)$ for all x and all positive integers k, hence

$$\lim_{n\to\infty} \int_{X_N} g_{k_n}\, d\lambda \otimes h = 0 \quad \text{for all } N.$$

Therefore, by diagonalization, we can find a subsequence $(k'_n)_{n=1}^{\infty}$ of $(k_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} g_{k'_n}(x, t) = 0$ a.e. on $\mathbb{R}/\mathbb{Z} \otimes G$. Let $A = \limsup_{n\to\infty} A_{k'_n}$. The set A has positive measure (condition iii)) and almost all elements of A belong to M (conditions i) and ii)). □

LEMMA 2: *Ergodicity of T_φ implies that β is either irrational or strictly non-q-adic.*

Proof: If T_φ is ergodic, so is the compact factor S_1, $S_1(x, t) = (Tx, (t + \varphi(x))$ mod 1). For every character χ of $G/\mathbb{Z}$ $\chi(\varphi(x)) = \chi(-\beta)$ is constant. Therefore Γ_1 is trivial (hence S_1 ergodic) if and only if there are no eigenfunctions of T to the eigenvaue $\chi(-\beta)$. The eigenvalues of T are known to be of the form $\exp(2\pi i\alpha)$, α q-adic. □

LEMMA 3: *If β is non-q-adic and γ is q-adic, then 1 is a period of T_φ.*

Proof: The q-adic representation of β is given by $\beta = \sum_{i=0}^{\infty} \beta_i/p(i+1)$ with digits $\beta_i \in \{0, 1, \ldots, q_{i+1} - 1\}$, infinitely many $\beta_i \neq q_{i+1} - 1$. Define $\beta(k) = \sum_{i=0}^{k-1} \beta_i/p(i+1)$, $k = 1, 2, 3 \ldots$ Then $0 < \beta - \beta(k) < 1/p(k)$ for all k. If

$\gamma = \sum_{i=0}^{\infty} \gamma_i/p(i+1)$, then $\gamma(k) = \gamma$ for all k sufficiently large. T is a bijection almost everywhere and maps elementary q-adic intervals $[a/p(k), (a+1)/p(k)[$, $0 \leqslant a \leqslant p(k)-1$, into elementary q-adic intervals of length $1/p(k)$. For any x in $\mathbb{R}/\mathbb{Z}$ exactly one point $T^j x$, $0 \leqslant j \leqslant p(k)-1$, belongs to a given elementary q-adic interval. For sufficiently large k the function $\varphi_{p(k)}$ takes only two values on $\mathbb{R}/\mathbb{Z}$, $\varphi_{p(k)}(x) \in \{\beta(k)p(k) - \beta p(k),\ 1 + \beta(k)p(k) - \beta p(k)\}$. Let $A_k = \{x : \varphi_{p(k)}(x) = (\beta(k) - \beta)p(k)\}$ and let B_k denote its complement. The integral of the function $\varphi_{p(k)}$ is zero, hence $\lambda(A_k) = 1 - (\beta - \beta(k))p(k)$ and $\lambda(B_k) = (\beta - \beta(k))p(k)$. β has infinitely many nonzero digits β_i, hence there is subsequence $(i_n)_{n=1}^{\infty}$ such that $0 < \beta_{i_n}$ and $\beta_{i_n+1} < q_{i_n+2} - 1$. This implies $1/K < (\beta - \beta(i_n))p(i_n) \leqslant 1 - 1/K^2$. Therefore we can find a subsequence $(k_n)_{n=1}^{\infty}$ of $(i_n)_{n=1}^{\infty}$ such that $0 < \lim_{n\to\infty} (\beta - \beta(k_n))p(k_n) < 1$. We apply lemma 1 to the sequences of sets $(A_{k_n})_{n=1}^{\infty}$ and $(B_{k_n})_{n=1}^{\infty}$ and obtain that 1 is a period of T_φ. □

COROLLARY: *If β is strictly non-q-adic then T_φ is ergodic for all q-adic γ.*

LEMMA 4: *If β is non-q-adic then the set $\{1, 2\}$ contains a period of T_φ for every γ.*

Proof: Let $A = \gamma + [0, \beta[\bmod 1$, $0 < \gamma < 1$, and $\varphi(x) = 1_A(x) - \beta$. In view of Lemma 3 we are only interested in non-q-adic γ. Let $\delta = \gamma + \beta \bmod 1$. We shall assume that δ is non-q-adic, otherwise Lemma 3 applies. Let

$$\gamma = \sum_{i=0}^{\infty} \gamma_i/p(i+1), \quad \delta = \sum_{i=0}^{\infty} \delta_i/p(i+1),$$

infinitely many digits $\gamma_i \neq q_{i+1} - 1$, infinitely many $\delta_i \neq q_{i+1} - 1$. We shall denote by $\gamma(k)$ and $\delta(k)$ the representations truncated at k (see Lemma 3). Elementary calculations as in Lemma 3 show that for all x, $\varphi_{p(k)}(x) \in \{y_k, y_k - 1, y_k + 1\}$, where $y_k = (\gamma - \gamma(k))p(k) - (\delta - \delta(k))p(k)$, $y_k \in \{(\beta(k) - \beta)p(k),\ 1 + (\beta(k) - \beta)p(k)\}$. Let $A_k = \{x \in \mathbb{R}/\mathbb{Z}\colon \varphi_{p(k)}(x) = y_k\}$, $B_k = \{x : \varphi_{p(k)} = y_k - 1\}$ and $C_k = \{x : \varphi_{p(k)} = y_k + 1\}$. As the integral of φ is zero, the relation $\lambda(B_k) = y_k + \lambda(C_k)$ holds. We shall check if conditions ii) and iii) of Lemma 1 can be satisfied.

- Condition ii): it is clear from the proof of Lemma 3 that there is a sequence $(k_n)_{n=1}^{\infty}$ such that $0 < \left| \lim_{n\to\infty} y_{k_n} \right| < 1$;
- Condition iii): due to the above relation between $\lambda(B_k)$ and $\lambda(C_k)$ we can always find a suitable subsequence $(k_n')_{n=1}^{\infty}$ of $(k_n)_{n=1}^{\infty}$ such that condition iii) holds for $(A_{k_n'})_{n=1}^{\infty}$ and one of $(B_{k_n'})_{n=1}^{\infty}$ or $(C_{k_n'})_{n=1}^{\infty}$ (which implies that 1 is a period) or $(B_{k_n'})_{n=1}^{\infty}$ and $(C_{k_n'})_{n=1}^{\infty}$ (which implies that 2 is a period). □

LEMMA 5: *If β is irrational, then T_φ is ergodic for all γ.*

Proof: We only have to show that S_2 is ergodic, i.e. that Γ_2 is trivial. Let χ be an arbitrary element of Γ_2 and let h be a nontrivial measurable solution of the functional equation $h \circ T = \chi(\varphi)\, h$ a.e.. Then $h^2 \circ T = \chi^2(\varphi)\, h^2$, hence χ^2 belongs to Γ_1. S_1 is ergodic for irrational β and thus the latter group is trivial. Hence χ is the trivial character of $\mathbb{R}/2\mathbb{Z}$. □

The argument employed in Lemma 5 is not valid for strictly non-q-adic β: for a nontrivial character χ in $\widehat{G/2\mathbb{Z}}$, χ^2 can be trivial in $\widehat{G/\mathbb{Z}}$. It is not difficult to see that Γ_2 would be trivial if the functional equation $h \cdot T = \Phi h$ a.e., $\Phi(x) = 1$ on A and $\Phi(x) = -1$ on the complement $\mathbb{R}/\mathbb{Z}$-A, had no nontrivial measurable solution h.

LEMMA 6: *Let β be rational and non-q-adic. If there is a strictly increasing sequence $(i_n)_{n=1}^{\infty}$ such that, for all n, $\beta_{i_n} \neq 0$, $\beta_{i_n+1} < q_{i_n+2} - 1$, $i_{n+1} - i_n \leqslant L$ with a constant L, then* 1 *is a period of T_φ for almost all γ in $\mathbb{R}/\mathbb{Z}$.*

Proof: We shall take as basis the proof of Lemma 4. Hence we assume that γ and $\delta = \delta(\gamma) = \gamma + \beta$ mod 1 are non-q-adic. It will be shown that for almost every γ there is a subsequence $(k_n)_{n=1}^{\infty}$ of $(i_n)_{n=1}^{\infty}$ such that conditions ii) and iii) of Lemma 1 are satisfied. It is then easy to deduce from Lemma 4 that 1 is a period of $T_\varphi = T_\varphi(\beta, \gamma)$.

Let I be an elementary q-adic interval of length $1/p(k)$, $I = [a/p(k), (a+1)/p(k)[$, $0 \leqslant a \leqslant p(k) - 1$. Choose i and j, $0 \leqslant i$, $j \leqslant p(k) - 1$ such that $\lambda(T^i I \Delta [\gamma(k), \gamma(k) + 1/p(k)[) = 0$, $\lambda(T^j I \Delta [\delta(k), \delta(k) + 1/p(k)[) = 0$. We write $D_I = T^{-i}]\gamma(k), \gamma[$, $E_I = T^{-j}]\delta(k), \delta[$. D_I and E_I are subsets of I and the following relations hold:

$$A_k \cap I = I - D_I \Delta E_I, \quad B_k \cap I = D_I - E_I, \quad C_k \cap I = E_I - D_I.$$

Therefore $\lambda(A_k \cap I) = 1/p(k) + 2\lambda(D_I \cap E_I) - \lambda(D_I) - \lambda(E_I)$. Let a_k and b_k be those integers with $\Phi(a_k) = \gamma(k)$ and $\Phi(b_k) = \delta(k)$. Then $0 < a_k$, $b_k \leqslant p(k) - 1$ for sufficiently large k and $a_{k+1} = a_k + \gamma_k p(k)$, $b_{k+1} = b_k + \delta_k p(k)$. Let a be an arbitrary integer such that $0 \leqslant a \leqslant \min(a_k, b_k)$. If we consider the elementary interval $I = I(a) = [\Phi(a), \Phi(a) + 1/p(k)[$, then the condition $a \leqslant \min(a_k, b_k)$ implies that D_I and E_I are intervals, $D_I =]\Phi(a), \Phi(a) + \gamma - \gamma(k)[$, $E_I =]\Phi(a), \Phi(a) + \delta - \delta(k)[$. This yields the following estimate for $\lambda(A_k)$:

$$\lambda(A_k) \geqslant (1 - |y_k|) \min(a_k, b_k)/p(k).$$

It is $1/K^2 \leqslant 1 - |y_{i_n}| \leqslant 1 - 1/K^2$ thus only $\min(a_{i_n}, b_{i_n})/p(i_n)$ requires further study. We see that $a_{i_{n+1}}/p(i_{n+1}) \geqslant \gamma_{i_n}/K^L$ and $b_{i_{n+1}}/p(i_{n+1}) \geqslant \delta_{i_n}/K^L$. We study $F = \{\gamma \text{ in } \mathbb{R}/\mathbb{Z} : \gamma_{i_n}\delta_{i_n} \neq 0 \text{ for infinitely many } n\}$. If γ belongs to F, then there is a subsequence $(k_n)_{n=1}^{\infty}$ of $(i_n)_{n=1}^{\infty}$ such that $\inf_n \lambda(A_{k_n}) \geqslant 1/K^{2+L} > 0$. Hence condition iii) of Lemma 1 is satisfied. Condition ii) will hold for a suitable subsequence.

It is elementary to show that the set F is invariant under T. Thus $\lambda(F)$ is either zero or one. One calculates $\lambda(\{\gamma : \gamma_{i_n}\delta_{i_n} \neq 0\}) \geqslant 1/3$ if $q_{i_n+1} \geqslant 3$ and that it is equal to $(\beta - \beta(i_n))p(i_n) - 1/2$ if $q_{i_n+1} = 2$. One deduces that $\lambda(F) = 1$. □

COROLLARY: *With the additional assumption that β be strictly non-q-adic Lemma* 6 *implies that T_φ is ergodic for almost all γ.*

COROLLARY: *Suppose that $q_i = q \geqslant 2$ for all i, q an integer. If β is strictly non-q-adic then T_φ is ergodic for almost all γ.*

Proof: The q-adic representation of rational numbers is periodic. As β is strictly non-q-adic, S_1 is ergodic and further there is a sequence $(i_n)_{n=1}^{\infty}$ which satisfies the conditions of Lemma 6. □

References

Anzai, H.: Ergodic skew product transformations on the torus. *Osaka Math. J.* 3 (1951) 83–99.

Conze, J.P.: Ergodicité d'une transformation cylindrique. *Bull. Soc. Math. France* 108 (1980) 441–456.

Hellekalek, P.: Regularities in the distribution of special sequences. *J. Number Th.* 18 (1984) 41–55.

Hewitt, E. and Ross, K.A.: *Abstract Harmonic Analysis,* Volume I. Springer-Verlag, Heidelberg-New York-Berlin (1963).

Liardet, P.: Résultats et problèmes de régularité de distributions. In: *Théorie élémentaire et analytique des nombres. Journées Mathématiques SMF-CNRS.* Valenciennes (1982).

Liardet, P.: *Regularities of distribution.* To appear in *Comp. Math.*

Oren, I.: Ergodicity of cylinder flows arising from irregularities of distribution. *Israel J. Math.* 44 (1983) 127–138.

Petersen, K.: On a series of cosecants related to a problem in ergodic theory. *Comp. Math.* 26 (1973) 313–317.

Petersen, K.: *Ergodic Theory.* Cambridge University Press, Cambridge, England (1983).

Schmidt, K.: Lectures on cocycles of ergodic transformation groups. Preprint, University of Warwick, Conventry, England (1976).