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# QUANTITATIVE REARRANGEMENT THEOREMS

Gerhard Larcher

## Introduction

The classical result in the theory of rearranging sequences, is that of Von Neumann [11] saying that any dense sequence in the unit-interval [0, 1] can be rearranged to an uniformly distributed sequence in [0, 1].

In general a rearrangement theorem may be defined as a result which says that, under suitable hypotheses, a sequence in a given space $X$ will attain a desired property after a suitable rearrangement of terms.

Rearrangement theorems in general compact metric spaces were considered for example by Hlawka [2], [3], Niederreiter [5], [6], Van der Corput [10] and Descovich [1]. (See also Kuipers and Niederreiter [4]). Recently Niederreiter [6] showed that all the standard results, which are not of a quantitative form, are simple consequences of the following theorem, which is stated here in an inessentially modified form.

THEOREM: *Let $(X, d)$ be a compact metric space without isolated points, and $(x_n)$, $(y_n)$ two sequences of elements of $X$, which are dense in $X$, then there exists a permutation $\tau$ of $N$, such that* $\lim_{n\to\infty} \mathrm{d}(x_n, y_{\tau(n)}) = 0$.

In this paper I give some quantitative variants of this theorem. The main result will be Theorem 2, from which all the standard results, even the quantitative, in compact metric spaces will follow in the form of easy corollaries, or can even be improved considerably, as for example Theorem 3 will show. Some of these considerations finally lead to the notion of the dispersion of a sequence and to a problem which is in some sense an analogon to a wellknown problem in the theory of irregularities of distribution.

## Notations and statement of the results

In everything what follows $(X, d)$ denotes a compact metric space without isolated points.

$(x_n)$, $(y_n)$, $(z_n)$ are always sequences of elements of $X$, which are dense in $X$.

$(\epsilon(n))$, $(\delta(n))$ are sequences of positive reals, and $\tau$ and $\sigma$ are permutations of $\mathbb{N}$.

By $K_\epsilon(x)$ we denote the closed $\epsilon$-ball with center $x$. Note that, because $X$ contains no isolated points, '$(x_n)$ dense in $X$' means, that in every neighbourhood $U$ of any $x \in X$, there are infinitely many $n \in \mathbb{N}$ with $x_n \in U$. By $d$ we denote the diameter of $X$: $d = \sup_{x,y \in X} \mathrm{d}(x, y) = \max_{x,y \in X} \mathrm{d}(x, y)$.
First it is shown that something like 'uniform convergence' on the set of all sequences is not possible:

THEOREM 1: *Let $(\epsilon(n))$ be given, then the following two assertions are equivalent*:

(*i*) *For all $(x_n)$, $(y_n)$ there is a $\tau$ with* $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ *for all* $n \in \mathbb{N}$.

(*ii*) $\epsilon(n) \geqslant d$ *for infinitely many $n$, or* $\liminf_{n\to\infty} \epsilon(n) > 0$.

As already mentioned, the main result is the following:

THEOREM 2: *Let $(\epsilon(n))$ be given, then for all $(x_n)$ and $(y_n)$ there are $\tau$ and $\sigma$ with*:

$$\mathrm{d}(x_{\tau(n)}, y_{\sigma(n)}) \leqslant \epsilon(n) \quad \textit{for all } n \in \mathbb{N}.$$

From this the result of Niederreiter follows easily, with any $(\epsilon(n))$ with $\lim_{n\to\infty} \epsilon(\mathrm{n}) = 0$ and because

$$\mathrm{d}(x_n, y_{\sigma\circ\tau^{-1}(n)}) \leqslant \epsilon(\tau^{-1}(n)) \quad \text{and} \quad \lim_{n\to\infty} \epsilon(\tau^{-1}(n)) = 0.$$

But from Theorem 2 we also get quantitative results and improvements of known results. One example is Theorem 3. (Compare for example with [4], page 187)

THEOREM 3: *Let $f_1, f_2, \ldots$ be a countable set of continuous functions from $X$ to the complex numbers, with* $\lim_{r\to\infty} \| f_r \| = 0$. $\left( \| f_r \| := \sup_{x\in X} | f(x) |. \right)$
*For every $\epsilon > 0$ and all $(x_n)$ and $(y_n)$ there is a $\tau$ with* $\sum_{k=1}^{\infty} \sup_{r=1,2,\ldots} | f_r(x_k) - f_r(y_{\tau(k)}) | \leqslant \epsilon$.

We shall give two applications:
Let $\mu$ be a nonnegative normed Borel measure on $X$. For a countable convergence-determining class of complex-valued continuous functions

$f_1, f_2, \ldots$ with $\lim_{r\to\infty} \| f_r \| = 0$, the maximal deviation $M_N$ of $(x_n)$ is defined by

$$M_N = \sup_{r=1,2,\ldots} \left| \frac{1}{N} \cdot \sum_{n=1}^{N} f_r(x_n) - \int_X f \, \mathrm{d}\mu \right|.$$

The sequence $(x_n)$ is $\mu$-uniformly distributed in $X$, if and only if $\lim_{N\to\infty} M_N = 0$. (For the result and the definitions see [4], Chapter 3).

COROLLARY 1: *For given $f_1, f_2, \ldots$ like above, for every $\epsilon > 0$ and for all $(x_n)$, $(y_n)$ there is a $\tau$, such that for the maximal deviations $M_N$ of $(x_n)$ and $M_N^*$ of $(y_{\tau(n)})$ we have*

$$| M_N - M_N^* | \leqslant \frac{\epsilon}{N} \quad \textit{for all } N.$$

PROOF: Follows immediately from Theorem 3.

COROLLARY 2: *For every sequence $(y_n)$ on the one-dimensional torus $T$ and every $\epsilon > 0$, there is a $\tau$, such that for all $N$, $h \in \mathbb{N}$:*

$$\frac{1}{h} \cdot \left| \sum_{k=1}^{N} \mathrm{e}^{2\pi i h y_{\tau(k)}} \right| < \frac{3+\sqrt{5}}{4} + \epsilon.$$

PROOF: $f_1, f_2, \ldots$ with $f_h(x) := \frac{1}{h} \cdot \mathrm{e}^{2\pi i h x}$ are like in Corollary 1. Take $(x_n) = \{n\alpha\}$ with $\alpha = \frac{1+\sqrt{5}}{2}$, then for all $N$: $M_N \leqslant \frac{3+\sqrt{5}}{4N}$, and by Corollary 1 the result follows.

However Theorem 2 doesn't say anything about the speed of the convergence of $\mathrm{d}(x_n, y_{\tau(n)})$. Theorem 1 shows that 'uniform convergence' is not possible, so we ask if for any given $(x_n)$, there is an $(\epsilon(n))$ with $\lim_{n\to\infty} \epsilon(n) = 0$, such that for all $(y_n)$ there is a $\tau$ with $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ for all $n$. An answer to the question is given by Theorem 4. (In the following we restrict ourselves to monotonically decreasing sequences $(\epsilon(n))$).

DEFINITION: For $(x_n)$ and $N \in \mathbb{N}$ let $d_N(x) := \min_{n \leqslant N} \mathrm{d}(x, x_n)$ be called the $N$-th dispersion-function of $(x_n)$ and $d_N := \sup_{x\in X} \mathrm{d}_N(x)$ the $N$-th dispersion of $(x_n)$ (or just the dispersion of the finite sequence $(x_n)$, $n = 1, 2, \ldots, N$).

We have: $\lim_{N\to\infty} d_N = 0$ if and only if $(x_n)$ is dense in $X$. (For further properties of the dispersion of a sequence see for example [7]).

THEOREM 4: *For a compact space $X$ without isolated points and which consists only of finitely many connected components, let $(x_n)$ be given with the sequence $(d_n)$ of dispersions. Is $(\epsilon(n))$ monotonically decreasing, with $\epsilon(n) \geqslant 3d_n$ for infinitely many $n$, then for every $(y_n)$ there is a $\tau$ with*

$$\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n) \quad \textit{for all } n.$$

It can easily be shown, that the result is (apart from the constant) best possible in the sense, that for example for every $(x_n)$ on the torus $T$, with dispersions $(d_n)$ we can find a $(z_n)$ with almost the same sequence of dispersions, and a $(y_n)$, such that for all $\tau$ we have $\mathrm{d}(z_n, y_{\tau(n)}) \geqslant 2d_n$ for infinitely many $n$.

However the result is not best possible in the sense, that for every monotonically decreasing $(\epsilon(n))$ with $\sum_{n=1}^{\infty} \epsilon(n) = \infty$, there is a sequence $(x_n)$ on the torus, such that for all $(y_n)$ there is a $\tau$ with $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ for all $n$. To show this, take for example $x_n := \sum_{k=1}^{n} \epsilon(k)$ modulo 1 and the assertion easily follows.

In fact it is also not difficult to see, that $\sum_{n=1}^{\infty} \epsilon(n) = \infty$ is a necessary condition, but it could be possible, that there is a $(x_n)$ such that for every monotonically decreasing $(\epsilon(n))$ with $\sum_{n=1}^{\infty} \epsilon(n) = \infty$ and all $(y_n)$ there is a $\tau$ with $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ for all $n$.

In the special case $X = I^s$ the $s$-dimensional unit cube, an answer to this follows from Theorem 5 in the form of Corollary 3. Theorem 5 is also of some interest for its own, because it may be considered as an analogon to some results on the discrepancyfunction of a given sequence. (See Schmidt [8], [9])

THEOREM 5: *For every function $f: \mathbb{N} \to R^+$ the following two assertions are equivalent*:

(*a*) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k \cdot f(k)} = \infty$

(*b*) *For all $(x_n)$ dense in $I^s$ we have for almost all $x \in I^s$*:

$$\sum_{k=1}^{\infty} \frac{d_k^s(x)}{f(k)} = \infty.$$

Note that it is well possible, that the sequence $(d_n(x))$ is for all $x \in X$ decreasing essentially faster than $(d_n)$.

COROLLARY 3: *For every* $(x_n)$ *dense in* $I^s$, *the measure of the* $(y_n)$ *in* $(I^s)^\infty$, *such that for every* $(\epsilon(n))$ *with* $\sum_{n=1}^{\infty} \epsilon(n) = \infty$ *there is a* $\tau$ *with* $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ *for all* $n$, *is equal to zero.*

## Proofs of the results

LEMMA: *For* $(x_n)$ *and* $(\epsilon(n))$ *given, the following two assertions are equivalent*:

(*i*) *For every* $(y_n)$ *there is a* $\tau$ *with* $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ *for all* $n \in \mathbb{N}$.
(*ii*) *For every* $y \in X$ *there are infinitely many* $n \in \mathbb{N}$ *with* $y \in K_{\epsilon(n)}(x_n)$.

PROOF: (i) $\to$ (ii): Let $y \notin K_{\epsilon(n)}(x_n)$ for $n \geqslant N_0$ and $y_n = y$ for $n = 1, 2, \ldots, N_0$ and $y_n$ arbitrary for $n > N_0$ such that $(y_n)$ is dense in $X$. For every $\tau$ there is a $k \geqslant N_0$ with $\tau(k) \leqslant N_0$ and so:

$$\mathrm{d}(x_k, y_{\tau(k)}) = \mathrm{d}(x_k, y) > \epsilon(k).$$

(ii) $\to$ (i): For $(y_n)$ given, let $\tau(n) := \min\{k \mid y_k \in K_{\epsilon(n)}(x_n),\ k \neq \tau(1)$ for every $1 < n\}$.

$\tau$ is well-defined, because $(y_k)$ is dense and so for every $n$ there are infinitely many $k$ with $\mathrm{d}(y_k, x_n) < \epsilon(n)$. $\tau$ is clearly injective. $\tau$ is also surjective, because for a given integer $k$, by (ii) we have: $y_k \in K_{\epsilon(n)}(x_n)$ for infinitely many $n$, say $y_k \in K_{\epsilon(n_i)}(x_{n_i})$ for $i = 1, 2, \ldots$. So by the definition of $\tau$ there is an $i$ with $1 \leqslant i \leqslant k$ and $\tau(n_i) = k$. Therefore $\tau$ is a permutation with $\mathrm{d}(x_n, y_{\tau(n)}) \leqslant \epsilon(n)$ for all $n$.

PROOF OF THEOREM 1: (i) $\to$ (ii): Let $\epsilon(n) < d$ for $n > N_0$ and $\liminf_{n\to\infty} \epsilon(n) = 0$. Let $x'$ and $y'$ be such, that $d = \mathrm{d}(x', y')$ and for a given dense sequence $(z_n)$ let $(x_n)$ be defined by:
$x_1 = y'$, $x_2, \ldots, x_{N_0}$ arbitrary and $x_k := z_{n(k)}$ for $k > N_0$ with $n(k) := \min\{j \mid \mathrm{d}(z_j, y') > \epsilon(k),\ j \neq n(1)$ for every $1 < k\}$. $n(k)$ is well-defined because $(z_n)$ is dense and because $\epsilon(k) < d$ for $k > N_0$. We have $\liminf \epsilon(n) = 0$, so for every $\epsilon > 0$ there are infinitely many $k$ with $\epsilon(k) < \epsilon$ and therefore for every integer $n$ with $\mathrm{d}(y', z_n) \geqslant \epsilon$ there is a $k$ with $n = n(k)$. Therefore for every $n \in \mathbb{N}$ with $z_n \neq y'$ there is a $k$ with $n = n(k)$ and so $(x_n)$ is dense in $X$. But $y' \notin K_{\epsilon(n)}(x_n)$ for $n > N_0$ and by the lemma the assertion follows.
(ii) $\to$ (i): This follows immediately by the lemma and because $X$ is compact.

PROOF OF THEOREM 2: We define functions $\pi, \iota : \mathbb{N} \to \mathbb{N}$ in the following way:

(A) $x_1 \to y_{\pi(1)}$ with: $\pi(1) := \min\{j \mid \mathrm{d}(x_1, y_j) \leqslant \epsilon(1)\}$

(B) $y_1 \to x_{\iota(1)}$ with:

$$\iota(1) := \begin{cases} 1 \text{ if } \pi(1) = 1 \\ \min\{ j \mid \mathrm{d}(y_1, x_j) \leqslant \epsilon(2),\ j > 1 \} & \text{if } \pi(1) \neq 1 \end{cases}$$

(C) If $n \geqslant 2$: $x_{\mathrm{n}} \to y_{\pi(n)}$ with:

$$\pi(n) := \begin{cases} k \text{ if there is a } k < n \text{ with } \iota(k) = n \\ \min\{ j \mid \mathrm{d}(x_n, y_j) \leqslant \epsilon(\rho(n)),\ j \geqslant n, \text{ and} \\ \quad j \neq \pi(k) \text{ for } k < n \} \quad \text{else} \end{cases}$$

(D) If $n \geqslant 2$: $y_n \to x_{\iota(n)}$ with:

$$\iota(n) := \begin{cases} k \text{ if there is a } k \leqslant n \text{ with } \pi(k) = n \\ \min\{ j \mid \mathrm{d}(y_n, x_j) \leqslant \epsilon(\psi(n)),\ j > n, \text{ and} \\ \quad j \neq \iota(k) \text{ for } k < n \} \quad \text{else} \end{cases}$$

(E) $\rho(1) = 1$, $\psi(1) := 2$ and for $n \geqslant 2$:

$$\rho(n) := \begin{cases} \psi(n-1) & \text{if in the } n-1\text{-th step of (D)} \\ & \text{the first case happens} \\ \psi(n-1) + 1 & \text{else} \end{cases}$$

$$\psi(n) := \begin{cases} \rho(n) & \text{if in the } n\text{-th step of (C) the first case} \\ & \text{happens} \\ \rho(n) + 1 & \text{else} \end{cases}$$

$\pi$, $\iota$ are well-defined because $(x_n)$ is dense in $X$. Clearly $\pi$ and $\iota$ are injective and we have: $\pi \circ \iota = \iota \circ \pi =$ identity. We define $M \subset \mathbb{N} \times \mathbb{N}$ by $M := \{(n, \pi(n)), (\iota(n), n) \mid n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ there is exactly one element $(a, b) \in M$ with $a = n$ and exactly one element $(c, d) \in M$ with $d = n$, because if for example there are integers $n$, $b$, $d$ with $(n, b) \in \mathbb{N}$ and $(n, d) \in M$, then by the injectivity of $\iota$ and $\pi$ we have: $b = \pi(n)$ and $n = \iota(d)$ and because of $\pi \circ \iota = id$, we have $b = d$.

Now we define a function $F: \mathbb{N} \to M$ in the following form: For $k \in \mathbb{N}$ let $n(k)$ be the maximal $n$ with $\rho(n) = k$ or $\psi(n) = k$. It is easy to see, that $n(k)$ is well-defined.

(I) If $\rho(n(k)) = k$ and $\psi(n(k)) > k$ then $F(k) := (n(k), \pi(n(k)))$

(II) If $\psi(n(k)) = k$ then $F(k) := (\iota(n(k)), n(k))$

If we can show that $F$ is a bijection, then because of the form of $M$, there are permutations $\tau$ and $\sigma$ with $F(k) = (\tau(k), \sigma(k))$ and $\mathrm{d}(x_{\tau(k)}, y_{\sigma(k)}) \leqslant \epsilon(k)$ for all $k \in \mathbb{N}$ and the proof of the theorem then is finished.

First we show, that $F$ is injective:
Assume that $k \neq j$ and $F(k) = F(j)$. It is clear that $F(k)$ and $F(j)$ must be defined one by (I) and one by (II), say: $F(k) = (n(k), \pi(n(k)))$ and $F(j) = (\iota(n(j)), n(j))$. If $n(k) > \pi(n(k))$, then by the definition of $\pi(n(k))$, in the $n(k)$-th step of C) the first case happens, so $\rho(n(k)) = \psi(n(k)) = k$ and this is a contradiction to the fact that $F(k)$ is defined by (I).
If $n(k) \leqslant \pi(n(k))$, that means $\iota(n(j)) \leqslant n(j)$, then the same argument, now for $\iota$ and $F(j)$ also gives a contradiction.

$F$ is also surjective because:
Let $(r, s) \in M$. Is $r \leqslant s$ then $s = \pi(r)$ is defined by the definition of $\pi$ in C) by the second case. Let $k = \rho(r)$, then $r = n(k)$, $\psi(r) > k$ and $F(k) = (r, \pi(r)) = (r, s)$.

Is $r > s$ then $r = \iota(s)$ is defined by the definition of $\iota$ in D) by the second case. Let $k = \psi(s)$ then $s = n(k)$, $\rho(s+1) > k$ and $F(k) = (\iota(s), s) = (r, s)$.

Proof of Theorem 3: Take any $(\delta(n))$ with $\sum_{n=1}^{\infty} \delta(n) = \epsilon$. For every $k \in \mathbb{N}$ there is a $h(k) \in \mathbb{N}$ with $\| f_r \| \leqslant \frac{\epsilon}{2^{k+1}}$ for all $r > h(k)$.
For $\delta > 0$ and $k \in \mathbb{N}$ let $M_k(\delta) := \max\limits_{r \leqslant h(k)} \max\limits_{\substack{x, y \in X \\ d(x,y) \leqslant \delta}} | f_r(x) - f_r(y) |$. Take $\epsilon(k) := M_k^{-1}(\delta(k)) > 0$ and permutations $\sigma$ and $\pi$ with:

$$d(x_{\sigma(k)}, y_{\pi(k)}) \leqslant \epsilon(k) \quad \text{for all } k.$$

For $N \in \mathbb{N}$ let $S(N) \in \mathbb{N}$ be so large, that $\sigma(k) > N$ for all $k > S(N)$. Then we have with $\tau := \pi \circ \sigma^{-1}$:

$$\sum_{k=1}^{N} \sup_{r \in \mathbb{N}} | f_r(x_k) - f_r(y_{\tau(k)}) | \leqslant \sum_{k=1}^{S(N)} \sup_{r \in \mathbb{N}} | f_r(x_{\sigma(k)}) - f_r(y_{\pi(k)}) |$$

$$\leqslant \sum_{k=1}^{S(N)} \max\left( \frac{\epsilon}{2^k}, \max_{r \leqslant h(k)} | f_r(x_{\sigma(k)}) - f_r(y_{\pi(k)}) | \right)$$

$$\leqslant \max\left( \epsilon, \sum_{k=1}^{S(N)} M_k(\epsilon(k)) \right) \leqslant \epsilon.$$

Proof of Theorem 4: By the lemma we have to show, that for every $y \in X$, there are infinitely many $n \in \mathbb{N}$ with $y \in K_{\epsilon(n)}(x_n)$. Let $n(1) < n(2) < \dots$ be an infinite sequence of integers with $\epsilon(n(k)) \geqslant 3 d_{n(k)}$ for all $k$. Let $k(0) := 1$ and for every $m \in \mathbb{N}$ let $k(m)$ be the smallest integer with:

(a) $\min_{\substack{i,j \leqslant n(k(m-1)) \\ x_i \neq x_j}} \mathrm{d}(x_i, x_j) > 2d_{n(k(m))}$ (this condition is also said to be satisfied, if $x_i = x_j$ for all $i$, $j \leqslant n(k(m-1))$) and

(b) such, that for every $x_i$ with $i \leqslant n(k(m-1))$ there is a $x \in X$ which is an element of the same component as $x_i$, and $\mathrm{d}(x, x_i) > d_{n(k(m))}$.

Now we show, that by $K_{\epsilon(n)}(x_n)$, $n = 1, 2, \ldots, n(k(m))$, $X$ will be covered $m+1$ times. $K_{d_{n(k(m))}}(x_n)$, $n = 1, 2, \ldots, n(k(m))$ and therefore $K_{\epsilon(n)}(x_n)$, $n = 1, 2, \ldots, n(k(m))$ are coverings of $X$ for all $m \geqslant 0$. Let $m \geqslant 1$, then because of the choice of $k(m)$, for every $i \leqslant n(k(m-1))$ there is a $j$ with $n(k(m-1)) < j \leqslant n(k(m))$ and $\mathrm{d}(x_t, x_y) \leqslant 2d_{n(k(m))}$, and so:

$K_{d_{n(k(m))}}(x_i) \subseteq K_{3d_{n(k(m))}}(x_j)$ and therefore $K_{3d_{n(k(m))}}(x_n)$, $n = n(k(m-1)) + 1, \ldots, n(k(m))$ covers the space $X$ for every $m$. Because $(\epsilon(n))$ is monotonically decreasing, and $\epsilon(n(k(i))) \geqslant 3d_{n(k(i))}$ for all $i$, the result follows.

Proof of Theorem 5: (a) $\to$ (b): $\lambda$ shall denote the $s$-dimensional Lebesgue-measure.

For any $L > 0$ let $P(L)$ be the set of all $x \in I^s$ with $\sum_{k=1}^{N} \frac{d_k^s(x)}{f(k)} \leqslant \mathrm{L}$ for all $N \in \mathbb{N}$. For $t \geqslant 0$ the measure of the $z \in I^s$ with $d_n(z) < t$ is less or equal to $\sum_{k=1}^{n} \lambda(K_t(x_k)) = c_s \cdot n \cdot t^s$ where $c_s$ is a positive constant, depending only on $s$. So the measure of the $z \in I^s$ with $d_n(z) \geqslant t$ is larger than or equal to $1 - c_s n t^s$. Let $t_n = \left(\frac{\lambda(P(L))}{2 \cdot c_s \cdot n}\right)^{1/s}$ and $M_n(L)$ be the set of the $x \in P(L)$ with $d_n(x) \geqslant t_n$, then $\lambda(M_n(L)) \geqslant \frac{\lambda(P(L))}{2}$ and $L \cdot \lambda(P(L)) \geqslant \int_{P(L)} \sum_{k=1}^{N} \frac{d_k^s(x)}{f(k)}\, \mathrm{d}x \geqslant \sum_{k=1}^{N} \int_{M_k(L)} \frac{d_k^s(x)}{f(k)} \mathrm{d}x$
$\geqslant \frac{(\lambda(P(L)))^2}{4 \cdot c_s} \cdot \sum_{k=1}^{N} \frac{1}{k \cdot f(k)}$ for all $N \in \mathbb{N}$.

Because $\sum_{k=1}^{\infty} \frac{1}{k \cdot f(k)} = \infty$, we have $\lambda(P(L)) = 0$ for all $L$ and from this (a) follows.

(b) $\to$ (a): It is not difficult to construct a sequence $(x_n)$ in $I^s$ with $d_n < c \cdot n^{-1/s}$ for all $n$ and a constant $c$. See for example [7]. Now by b) there is a $x \in I^s$ with $\sum_{k=1}^{\infty} \frac{d_k^s(x)}{f(k)} = \infty$ and therefore:

$$\infty = \sum_{k=1}^{\infty} \frac{d_k^s(x)}{f(k)} \leqslant \sum_{k=1}^{\infty} \frac{d_k^s}{f(k)} \leqslant c^s \cdot \sum_{k=1}^{\infty} \frac{1}{k \cdot f(k)}$$

and the proof is finished.

PROOF OF COROLLARY 3: Is $x$ such that $\sum_{k=1}^{\infty} d_k(x) = \infty$, $\epsilon(n) := \frac{d_n(x)}{2}$ and $(z_n)$ a sequence with $z_n = x$ for a $n \in \mathbb{N}$, then there is no $\tau$ with $d(x_n, z_{\tau(n)}) < \epsilon(n)$ for all $n$. So if there is a $\tau$ with $d(x_n, y_{\tau(n)}) < \epsilon(n)$ for all $n$, then every element $y_n$ of $(y_n)$ must not be equal to a $x \in I^s$ with $\sum_{n=1}^{\infty} d_n(x) = \infty$. So by Theorem 5 and by the definition of the usual product-measure in $(I^s)^\infty$ the result follows.

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